Spectral properties of materials cut in half

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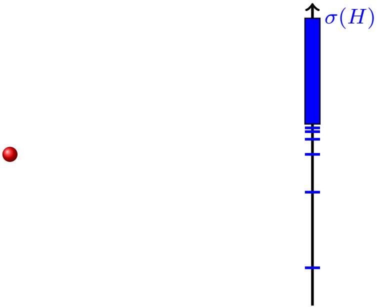
Goal of the talk

- Make a connection between spectral properties of materials, and electronic transport
- The case of periodic materials.
- The case of periodic materials, cut in half.

Start with a single atom in $\mathbb{R}^d$. We study the spectrum of the Schrödinger operator

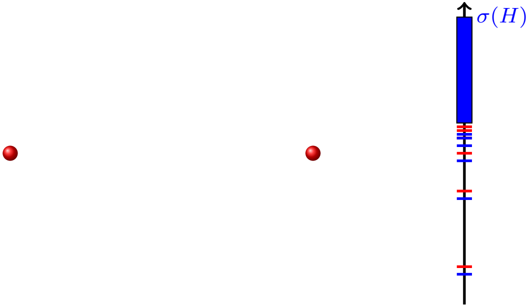
$$H = -\Delta + V(\mathbf{x}), \quad \text{e.g.} \quad V(\mathbf{x}) = \frac{-Z}{|\mathbf{x}|}.$$



- Discrete spectrum (= eigenvalues), and continuous/essential spectrum.
- lowest part of the spectrum = ground state energy, then excited state energy.
- An electron needs energy to *jump* from one level to the next (*quantum*).

Then take two atoms in $\mathbb{R}^d$.

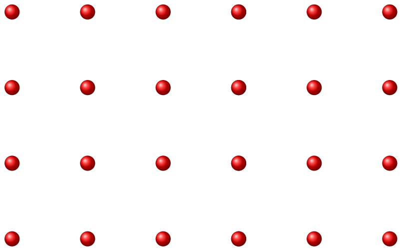
$$H = -\Delta + V\left(\mathbf{x} - \frac{R}{2}\right) + V\left(\mathbf{x} + \frac{R}{2}\right).$$

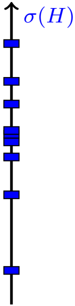


- When $R = \infty$, the spectrum is copied twice (each eigenvalue doubles its multiplicity);
- When $R \gg 1$, *tunnelling* effect = interaction of eigenvectors $\implies$ splitting of the eigenvalues;
- The eigenvectors are delocalized between the two atoms;

Now take an infinity of atoms in $\mathbb{R}^d$, located along a lattice (= material)

$$H = -\Delta + \sum_{\mathbf{v} \in R\mathbb{Z}^d} V(\mathbf{x} - \mathbf{v})$$





- When $R = \infty$, each eigenvalue is of infinite multiplicity;
- When $R \gg 1$, each eigenvalue becomes a **band of essential spectrum**;
- Each band represents « *one electron per unit cell* »;
- When R decreases, the bands may overlap.

The spectrum of $-\Delta + V$ with V-periodic has a band-gap structure!

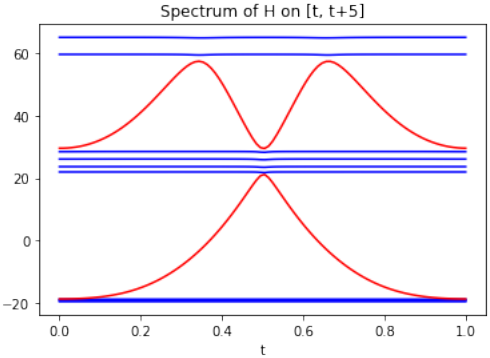
Rigorous proof using the *Bloch transform* ($\sim$ discrete version of the Fourier transform).

Motivation: Spectral pollution

Let's compute numerically the spectrum of the (simple, one-dimensional) operator

$$H := -\partial_{xx}^2 + V(x), \quad \text{with} \quad V(x) = 50 \cdot \cos(2\pi x) + 10 \cdot \cos(4\pi x).$$

The potential V is 1-periodic. We expect a band-gap structure for the spectrum.
We study H in a box $[t, t + L]$ with Dirichlet boundary conditions, and with finite difference.



Depending on where we fix the origin t, the spectrum differs...
There are branches of spurious eigenvalues = spectral pollution (they appear for all L).
The corresponding eigenvectors are edge modes: they are localized near the boundaries.

In this talk: understand why edge modes *must* appear.

Setting

Let V be a 1-periodic potential, and consider the cut (one-dimensional) Hamiltonian

$$H_t^\sharp = -\partial_{xx}^2 + V(x-t) \quad \text{on} \quad L^2(\mathbb{R}^+),$$

with Dirichlet boundary conditions, that is with domain $H^2(\mathbb{R}^+) \cap H_0^1(\mathbb{R}^+)$.
Since V is 1-periodic, the map $t \mapsto H_t^\sharp$ is also 1-periodic.

Theorem (Korotyaev 2000, Hempel Kohlmann 2011, DG 2020)

In the n-th essential gap, there is a flow of n eigenvalues going downwards as t goes from 0 to 1.
In addition, these eigenvalues are simple, and their associated eigenvectors are exponentially localised.

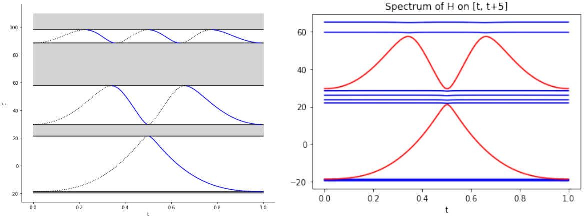


Figure: (left) Spectrum of $H^\sharp(t)$ for $t \in [0, 1]$. (right) Spectrum of the operator on $[t, t+L]$.

We provide here two proofs, applications, and extensions of this theorem.

E. Korotyaev, Commun. Math. Phys., 213(2):471–489, 2000.
R. Hempel and M. Kohlmann, J. Math. Anal. Appl., 381(1):166–178, 2011.
D. Gontier, J. Math. Phys. 61, 2020.

First proof: «compute» everything

Preliminaries.

Potential: Let $V \in C^1(\mathbb{R},\mathbb{R})$ be any potential (not necessarily 1-periodic).

Hamiltonian: $H := -\partial_{xx}^2 + V$ as an operator on $L^2(\mathbb{R})$.
Associated ODE: $-u'' + V(x)u = Eu, \quad \text{on} \quad \mathbb{R}$.
Vector space of solutions: Let $\mathcal{L}_V(E)$ denote the vectorial space of solutions of the ODE.
Since it is a second order ODE, $\dim \mathcal{L}_V(E) = 2$, and

$$\mathcal{L}_V(E) = \mathrm{Ran}\,\{c_E, s_E\}, \quad \begin{cases} -c_E'' + Vc_E = Ec_E \\ c_E(0) = 1,\ c_E'(0) = 0 \end{cases}, \quad \begin{cases} -s_E'' + Vs_E = Es_E \\ s_E(0) = 0,\ s_E'(0) = 1 \end{cases}.$$

Lemma (definition?)

$E \in \mathbb{R}$ is an eigenvalue of H iff $\mathcal{L}_V(E) \cap L^2(\mathbb{R}) \neq \emptyset$.

Transfer matrix

$$T_E(x) := \begin{pmatrix} c_E(x) & c_E'(x) \\ s_E(x) & s_E'(x) \end{pmatrix}.$$

Lemma

For all $x \in \mathbb{R}$, we have $\det T_E(x) = 1$

Indeed, $\det T_E$ is the Wronskian of the ODE. At $x = 0$, we have $T_E(0) = \mathbb{I}_2$, and

$$(\det T_E)' = \left(c_E s_E' - s_E c_E'\right)' = c_E s_E'' - s_E c_E'' = c_E(V-E)s_E - s_E(V-E)c_E = 0.$$

Case of periodic potentials.

We now assume that V is 1-periodic.

If $u(x)$ is solution to the ODE, then so is $u(\cdot + 1)$. In particular there are constants $\alpha, \beta, \gamma, \delta$ such that

$$\begin{cases} c_E(x+1) = \alpha c_E(x) + \beta s_E(x) \\ s_E(x+1) = \gamma c_E(x) + \delta s_E(x). \end{cases} \quad \text{or equivalently} \quad T_E(x+1) = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} T_E(x).$$

At $x = 0$, we recognise $T_E(x = 1)$, so $\boxed{T_E(x+1) = T_E(1) T_E(x)}$.

So for any solution $u \in \mathcal{L}_E$, we have

$$\begin{pmatrix} u(x+n) \\ u'(x+n) \end{pmatrix} = [T_E(1)]^n \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix}.$$

$\Longrightarrow$ **The behaviour of solutions at infinity is given by the singular values of $T_E(1)$.**

If λ_1 and λ_2 are the singular values of $T_E(1)$, then

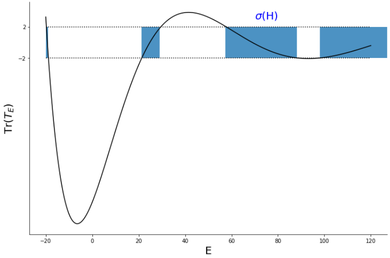
- $\lambda_1 \lambda_2 = \det T_E(1) = 1$.
- $\lambda_1 + \lambda_2 = \mathrm{Tr}(T_E) \in \mathbb{R}$.

Two cases.

- if $|\lambda_1| > 1$, then $|\lambda_2| < 1$. This implies $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\boxed{|\mathrm{Tr}(T_E)| > 2}$.
  There is one mode exponentially increasing at $+\infty$ and exponentially decreasing at $-\infty$.
  There is one mode exponentially increasing at $-\infty$ and exponentially decreasing at $+\infty$.
  The elements of $\mathcal{L}_E$ cannot be approximated in L^2, which implies $E \notin \sigma(H)$.
- if $|\lambda_1| = 1$, then $|\lambda_2| = 1$. This implies $|\lambda_1| = 1$, $\lambda_2 = \overline{\lambda_1}$ and $\boxed{|\mathrm{Tr}(T_E)| \leq 2}$.
  All solutions in $\mathcal{L}_E$ are bounded (quasi-periodic).
  All solutions in $\mathcal{L}_E$ can be approximated in L^2, which implies $E \in \sigma_{\mathrm{ess}}(H)$.

The spectrum of H can be read from the (continuous) map $E \mapsto \mathrm{Tr}(T_E)$.

Example: for $V(x) := 50 \cdot \cos(2\pi x) + 10 \cdot \cos(4\pi x)$,



Theorem (Spectrum of 1-dimensional periodic operators)

If V is 1-periodic, the spectrum $H := -\partial_{xx}^2 + V(x)$ is purely essential (no eigenvalues). It is composed of bands:

$$\sigma(H) = \sigma_{\text{ess}}(H) = \bigcup_{n \geq 1} [E_n^-, E_n^+].$$

Essential gap: The interval $g_n := (E_n^+, E_{n+1}^-)$ is called the n-th essential gap of the operator H.

Physical interpretation:

- If $E \in \sigma(H)$, electrons with energy E can travel through the medium (quasi-periodic solutions);
- If $E \notin \sigma(H)$, electrons cannot propagate: they are exponentially attenuated in the medium.

Example: If $V = 0$, then $H = -\partial_{xx}^2$. We have $-u'' = Eu$ if $u = \alpha \mathrm{e}^{\mathrm{i}\sqrt{E}} + \beta \mathrm{e}^{-\mathrm{i}\sqrt{E}}$.

- If $E \geq 0$, $\sqrt{E} \in \mathbb{R}$, and we have *travelling waves*;
- If $E < 0$, $\sqrt{E} \in \mathrm{i}\mathbb{R}$, and we have *exponential waves*.
- The spectrum of $-\partial_{xx}^2$ is $[0, \infty)$.

How about the half system?

Let $E \notin \sigma(H)$. The set of solutions can be split as

$$\mathcal{L}_V(E) = \mathcal{L}_V^+(E) \oplus \mathcal{L}_V^-(E), \quad \mathcal{L}_V^\pm(E) := \left\{ u \in \mathcal{L}_V(E), \quad u \in L^2(\mathbb{R}^\pm) \right\}.$$

They are both of dimension 1.

We define the discrete set $\mathcal{Z}_V^+[u] := u^{-1}(\{0\})$ for $u \in \mathcal{L}_V^+(E)$.

- The set $\mathcal{Z}_V^+ \subset \mathbb{R}$ depends only on $\mathcal{L}_V^+$ (not on u).
- The set $\mathcal{Z}_V^+$ is 1-periodic (because if $u \in \mathcal{L}_V^+(E)$, then $u(\cdot - 1) \in \mathcal{L}_V^+(E)$, hence $u(\cdot - 1) = \alpha u$).

Key remark: If $0 \in \mathcal{Z}^+$, then E is an eigenvalue of $H^\sharp$ (with corresponding eigenspace $\mathcal{L}_V^+$).

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Key remark: If $0 \in \mathcal{Z}^+$, then E is an eigenvalue of $H^\sharp$ (with corresponding eigenspace $\mathcal{L}_V^+$).

Consider now $V_t(x) = V(x-t)$, $H_t = -\partial^2_{xx} + V_t$, $\mathcal{L}_t^\pm(E) = \mathcal{L}_{V_t}^\pm(E)$, $\mathcal{Z}_t^+ := \mathcal{Z}_{V_t}^+$, ...

- We have $\mathcal{Z}_t^+ = \mathcal{Z}_0^+ + t$ (the set of roots is shifted);
- If $0 \in \mathcal{Z}_t^+$, then $E \in \sigma\left(H_t^\sharp\right)$.

So, the number of $t \in [0,1)$ so that $E \in \sigma\left(H_t^\sharp\right)$ equals the number of points of $\mathcal{Z}_V^+$ in $(-1, 0]$.

Lemma

If E is in the n-th gap, and if $u \in \mathcal{L}_V(E)$ is any non null solution, then u has n zeros in $(-1, 0]$.

Proof.

Step 1. If $x_0 \in \mathcal{Z}_0$, then $x_0 + 1 \in \mathcal{Z}_0$.
In particular, $(E, u_{t=0}|_{[x_0, x_0+1]})$ is an eigenpair of the Dirichlet problem

$$\begin{cases} \left(-\partial_{xx}^2 + V(x)\right) u = Eu, \quad \text{on} \quad (x_0, x_0+1) \\ u(x_0) = u(x_0+1) = 0. \end{cases}$$

We want to evaluate $\mathcal{M}$, the number of roots of u in $[x_0, x_0+1)$

Step 2 (deformation). For $0 \le s \le 1$, we introduce $(E(s), \widetilde{u_s})$ the Dirichlet eigenpair of

$$\begin{cases} \left(-\partial_{xx}^2 + sV(x)\right) \widetilde{u_s} = E_s \widetilde{u_s}, \quad \text{on} \quad (x_0, x_0+1) \\ \widetilde{u_s}(x_0) = \widetilde{u_s}(x_0+1) = 0. \end{cases}$$

which is a continuation of (E, u) at $s = 1$, and by $\mathcal{M}_s$ the number of zeros of $\widetilde{u_s}$ in the interval $[x_0, x_0+1)$.

By continuity, $E(s)$ cannot cross the essential spectrum, so $E(s)$ is always in the n-th gap.
By Cauchy-Lipschitz, two zeros cannot merge, so $\mathcal{M}_s$ is independent of s, and $\mathcal{M} = \mathcal{M}_{s=1}$.
At $s = 0$, we recover the usual Laplacian (hence $u_{s=0}(x) \approx \sin(\pi(n+1)x)$)

We deduce that $E(s)$ is the branch of n-th eigenvalues, and that $\mathcal{M} = n$.

Lemma

If $\widetilde{E}(t)$ is a branch of eigenvalues of $H_t^\sharp$ in the gap, then $E'(t) < 0$ (all branches go downwards).

If $\left(\widetilde{E}(t), \widetilde{u}(t)\right)$ is a branch of eigenpair for $H_t^\sharp$ with $\|\widetilde{u}_t\|^2 = 1$. We have $H(t)\widetilde{u}(t) = \widetilde{E}(t)$, and $\widetilde{E}(t) = \langle \widetilde{u}(t), H(t)\widetilde{u}(t)\rangle$. Differentiating in t gives (Hellmann-Feynman argument)

$$\begin{aligned}\widetilde{E}'(t) &= \langle \widetilde{u_t}, \partial_t H_t \widetilde{u_t}\rangle + \langle \partial_t \widetilde{u_t}, H_t \widetilde{u_t}\rangle + \langle \widetilde{u_t}, H_t \partial_t \widetilde{u_t}\rangle \\ &= \langle \widetilde{u_t}, (\partial_t V_t)\, \widetilde{u_t}\rangle + \widetilde{E}(t) \underbrace{\left(\langle \partial_t \widetilde{u_t}, \widetilde{u_t}\rangle + \langle \widetilde{u_t}, \partial_t \widetilde{u_t}\rangle\right)}_{=\partial_t \|\widetilde{u_t}\|^2 = 0} = \int_0^\infty (\partial_t V_t)\, |\widetilde{u}_t|^2 \mathrm{d}x.\end{aligned}$$

On the other hand, if $u(t) = u(x-t)$ is a branch of functions in $\mathcal{L}_t^+(E)$ (E is fixed now), then

$$(-\partial_{xx}^2 + V_t - E)u_t = 0.$$

These functions do not satisfy Dirichlet in general! Differentiating in t gives

$$(-\partial_{xx}^2 + V_t - E)\partial_t u_t + (\partial_t V_t)\, u_t = 0.$$

We multiply by u_t and integrate on $\mathbb{R}^+$. We integrate by part and obtain (now we have boundary terms)

$$\int_0^\infty (\partial_t V_t)\, |u_t|^2 = \partial_x u_t(0) \partial_t u_t(0).$$

Of course, at the point t, we have $u_t = \widetilde{u_t}$. Since $u_t(x) = u(x-t)$, we obtain

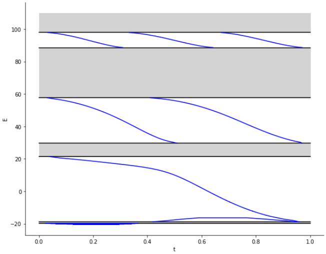
$$\widetilde{E}'(t) = -|\partial_t u_t|^2(0) < 0.$$

An application: junctions and dislocations

The Spectral flow

If $t \mapsto A_t$ is a 1-periodic and *continuous* family of self-adjoint operators, and if $E \notin \sigma_{\rm ess}(A_t)$ for all t, we can define the Spectral flow

$$\mathrm{Sf}\,(A_t, E) := \text{number of eigenvalues going } \textbf{downwards} \text{ in the essential gap where } E \text{ lies.}$$



The previous result can be formulated as:

$$\mathrm{Sf}\left(H_t^\sharp, E\right) = \mathcal{N}(E), \quad \mathcal{N}(E) := \text{number of bands below } E.$$

Facts :

- If $t \mapsto K_t$ is a 1-periodic continuous family of **compact** operators, then

$$\mathrm{Sf}\,(A_t, E) = \mathrm{Sf}\,(A_t + K_t, E)\,.$$

- If $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then

$$\mathrm{Sf}\,(f\,(A_t)\,, f(E)) = \mathrm{Sf}\,(A_t, E)\,.$$

Application: Junctions between two materials

Let V_L and V_R be two 1-periodic potentials. We consider the junction operator

$$H_t^{\text{junction}} := -\partial_{xx}^2 + [V_L(x)\mathbb{1}(x<0) + V_R(x-t)\mathbb{1}(x>0)] \quad \text{on} \quad L^2(\mathbb{R})$$

Theorem

If $E \in \mathbb{R}$ is in the resolvent set of all left and right bulk operators, then

$$\text{Sf}(H_t^{\text{junction}}, E) = \mathcal{N}^+(E).$$

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Idea of the proof

Consider the cut Hamiltonian

$$H_t^{\text{cut}} := -\partial_{xx}^2 + [V_L(x)\mathbb{1}(x<0) + V_R(x-t)\mathbb{1}(x>0)] \quad \text{on} \quad L^2(\mathbb{R}) = L^2(\mathbb{R}^-) \cup L^2(\mathbb{R}^+),$$

and with Dirichlet boundary conditions at $x = 0$.

For any Σ negative enough (below the essential spectra of all operators), we have

$$K_t := \left(\Sigma - H_t^{\text{cut}}\right)^{-1} - \left(\Sigma - H_t^{\text{junction}}\right)^{-1} \quad \text{is compact (here, it is finite rank).}$$

So

$$\mathrm{Sf}\left(\left(\Sigma - H_t^{\text{junction}}\right)^{-1}, (\Sigma - E)^{-1}\right) = \mathrm{Sf}\left(\left(\Sigma - H_t^{\text{cut}}\right)^{-1}, (\Sigma - E)^{-1}\right).$$

Since $f(x) := (\Sigma - x)^{-1}$ is strictly increasing on $x > \Sigma$, we have

$$\mathrm{Sf}\left(H_t^{\text{junction}}, E\right) = \mathrm{Sf}\left(H_t^{\text{cut}}, E\right) = \mathrm{Sf}\left(H_t^{\sharp,+}, E\right) = \mathcal{N}^+(E).$$

Remarks on this first proof

Good points

Can be generalized in different settings.
Instead of a *flow* of roots (the set $\mathcal{Z}_V^+$), we use the notion of *Maslov index = crossings of Lagrangian planes* (tools of symplectic geometry).

- Vector valued operators
$$H_t := -\Delta + \mathbb{V}_t(x), \quad \text{on} \quad L^2(\mathbb{R}, \mathbb{C}^N).$$
We prove that if $E \notin \sigma(H)$, then $\dim(\mathcal{L}_V^\pm)$ are both of dimension N.
- We can change the boundary conditions (and have a t-dependent boundary conditions). For instance, we prove that for the family of operators
$$H_t^\sharp := -\Delta + V(x), \quad \text{with Robin domain} \quad \sin(\pi t)u(0) = \cos(\pi t)u'(0),$$
we have $\mathrm{Sf}(H_t^\sharp, E) = -1$ in all gaps (including the 0-th one!)

Bad point

Not really adapted to the two-dimensional setting...

Second proof (by Hempel and Kohlmann)

Idea of the proof

Idea: Prove the result in the dislocated case.

Let $L \in \mathbb{N}$ be a (large) integer. Consider the family of operators

$$\mathcal{H}_{L,t}^{\text{junction}} := -\partial_{xx}^2 + \left[V(x)\mathbb{1}(x<0) + V(x-t)\mathbb{1}(x>0)\right], \quad \text{on} \quad L^2([-\tfrac{1}{2}L, \tfrac{1}{2}L + t])$$

with periodic boundary conditions.

- The branches of eigenvalues of $t \mapsto \mathcal{H}_{L,t}^{\text{junction}}$ are continuous;
- At $t=0$, the system is 1-periodic, on a box of size L. Each «band» contributes to L eigenvalues.
- At $t=1$, the system is 1-periodic, on a box of size $L+1$. Each «band» contributes to $L+1$ eigenvalues.

$\Longrightarrow$ The extra eigenvalue must come from an upper band...

$\Longrightarrow$ There is a «spectral flow» of 1 between the second band and the first one
There is a «spectral flow» of 2 between the third band and the second one, ...

A «fun» analogy

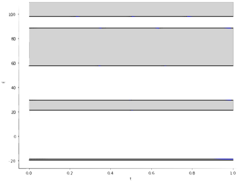
The *«Grand Hilbert Hotel»*
An infinity of floors, an infinity of rooms in each floor.

Idea: each period represents 1 room (per floor), each spectral band represents one floor.

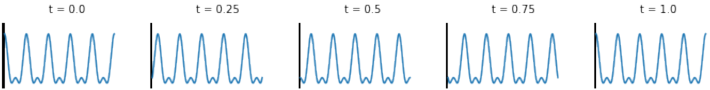
… Floor 3.

… Floor 2.

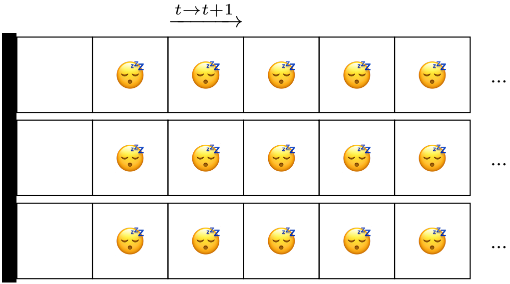
… Floor 1.



As t moves from 0 to 1...



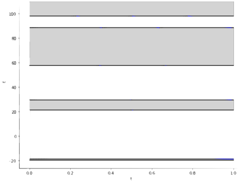
... a new room is created on each floor!



Floor 3.

Floor 2.

Floor 1.



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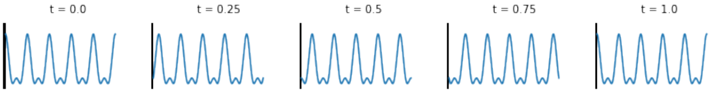

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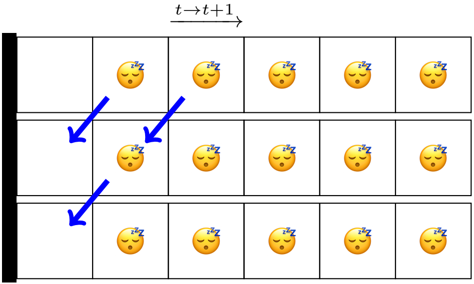

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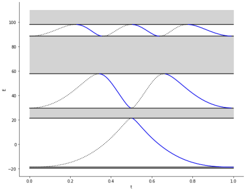


In order to fill the new rooms,

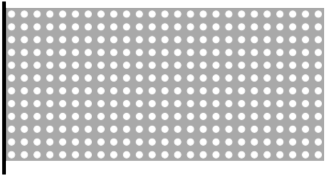
- 1 person from floor 2 must come down to floor 1;
- 2 persons from floor 3 must come down to floor 2;
- and so on.

Remark: the proof can be generalized to higher dimensions!

The two-dimensional case

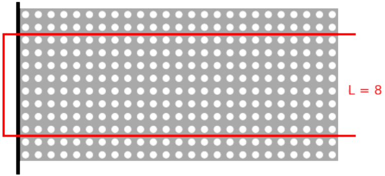
Let V be a $\mathbb{Z}^2$-periodic potential, and we study the edge operator

$$H^\sharp(t) = -\Delta + V(x-t, y), \quad \text{on} \quad L^2(\mathbb{R}_+ \times \mathbb{R}), \quad \text{with Dirichlet boundary conditions.}$$



Let V be a $\mathbb{Z}^2$-periodic potential, and we study the edge operator

$$H^\sharp(t) = -\Delta + V(x - t, y), \quad \text{on} \quad L^2(\mathbb{R}_+ \times \mathbb{R}), \quad \text{with Dirichlet boundary conditions.}$$

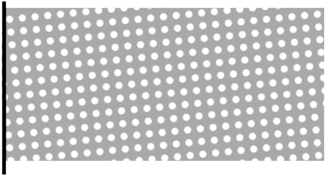


- For $L \in \mathbb{N}$, consider the model in the **tube** $\mathbb{R}_+ \times [0, L]$ with **periodic boundary conditions** in x_2.
- Consider the **«Two-dimensional Grand Hilbert Hotel»**.
- As t moves from 0 to 1, L new rooms are created on each floor.
- Let $L \to \infty$...

There is a spectral flow of **essential spectrum** appearing in each gap.
The corresponding modes can only propagate along the boundary.

The two-dimensional twisted case.

We rotate V by θ.



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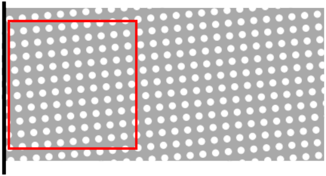


Commensurate case ($\tan\theta = \frac{p}{q}$)

Considering a **Supercell** of size $L = \sqrt{p^2+q^2}$, we recover a $L\mathbb{Z}^2$-periodic potential.

« As t moves from 0 to L, L^2 new rooms are created»

Key remark:

- The map $t \mapsto H_\theta^\sharp(t)$ is now $1/L$-periodic (up to some x_2 shifts)
- So the map $t \mapsto \sigma(H_\theta^\sharp(t))$ is $1/L$ periodic.

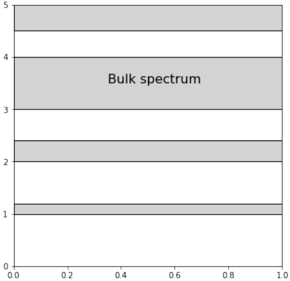
«As t moves from 0 to $\frac{1}{L}$, 1 new room is created»

In-commensurate case ($\tan\theta \notin \mathbb{Q}$, corresponds to $L \to \infty$)

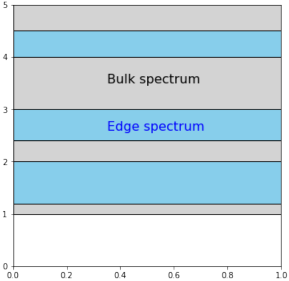
- The spectrum of $H^\sharp(t)$ is independent of t (ergodicity);
- All bulk gaps are filled with edge spectrum!

Theorem (DG, Comptes Rendus. Mathématique, Tome 359 (2021))

If $\tan\theta \notin \mathbb{Q}$, the spectrum of $H^\sharp_\theta$ is of the form (Σ, ∞).



(a) Uncut two-dimensional material



(b) Two-dimensional material with **incommensurate** cut

Idea of the proof

Remark: The map $\theta \mapsto H_\theta$ is not *norm-resolvent* continuous...
so the convergence of the spectrum is not guaranteed, and we need to prove it *by hand*.

Limiting procedure

Consider a sequence $\theta_n \to \theta$, with $\tan(\theta_n) = \frac{p_n}{q_n} \in \mathbb{Q}$, and set $L_n := \sqrt{p_n^2 + q_n^2}$.
By the commensurate case result, there is $t_n \in [0, \frac{1}{L_n}]$ and $\phi_n \in L^2_{\text{per}}(\mathbb{R}^+ \times [0, L_n])$ so that

$$(-\Delta + V_{\theta_n}(t - t_n) - E)\phi_n = 0, \qquad \int_{\mathbb{R}^+ \times [0, L_n]} |\phi_n|^2 = 1.$$

It is tempting to extract a weak-limit of ϕ_n, but this will fail (we would get $\phi_* = 0$ at the end)...

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Idea: Normalize the functions in L^∞

Consider the functions

$$\Psi_n := \frac{\phi_n}{\|\phi_n\|_{L^\infty}}, \quad \text{so that} \quad (-\Delta + V_{\theta_n}(t - t_n) - E)\Psi_n = 0, \quad \|\Psi_n\|_{L^\infty} = 1.$$

(the parameter L_n is no longer here).

$$(-\Delta + V_{\theta_n}(t - t_n) - E)\Psi_n = 0, \quad \|\Psi_n\|_{L^\infty} = 1.$$

Step 1: Control the mass

Consider $x_n \in \mathbb{R}^2$ so that $\Psi_n(x_n) > \frac{1}{2}$.

- Upon shifting the whole system in the x_2-direction (which effectively corresponds to changing t_n), we may assume $x_{n,2} = 0$.
- Since $E \notin \sigma_{\text{ess}}(H)$, the function Ψ_n is exponentially decaying away from the boundary (the bulk is an insulator). So there is $C > 0$ independent of n so that $0 < x_{n,1} < C$ (the full proof uses Combes-Thomas estimates).

Step 2: Regularity and taking the limit

- Since $\|(-\Delta\Psi_n)\| \leq C$, there is $\delta > 0$ so that $\Psi_n(x) > \frac{1}{4}$ for all $x \in \mathcal{B}(x_n, \delta)$.
- Take the limit $n \to \infty$, and sub-sequences. $\Psi_n \to \Psi_*$ weakly-* in L^∞.
- We have, in the distributional sense

$$(-\Delta + V_\theta(x - t^*) - E)\Psi_* = 0.$$

- We have $\|\Psi_*\|_\infty \leq 1$, and since $\int_{\mathcal{B}(0,\delta)} \Psi_* \neq 0$, we have $\Psi_* \neq 0$.
- This implies that $E \in \sigma(H_\theta)$.

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Open question

Is E an eigenvalue of H_θ ($\sim$ Anderson localization), or in the essential spectrum (travelling waves).

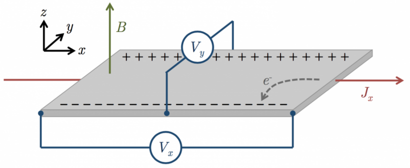
Another application: the definition of the kilo

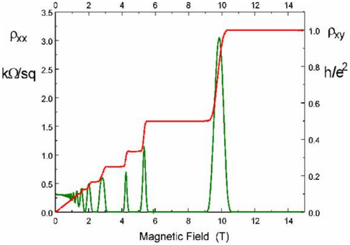
May 20, 2019: New definition of the kg by the *Bureau International des Poids et Mesures* (BIPM)[1] :
"Le kilogramme, symbole kg, est l'unité de masse du SI. Il est défini en prenant la valeur numérique fixée de la constante de Planck, h, égale à $6,626\,070\,15 \times 10^{-34}$ *J.s."*

Question: How do you measure h? How do you measure h with 10^{-9} accuracy?

Comments by von Klitzing[2]: *"The discovery of the QHE led to a new type of electrical resistor [...]. This resistor is universal for all 2D electron systems in strong magnetic fields with an uncertainty of less than one part in* 10^{10}*."*

QHE = Quantum Hall Effect[3] (von Klitzing got Nobel prize in 1985 for discovery of Quantum Hall Effect).





[1] https://www.bipm.org/fr/measurement-units/
[2] von Klitzing, Nature Physics 13, 2017
[3] K. von Klitzing; G. Dorda; M. Pepper, Phys. Rev. Lett. 45 (6): 494–497, 1980.

In this setting, the magnetic field A plays the role of the *pump*.

$$H_B = -\partial_{xx}^2 + (-\mathrm{i}\partial_y + Bx)^2.$$

After a Fourier transform in y, we get

$$H_{B,k_y} = -\partial_{xx}^2 + (k_y + Bx)^2 = -\partial_{xx}^2 + B^2(x-t)^2, \quad \text{with} \quad t = \frac{-k_y}{B}.$$

Lemma

If $B \neq 0$, the bulk Hamiltonian has discrete spectrum. $\sigma(H_B) = |B|(2\mathbb{N}+1)$. (Landau operator).
The edge Hamiltonian $H_{B,t}^\sharp$ has flows of eigenvalues between the Landau levels.
In particular $\sigma(H_B^\sharp) = [|B|, \infty)$.

The *plateaus* observed by von Klitzing correspond to these spectral flows.

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Thank you for your attention!