Sphere packing
The work of Maryna Viazovska

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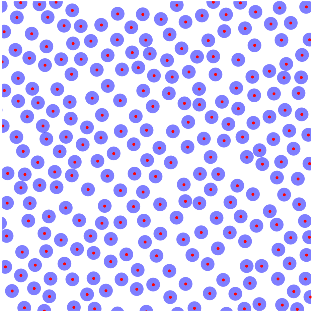
Main goal: find the optimal packing for spheres/balls

Find the densest disposition of (non–overlapping) balls of the same size in $\mathbb{R}^d$.

Consider a collection of centers $\mathscr{C} = \{\mathbf{x}_i\} \subset \mathbb{R}^d$ with $\|\mathbf{x}_i - \mathbf{x}_j\| \geq 2r$ for $i \neq j$, and consider the corresponding union of balls of radius r,

$$\Omega_{\mathscr{C}} := \bigcup_i \mathcal{B}(\mathbf{x}_i, r)$$



The **density** of this disposition is

$$\boxed{\rho(\mathscr{C}) := \liminf_{R \to \infty} \frac{\mathrm{Vol}_d(\Omega_{\mathscr{C}} \cap \mathcal{B}(\mathbf{0}, R))}{\mathrm{Vol}_d(\mathcal{B}(\mathbf{0}, R))}.}$$

Remarks

- We have $\rho(\mathscr{C}) \leq 1$.
- By **scaling**, we may consider only the case $r = 1$.

We want to solve the optimization problem

$$\boxed{\rho_d := \sup \left\{ \rho(\mathscr{C}), \quad \mathscr{C} = \{\mathbf{x}_i\} \subset \mathbb{R}^d, \ \forall i \neq j, \ \|\mathbf{x}_i - \mathbf{x}_j\| \geq 2 \right\}.}$$

What we really want is the optimal *configuration* (if it exists).

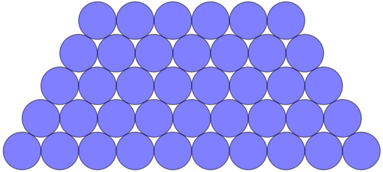
State of the art

Dimension $d = 1$ $\boxed{\rho_1 = 1}$

Trivial!

In dimension 1, the *balls* of radius r are the **intervals** $(a, a+2r)$, and we can **cover** $\mathbb{R}$ with such intervals.

Dimension $d = 2$ $\boxed{\rho_2 = \dfrac{\pi}{\sqrt{12}} \approx 0.907...}$ and the optimal configuration is the **triangular lattice**.



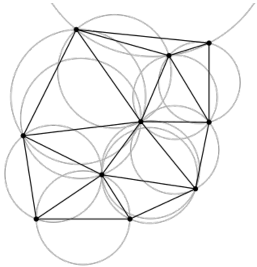
They did not get it right at Jules Destrooper...

Proof by **Lagrange** (1773, among lattices only), **Thue** (1910, incomplete) and **Tóth** (1940).

Short proof by **Chang/Wang** (2010).

The proof by **Chang/Wang**

Consider a *saturated* configuration $\mathscr{C}$ with radius r (one cannot add another point).
Consider a **Delaunay triangulation** of these points (= there are no points inside each circumcircle).



Step 1. In each triangle $T = (ABC)$, the largest angle satisfies $\boxed{\frac{\pi}{3} \le \theta < \frac{2\pi}{3}}$.

Assume $\theta_A \ge \theta_B \ge \theta_C$. The lower bound comes from $\theta_A + \theta_B + \theta_C = \pi$.
Assume $\theta_1 \ge \frac{2\pi}{3}$, so $\theta_C \le \frac{\pi}{6}$, and $\sin(\theta_C) \le \frac{1}{2}$. The radius R of the circumcircle satisfies

$$2R = \frac{AB}{\sin(\theta_C)} \ge \frac{2r}{\frac{1}{2}} \ge 4r.$$

So the **center** of this circle can be added to the configuration $\mathscr{C}$ (contradicts *saturation*).

Step 2. For each triangle $T = (ABC)$, we have $\boxed{\mathrm{Vol}(\Omega_{\mathscr{C}} \cap T) \leq \frac{\pi}{\sqrt{12}} \mathrm{Vol}(T)}$.

We have $\mathrm{Vol}(\Omega_{\mathscr{C}} \cap T) = \frac{1}{2}\pi r^2$ and, since $\frac{\pi}{3} \leq \theta_A \leq \frac{2\pi}{3}$, $\sin(\theta_A) \geq \frac{\sqrt{3}}{2}$, so

$$\mathrm{Vol}(T) = \frac{1}{2} \cdot AB \cdot AC \cdot \sin(\theta_A) \geq \frac{1}{2} \cdot 2r \cdot 2r \cdot \frac{\sqrt{3}}{2} = \sqrt{3} r^2 = \frac{\sqrt{12}}{\pi} \left(\frac{1}{2} \pi r^2 \right).$$

Step 3. For any **finite** collection of triangles $\mathcal{T} := \bigcup T_j$, we have

$$\mathrm{Vol}\left(\Omega_{\mathscr{C}} \cap \mathcal{T}\right) = \sum_j \mathrm{Vol}\left(\Omega_{\mathscr{C}} \cap T_j\right) \leq \frac{\pi}{\sqrt{12}} \sum_j \mathrm{Vol}(T_j) = \frac{\pi}{\sqrt{12}} \mathrm{Vol}(\mathcal{T}).$$

Step 4. Equality iff $\sin(\theta_A) = \frac{\pi}{3}$ for all triangles $\Longleftrightarrow$ all triangles are equilateral.
The "*unique*" optimal configuration is the triangular latttice. □

Simple argument, which proves the result locally, and extends it globally.

Dimension $d = 3$ (*aka* the Kepler conjecture, 1611)

$$\rho_3 = \frac{\pi}{\sqrt{18}} \approx 0.741$$

The optimal configuration is **face centered cubic (fcc) lattice**.

Nope, this is bcc...

Proof by **Gauss** (1831, among lattices only).
Hilbert put this problem in its 18th problem.
Tóth (1953) shows that one only needs to check that fcc is optimal among a huge (but finite) number of configurations.

First tentative by **Hales** in 1998. Submission to *Annals of Mathematics*, but after 4 years and a team of 12 reviewers, the report states that: *they are 99% sure that the proof is correct...*

Hales decides to create a computer program to check the proof (= **Flyspeck**), and completes the proof in 2014 with his team (22 people during $\sim$ 10 years).

The proof is accepted in 2017. It is one of the first *computer assisted proof*.

Dimensions $d \geq 4$?

Nothing is known! (except in dimensions $d = 8$ and $d = 24$, thanks to **Viazowska**, see below).

Theorem (Minkowski bound 1911, proved by Hlawka 1943)

$$\rho_d \geq \frac{\zeta(d)}{2^{d-1}} = O\left(\frac{1}{2^d}\right).$$

"With high probability, a random lattice satisfies this bound".

Remark: for the configuration $\mathbb{Z}^d$ (with $r = \frac{1}{2}$), one finds

$$\rho(\mathbb{Z}^d) = \mathrm{Vol}(\mathcal{B}(\mathbf{0}, \frac{1}{2})) = \omega_d \cdot \frac{1}{2^d} \quad \text{with} \quad \omega_d := \mathrm{Vol}_d(\mathcal{B}(\mathbf{0}, 1)) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \approx \left(\frac{2e\pi}{d}\right)^{d/2} \frac{2}{\sqrt{\pi} d^{3/2}}$$

This is **terrible** lower bound.

Torquato's conjecture. For *random configurations*, it seems that

$$\rho_d \geq O\left(\frac{1}{2^{0.77\, d}}\right).$$

The upper bound by Cohn–Elkies

Definition (Lattice)

A **lattice** of $\mathbb{R}^d$ is a discrete set $\mathbb{L} \subset \mathbb{R}^d$ of the form

$$\mathbb{L} := \mathbf{a}_1\mathbb{Z} \oplus \mathbf{a}_2\mathbb{Z} \oplus \cdots \oplus \mathbf{a}_d\mathbb{Z}, \qquad \text{with} \quad \det(A) \neq 0, \quad \text{where} \quad A := (\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_d).$$

The **unit cell** of a lattice $\mathbb{L}$ is the set $\Lambda := \mathbf{a}_1[0,1) \times \mathbf{a}_2[0,1) \times \cdots \times \mathbf{a}_d[0,1)$.
The **volume** of a lattice $\mathbb{L}$ is $\mathrm{Vol}_d(\Lambda) := \mathrm{Vol}(\mathbb{L}) := |\det(A)|$.
The **radius** of a lattice $\mathbb{L}$ is $r(\mathbb{L}) := \inf\left\{\frac{1}{2}\|\mathbf{x}_i - \mathbf{x}_j\|,\ \mathbf{x}_i \neq \mathbf{x}_j \in \mathbb{L}\right\}$.

The **dual** lattice is

$$\mathbb{L}^* := \mathbf{a}_1^*\mathbb{Z} \oplus \mathbf{a}_2^*\mathbb{Z} \oplus \cdots \oplus \mathbf{a}_d^*\mathbb{Z}, \qquad \text{with} \quad \mathbf{a}_i^* \cdot \mathbf{a}_j = \delta_{ij}, \qquad (A_* = A^{-T}).$$

Fourier transform

For $f : \mathbb{R}^d \to \mathbb{C}$, we denote by $\mathcal{F}_d(f)(\mathbf{k})$ or $\widehat{f}(\mathbf{k})$ its **Fourier transform**, with the convention

$$\boxed{\widehat{f}(\mathbf{k}) = \int_{\mathbb{R}^d} f(\mathbf{x}) \mathrm{e}^{-2\mathrm{i}\pi \mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x}.}$$

In what follows, we focus on nice (Schwartz) functions

$$\mathscr{S} := \mathscr{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d), \quad \widehat{f} \in C^\infty(\mathbb{R}^d) \right\}, \qquad \mathscr{S}_{\mathrm{rad}} := \{ f \in \mathscr{S}, \ f \text{ radial} \}.$$

Lemma (Recap on Fourier transforms)

- **Inverse Fourier transform**. *If* $f \in \mathscr{S}$, *then* $\widehat{\widehat{f}}(\mathbf{x}) = f(-\mathbf{x})$. *(In particular,* $(\mathcal{F}_d)^4 = \mathbb{I}_{\mathscr{S}}$*).*
- **Poisson summation formula**. *For all* $f \in \mathscr{S}(\mathbb{R}^d)$, *and all lattices* $\mathbb{L} \subset \mathbb{R}^d$, *we have*

$$\sum_{\mathbf{R} \in \mathbb{L}} f(\mathbf{x} + \mathbf{R}) = \frac{1}{|\Lambda|} \sum_{\mathbf{K} \in \mathbb{L}^*} \widehat{f}(\mathbf{K}) \mathrm{e}^{2\mathrm{i}\pi \mathbf{K}\cdot\mathbf{x}}.$$

Cohn–Elkies upper bound

Theorem (Cohn–Elkies, *Annals of Mathematics*, 2003)

Assume there is $f : \mathbb{R}^d \to \mathbb{R}$ *(for instance in* $\mathscr{S}$*) so that:*

- $f(\mathbf{x}) \leq 0$ *for all* $|\mathbf{x}| \geq 2r_0$;
- $\widehat{f}(\mathbf{k}) \geq 0$ *for all* $\mathbf{k} \in \mathbb{R}^d$;
- $\widehat{f}(\mathbf{0}) = f(\mathbf{0}) = 1$

Then, for all lattice $\mathbb{L} \subset \mathbb{R}^d$ *with* $r(\mathbb{L}) \geq r_0$*, we have* $|\Lambda| \geq 1$*. In particular,* $\boxed{\rho_d \leq \omega_d \cdot r_0^d}$.

Remark: Taking the *radial average* shows that one can always look for f radial.

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Remark: Taking the *radial average* shows that one can always look for f radial.

Proof of the Theorem

$$1 = f(\mathbf{0}) \geq \sum_{\mathbf{R} \in \mathbb{L}} f(\mathbf{R}) \overset{\text{Poisson}}{=} \frac{1}{|\Lambda|} \sum_{\mathbf{K} \in \mathbb{L}^*} \widehat{f}(\mathbf{K}) \geq \frac{1}{|\Lambda|} \widehat{f}(\mathbf{0}) = \frac{1}{|\Lambda|}. \quad \square$$

We say that the pair $(f, \mathbb{L})$ is **magical** if they satisfy the equality in the previous computation. It implies in particular that $\mathbb{L}$ is an optimal configuration.

The authors *optimized* the function f in all dimensions, and found a **numerical upper bound** which matches the best known lower ones in dimensions $d = 1, 2, 8, 24$, up to 100 digits.

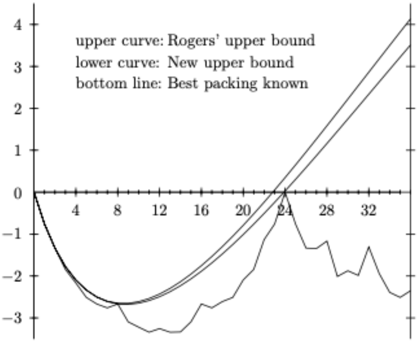


Figure 1. Plot of $\log_2 \delta + n(24 - n)/96$ vs. dimension n.

Magic functions must exist in dimensions 1, 2, 8 and 24...

Remarkable paper! In *Annals of Mathematics*, but almost no mathematical proofs... only numerical simulations!

Magic functions?

If f is radial, then $\widehat{f}$ is also radial, and the previous one–line computation reads

$$1 = f(0) \geq \sum_{\mathbf{R}\in\mathbb{L}} f(|\mathbf{R}|) = \frac{1}{|\Lambda|}\sum_{\mathbf{K}\in\mathbb{L}^*} \widehat{f}(|\mathbf{K}|) \geq \frac{1}{|\Lambda|}\widehat{f}(0) = \frac{1}{|\Lambda|} \qquad \text{with} \qquad \begin{cases} f(r) \leq 0 & \text{for all} \quad r > 2r_0 \\ \widehat{f}(k) \geq 0 & \text{for all} \quad k \in \mathbb{R} \\ f(0) = \widehat{f}(0) = 1. \end{cases}$$

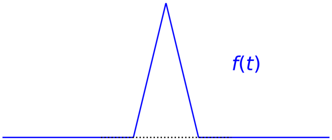
We set $Z_{\mathbb{L}} := \{\|\mathbf{R}\|,\ \mathbf{R} \in \mathbb{L}\}$. If $(f, \mathbb{L})$ is **magical**, then $2r_0 \in Z_{\mathbb{L}}$, and

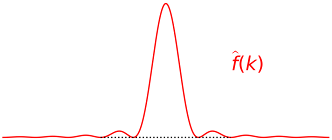
$$\boxed{f(0) = \widehat{f}(0) = 1, \quad f(2r_0) = 0, \quad \begin{cases} \forall r \in Z_{\mathbb{L}} \setminus \{0, 2r_0\}, & f(r) = f'(r) = 0, \\ \forall k \in Z_{\mathbb{L}^*} \setminus \{0\}, & \widehat{f}(k) = \widehat{f}'(k) = 0. \end{cases}}$$

Example in dimension $d = 1$. Consider $g(x) := \mathbb{1}(|x| \leq \frac{1}{2})$ and

$$f(x) := g * g(x) = \begin{cases} 1 - |x| & \text{if} \quad |x| \leq 1 \\ 0 & \text{else} \end{cases}. \qquad \text{Then} \quad \widehat{f}(k) = |\widehat{g}|^2(k) = \left(\frac{\sin(\pi k)}{\pi k}\right)^2.$$

So $(f, \mathbb{Z})$ is **magical** in dimension $d = 1$.





Speech of Henry Cohn in his presentation of Maryna Viazovska for her Fields medal (2022):

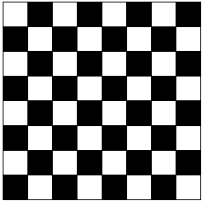
> *I imagined that we had almost solved the sphere packing problem in eight and twenty-four dimensions, and our inability to find the magic functions was extremely frustrating.*
>
> *At first, I worried that someone else would find an easy solution and leave me feeling foolish for not doing it myself. Over time I became convinced that obtaining these functions was in fact difficult, and others also reached the same conclusion. For example, Thomas Hales has said that I felt that* **it would take a Ramanujan to find it**. *Eventually, instead of worrying that someone else would solve it, I began to fear that nobody would solve it, and that I would someday die without knowing the outcome.*
>
> *I am grateful that Viazovska found such a satisfying and beautiful solution, and that she introduced wonderful new ideas for the mathematical community to explore.*

The work of Maryna Viazovska

Maryna Viazovska

- Born in 1984 in Kiev. Study at Kiev until 2010.
- PhD in Bonn under the supervision of Don Zagier and Werner Müller.
- Professor at EPFL since 2018.
- Fields medalist in 2022.

The E_8 lattice?



Consider first the chessboard lattice

$$D_d := \left\{ (x_1, \cdots, x_d) \in \mathbb{Z}^d, \quad \sum_{i=1}^d x_i \in 2\mathbb{Z} \right\}.$$

Density

$$\rho_d(\mathbb{Z}^d) = \omega_d \frac{1}{2^d} \quad \text{and} \quad \rho_d(D_d) = \omega_d \frac{1}{2^{\frac{d}{2}+1}}.$$

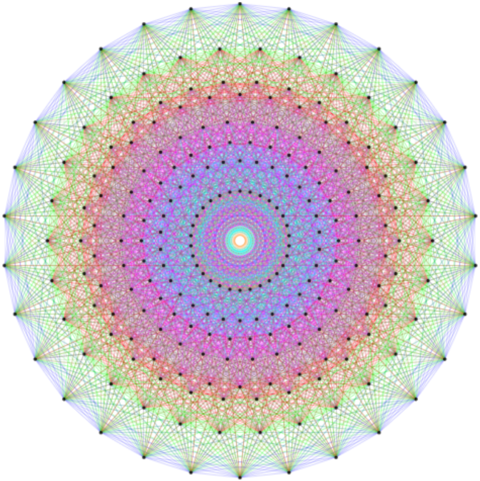
So D_d is a better lattice than $\mathbb{Z}^d$ in dimension $d \geq 3$. (Still a terrible lower bound).

Facts

- In dimension $d = 3$, the optimal fcc lattice (Kepler's conjecture) corresponds to $D_{d=3}$.
- In dimension $d = 4, 5$, it is conjectured that $D_{d=4}$ and $D_{d=5}$ are optimal.
- In dimension $d = 8$, the packings constructed from D_8 and $D_8 + (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ "touches" without overlap.

Note that $r(D_d) = \sqrt{2}$ and $\|(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})\|_d = \frac{\sqrt{d}}{2} = \sqrt{2}$ in dimension $d = 8$.

$$E_8 := D_8 \bigcup \left(D_8 + (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \right).$$



Each point has 240 nearest neighbours.... (Image from Wikipedia).

Proposition

- *The E_8 lattice satisfies* $|\Lambda_{E_8}| = 1$
- $(E_8)^* = E_8$.
- $Z_{E_8} := \{\|\mathbf{R}\|,\ \mathbf{R} \in E_8\} = 2\mathbb{N}$.

Idea of the Proof

If $\mathbf{x} \in D_8$, then $\sum x_i \in 2\mathbb{Z}$. There is an even number of odd coefficients. So $\sum x_i^2 \in 2\mathbb{N}$.
If $\mathbf{y} = \mathbf{x} + (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ with $\mathbf{x} \in D_8$, then

$$\sum_{i=1}^{8} y_i^2 = \sum_{i=1}^{8} \left(x_i + \frac{1}{2}\right)^2 = \sum_{i=1}^{8} x_i^2 + \sum_{i=1}^{8} x_i + 8\frac{1}{4} \in 2\mathbb{N}.$$

Conversely, for $N = 2n \in 2\mathbb{N}$, the **Lagrange "four square" theorem** states that $n = a^2 + b^2 + c^2 + d^2$.
So $N = \|\mathbf{x}\|^2$ with $\mathbf{x} = (a, a, b, b, c, c, d, d)$. □

The magic function in dimension $d = 8$?

Any magical function $f \in \mathscr{S}_{\rm rad}(\mathbb{R}^8)$ must satisfy

$$f(0) = \widehat{f}(0) = 1, \quad f(\sqrt{2}) = f'(\sqrt{2}) = \widehat{f}(\sqrt{2}) = 0, \quad \forall n \geq 2,\ f(\sqrt{2n}) = f'(\sqrt{2n}) = \widehat{f}(\sqrt{2n}) = \widehat{f}'(\sqrt{2n}) = 0.$$

Theorem (Cohn, Kumar, Miller, Radchenko et Viazovska, *Annals of Mathematics* 2022)

There are radial Schwartz functions (a_n, b_n) *so that, for all* $f \in \mathscr{S}_{\rm rad}(\mathbb{R}^8)$, *we have*

$$f(x) = \sum_{n=1}^{\infty} f(\sqrt{2n}) a_n(x) + f'(\sqrt{2n}) b_n(x) + \widehat{f}(\sqrt{2n}) \widehat{a}_n(x) + \widehat{f}'(\sqrt{2n}) \widehat{b}_n(x).$$

Remarks

- One can reconstruct f solely from the data $\left(f(\sqrt{2n}), f'(\sqrt{2n}), \widehat{f}(\sqrt{2n}), \widehat{f}'(\sqrt{2n}) \right)_{n \geq 1}$.
- The only possible magic function in $d = 8$ is $f_8 := \text{cst} \cdot b_1$.
- Turns out that b_1 **is magical** (computer assisted proof for this part).

In dimension $d = 8$, the lattice E_8 is optimal, and $\rho_8 = \frac{\pi^4}{384} \approx 0.254$.
In dimension $d = 24$, the lattice Λ_{24} (Leech 24) is optimal, and $\rho_{24} = \frac{\pi^{12}}{12!} \approx 0.0019$.

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Theorem (Radchenko, Viazovska, *Publications mathématiques de l'IHÉS*, 2019)

There are even Schwartz functions (a_n) *so that, for all* $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$*, we have*

$$f(x) = \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) + \widehat{a}_n(x)\widehat{f}(\sqrt{n}).$$

The *magical* Ansatz of Maryna Viazovska

Fourier transform of the Gaussian. For $a \in \mathbb{C}$ with $\mathrm{Re}\,(a) > 0$,

$$G_a(\mathbf{x}) := \mathrm{e}^{-\pi a |\mathbf{x}|^2} \text{ (Gaussian)} \quad \text{has Fourier transform} \quad \mathcal{F}_d(G_a)(\mathbf{k}) = \widehat{G}_a(\mathbf{k}) = \frac{1}{\sqrt{a}^d} G_{\frac{1}{a}}(\mathbf{k}).$$

In particular, $G_{a=1}$ satisfies $\widehat{G}_1 = G_1$.

Warm-up: the Jacobi Theta function.

$$\boxed{\theta(z) := \sum_{n \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi n^2 z}.}$$

Well defined for $z \in \mathbb{C}_+ := \{z \in \mathbb{C},\ \mathrm{Im}\,(z) > 0\}$. Clearly, $\theta(z+2) = \theta(z)$.
In addition, the Poisson summation formula shows that

$$\theta(z) = \sum_{n \in \mathbb{Z}} G_{-\mathrm{i}z}(n) = \sum_{R \in \mathbb{Z}} \frac{1}{\sqrt{-\mathrm{i}z}} G_{\frac{1}{-\mathrm{i}z}}(R) = \frac{1}{\sqrt{-\mathrm{i}z}} \theta\left(\frac{-1}{z}\right).$$

The map $z \mapsto \frac{-1}{z}$ leaves $\mathbb{C}_+$ invariant.
It also leaves $\mathbb{S}^1_+ := \{z \in \mathbb{C}_+, |z| = 1\}$ (but reverse orientation).

Construct a function $g : \mathbb{R} \to \mathbb{R}$ even (radial, in dimension $d = 1$) so that

$$\widehat{g} = g, \quad g(0) = 1, \qquad \forall n \geq 1, \quad g(\sqrt{n}) = 0.$$

We search g of the form

$$\boxed{g(x) = \frac{1}{2} \int_{-1}^{1} \psi(t) \mathrm{e}^{\mathrm{i}\pi t x^2} \mathrm{d}t,} \qquad = -\frac{1}{2} \int_{\mathbb{S}_1^+} \psi(z) \mathrm{e}^{\mathrm{i}\pi z x^2} \mathrm{d}z.$$

with ψ a 2–periodic function. Clearly $g(-x) = g(x)$. If $\psi(t) = \sum_{n \in \mathbb{Z}} \alpha_n \mathrm{e}^{\mathrm{i}\pi n t}$ (Fourier decomposition), then

$$g(\sqrt{n}) = \frac{1}{2} \int_{-1}^{1} \psi(t) \mathrm{e}^{\mathrm{i}\pi t n} \mathrm{d}t = \alpha_{-n}. \qquad \text{So we must have } \alpha_m = 0 \text{ for all } m < 0.$$

In particular, $\psi(\cdot)$ admits an analytic extension on $\mathbb{C}_+$. With the change of variable $u = \frac{1}{z}$ so $\mathrm{d}z = -\frac{\mathrm{d}u}{u^2}$.

$$\widehat{g}(x) = -\frac{1}{2} \int_{\mathbb{S}_1^+} \psi(z) \frac{1}{\sqrt{-\mathrm{i}z}} \mathrm{e}^{\pi x^2 \frac{1}{\mathrm{i}z}} \mathrm{d}z = -\frac{1}{2} \int_{\mathbb{S}_1^+} \left[\psi\left(\frac{-1}{u} \right) \frac{1}{\sqrt{-\mathrm{i}u}^3} \right] \mathrm{e}^{\mathrm{i}\pi x^2 u} \mathrm{d}u.$$

Conclusion: g satisfies the conditions if:

- ψ is an analytic function on $\mathbb{C}_+$;
- ψ is 2–periodic $\psi(z+2) = \psi(z)$;
- ψ satisfies $\psi(z) = \left(\frac{1}{\sqrt{-\mathrm{i}z}} \right)^3 \psi\left(\frac{-1}{z} \right)$...

$$\boxed{\psi(z) = \theta^3(z).}$$

A word on *modular* forms

Definition

Let $k \in \mathbb{Z}$. A function ϕ is **weakly modular with weight** $2k$ if ϕ is meromorph on $\mathbb{C}_+$, and satisfies

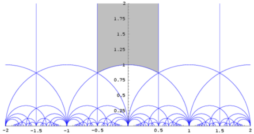
$$\phi(z+1) = \phi(z), \qquad \phi\left(\frac{-1}{z}\right) = z^{2k}\phi(z).$$

Writing $\phi(z) = \sum_{n\in\mathbb{Z}} \alpha_n \mathrm{e}^{2\mathrm{i}\pi z}$, we say that ϕ is **modular** if $\alpha_n = 0$ for all $n < 0$.

Remark

The maps $z \mapsto z+1$ and $z \mapsto \frac{-1}{z}$ generates the **modular group** acting on $\mathbb{C}_+$:

$$T \in \mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det(ab - cd) = 1 \right\} \quad \mapsto \quad (T\phi)(z) := \phi\left(\frac{az+b}{cz+d}\right).$$



A word on *modular* forms

Definition

Let $k \in \mathbb{Z}$. A function ϕ is **weakly modular with weight** $2k$ if ϕ is meromorph on $\mathbb{C}_+$, and satisfies

$$\phi(z+1) = \phi(z), \qquad \phi\left(\frac{-1}{z}\right) = z^{2k}\phi(z).$$

Writing $\phi(z) = \sum_{n\in\mathbb{Z}} \alpha_n \mathrm{e}^{2i\pi z}$, we say that ϕ is **modular** if $\alpha_n = 0$ for all $n < 0$.

Remark

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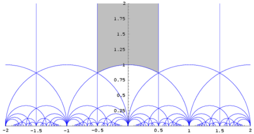


Maryna Viazovska got the Field's Medal for exhibiting a deep link between the sphere packing problem, Fourier theory, and modular forms

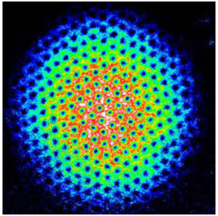
What's next?

The *sphere packing* problem has been extended to any *repulsive interaction* among particles (**universal optimality**).

- Cohn, Kumar, Miller, Radchenko, Viazovska (2022, for Gaussian interactions $W(\mathbf{x}-\mathbf{y}) = G_a(\|\mathbf{x}-\mathbf{y}\|)$);
- Petrache, Serfaty (2020, for Riez's interactions $W(\mathbf{x}-\mathbf{y}) = \frac{1}{\|\mathbf{x}-\mathbf{y}\|^s}$) ;
- Luo, Wei (2024, for Lennard–Jones potentials).

What about the two dimensional case?

- Does there exist a **magical** function in dimension 2 (for the triangular lattice)?
- Is the triangular lattice **universal optimal**?



Rotating Bose-Einstein Condensate.