Path-Dependent Volatility: Practical Examples

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Motivation: what we can achieve with path-dependent volatility

$$\frac{dS_t}{S_t} = \sigma(t, (S_u, u \leq t))\, dW_t$$

- Calibrate to SPX smile
- Produce rich joint dynamics of spot, implied vol, and implied vol of vol
- Capture historical joint behavior of SPX trend, VIX, and VVIX
- Naturally generate mean reverting volatility
- Jointly calibrate to SPX and VIX smiles? (work in progress)

Dynamics of everything driven by a single Brownian motion:

- Complete model: unique price, perfect delta-hedging
- Instantaneous correlations of all quantities are $\pm 100\%$

A look at SPX data: SPX, VIX and VVIX time series

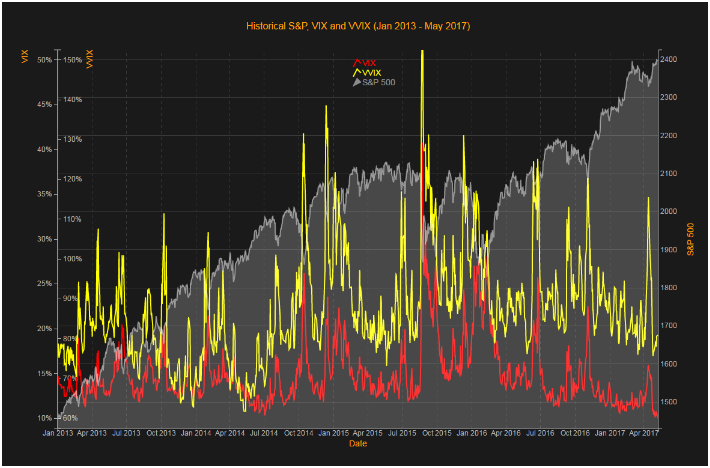


Figure: January 1, 2013 to May 8, 2017

A look at SPX data: VIX vs SPX

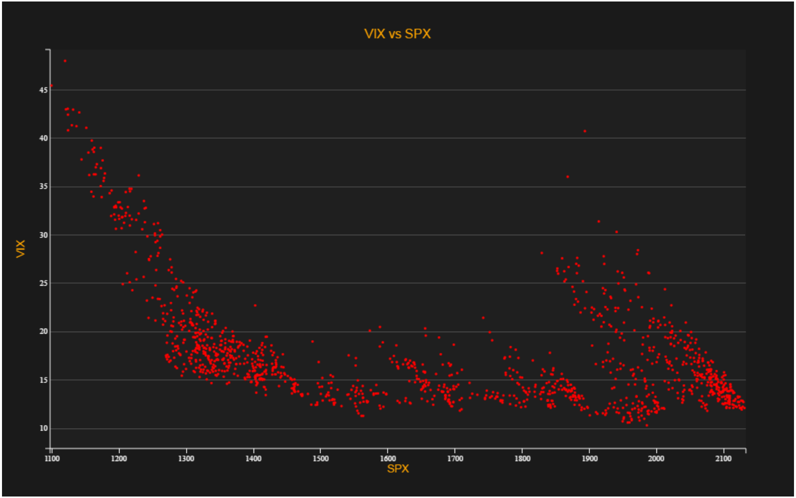


Figure: January 1, 2011 to March 31, 2016

A look at SPX data: VIX vs recent trend of SPX

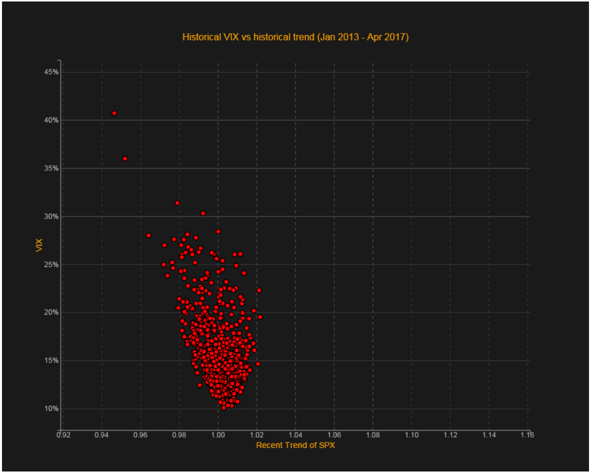


Figure: January 1, 2013 to May 3, 2017; trend = spot/moving average; window $h = 5, 30, 60, 100, 150, 200, 300$ days

A look at SPX data: VVIX vs VIX

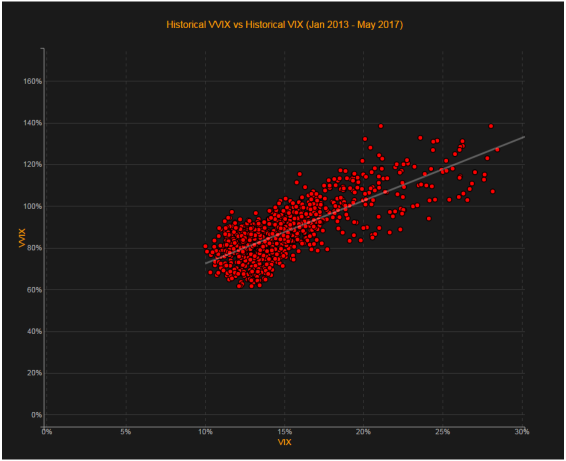


Figure: January 1, 2013 to May 8, 2017

A look at SPX data: VVIX vs VIX

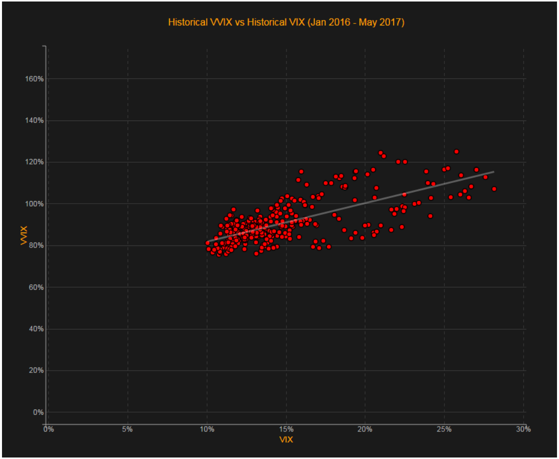


Figure: January 1, 2016 to May 8, 2017

A look at SPX data: historical vol of VS vol vs VS vol

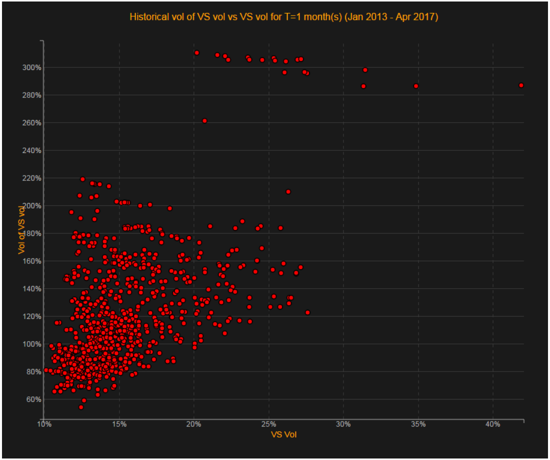


Figure: January 1, 2013 to April 27, 2017; $T = 1$ month

A look at SPX data: historical vol of VS vol vs VS vol

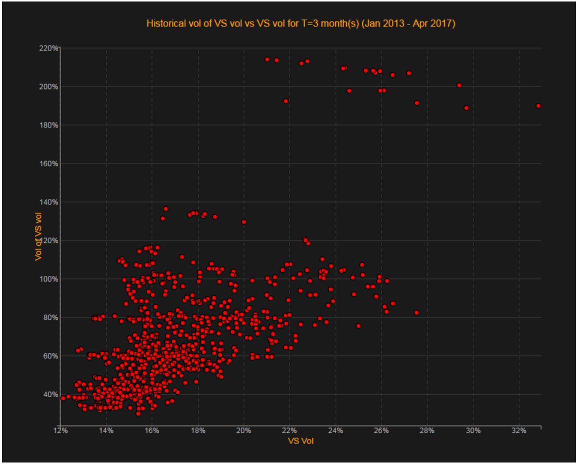


Figure: January 1, 2013 to April 27, 2017; $T = 3$ months

A look at SPX data: historical vol of VS vol vs VS vol

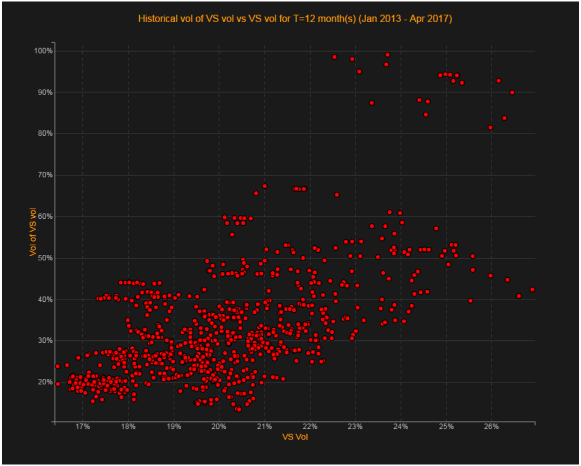


Figure: January 1, 2013 to April 27, 2017; $T = 1$ year

A look at SPX data: historical vol of VS vol vs VS vol

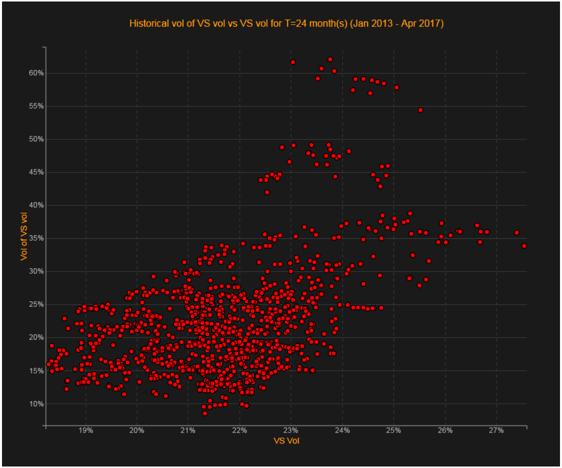
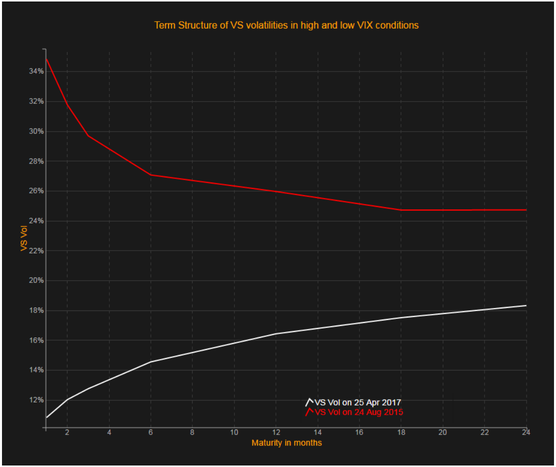
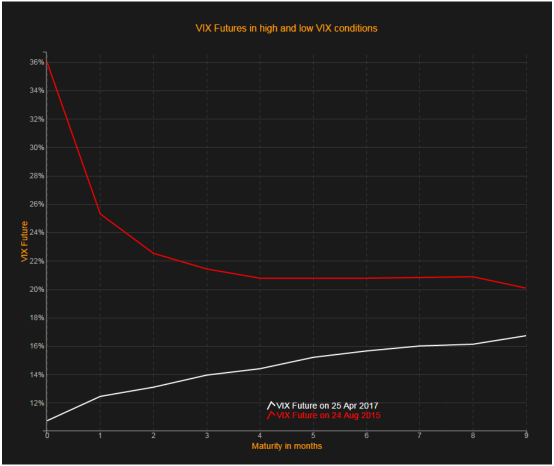


Figure: January 1, 2013 to April 27, 2017; $T = 2$ years

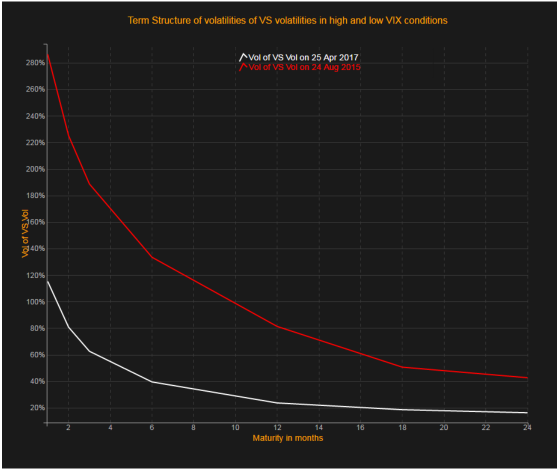
A look at SPX data: Term-structure of VS vol



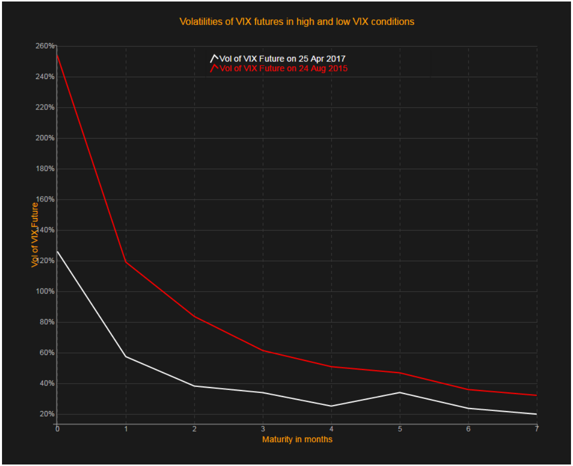
A look at SPX data: Term-structure of VIX futures



A look at SPX data: Term-structure of vol of VS vol



A look at SPX data: Term-structure of vol of VIX futures



A natural PDV model

$$\frac{dS_t}{S_t} = \sigma(X_t)\, dW_t, \qquad X_t = \frac{S_t}{\bar{S}_t^h}$$

- $\bar{S}_t^h =$ moving average of S over time window h, e.g.:

$$\text{classical MA:} \qquad \bar{S}_t^h = \frac{1}{h}\int_{t-h}^{t} S_u\, du$$

$$\text{exponentially weighted MA:} \qquad \bar{S}_t^h = \frac{1}{h}\int_{-\infty}^{t} e^{-\frac{t-u}{h}} S_u\, du$$

- $X_t =$ **recent trend of the SPX** over time window h
- Instantaneous vol $=$ a function of recent SPX trend
- Function σ is decreasing: negative SPX trends $\longleftrightarrow$ high SPX vol
- $h = ?$ Multiple time scales?
- $\sigma = ?$

Trend is mean-reverting

CMA:

$$d\bar{S}_t^h = \frac{1}{h}(S_t - S_{t-h})\,dt, \qquad \frac{dX_t}{X_t} = -\frac{1}{h}\frac{S_t - S_{t-h}}{\bar{S}_t^h}\,dt + \frac{dS_t}{S_t}$$

- Statistical mean-reversion of X towards 1

EWMA:

$$d\bar{S}_t^h = \frac{1}{h}(S_t - \bar{S}_t^h)\,dt, \qquad \frac{dX_t}{X_t} = -\frac{1}{h}(X_t - 1)\,dt + \frac{dS_t}{S_t}$$

- X mean-reverts towards 1
- X is Markov: One-factor stochastic volatility model with perfectly correlated Brownian motions

In those PDV models, **vol is mean-reverting**.

Trend is mean-reverting: CMA

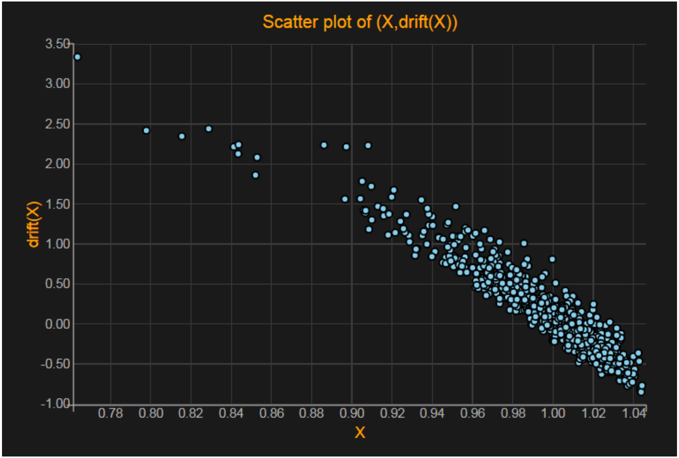


Figure: Drift of X_t (CMA), $t = 20$ days

Vol of vol

$$\begin{aligned}\frac{dS_t}{S_t} &= \sigma(X_t)\,dW_t, \qquad X_t = \frac{S_t}{\bar{S}_t^h}\\ \frac{dX_t}{X_t} &= \{\text{mean-reversion}\}\,dt + \sigma(X_t)\,dW_t\\ \Longrightarrow \frac{d\sigma(X_t)}{\sigma(X_t)} &= \{\text{mean-reversion}\}\,dt + X_t\sigma'(X_t)\,dW_t\end{aligned}$$

- Inst. vol of inst. vol $= -X_t\sigma'(X_t)$ (σ decreasing)
- Mean-reversion $\Longrightarrow$ inst. vol of VIX future $<$ inst. vol of inst. vol
- For mean-reversion time scale $h \gg 30$ days, VVIX $\approx$ inst. vol of inst. vol
- Moreover VIX $\approx$ inst. vol
- VVIX $\approx a + b\,$VIX translates into $-X_t\sigma'(X_t) = a + b\sigma(X_t)$
- For smaller h, find (α, β) s.t. $-X_t\sigma'(X_t) = \alpha + \beta\sigma(X_t)$ can reproduce VVIX $\approx a + b\,$VIX

$$\Longrightarrow \sigma(x) = -\frac{\alpha}{\beta} + \gamma x^{-\beta}$$

Path-dependent volatility $\sigma(X)$

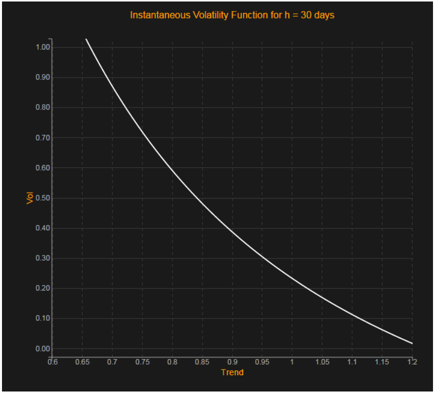


Figure: Graph of $x \mapsto \sigma(x)$ using the values (α, β, γ) corresponding to $h = 30$ days

VVIX vs VIX: $h \gg 30$ days

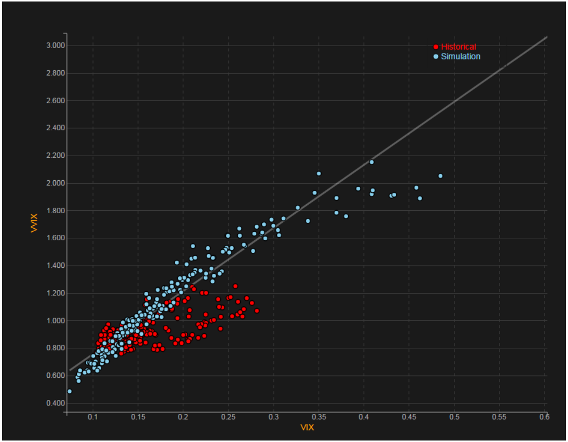


Figure: $h = 180$ days, $\alpha = 1$, $\beta = 1.5$, $\gamma = 0.9$

VVIX vs VIX: $h \approx 30$ days

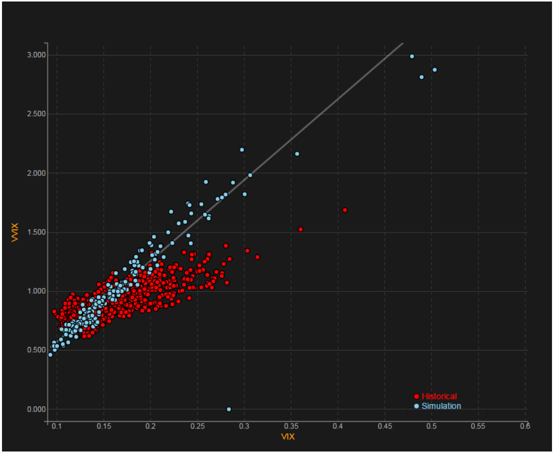


Figure: $h = 30$ days, $\alpha = 2.5$, $\beta = 1.2$, $\gamma = 2.3$

VVIX vs VIX: $h \ll 30$ days

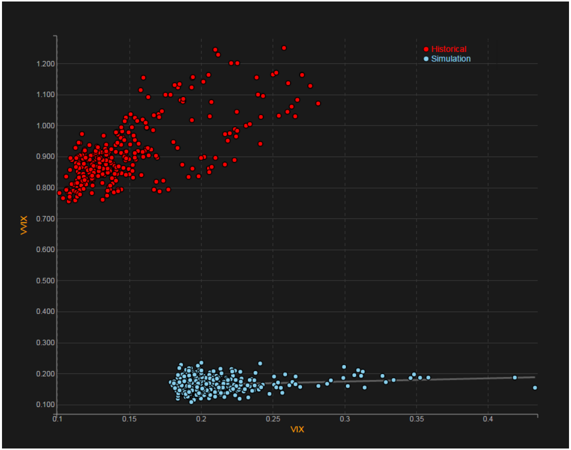


Figure: $h = 5$ days, $\alpha = 5$, $\beta = 6$, $\gamma = 1$

VIX vs recent trend of SPX: $h \gg 30$ days

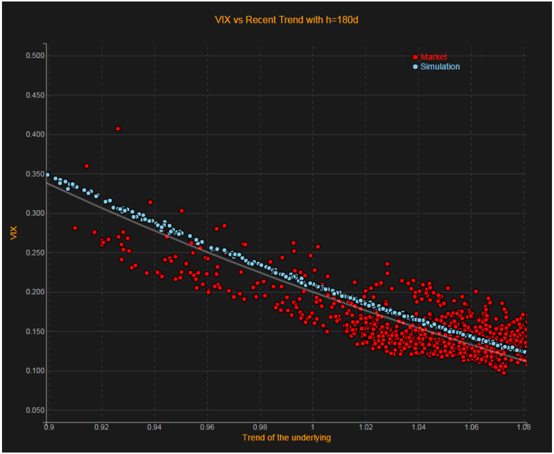


Figure: $h = 180$ days, $\alpha = 0.9$, $\beta = 1.5$, $\gamma = 0.8$

VIX vs recent trend of SPX: $h \approx 30$ days

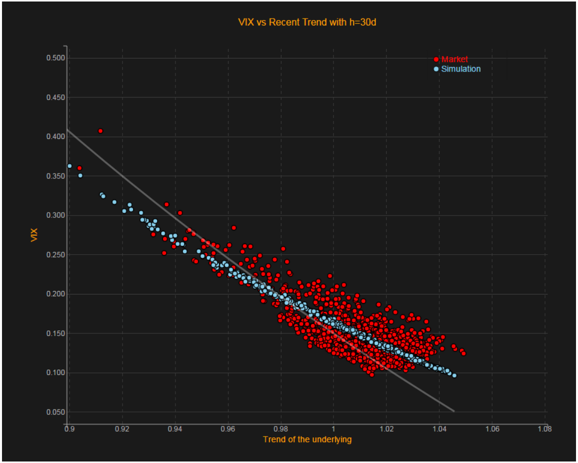


Figure: $h = 30$ days, $\alpha = 2.1$, $\beta = 1.2$, $\gamma = 1.9$

VIX vs recent trend of SPX: $h \ll 30$ days

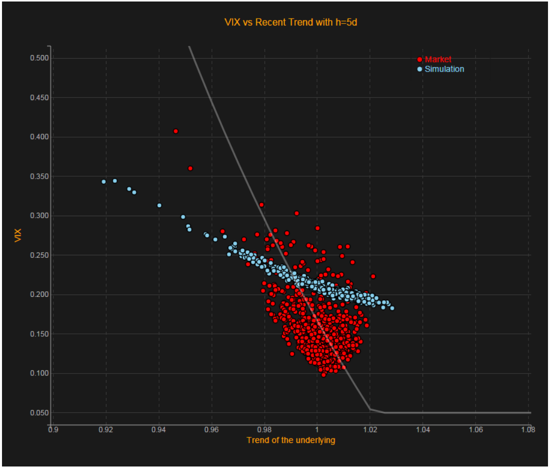


Figure: $h = 5$ days, $\alpha = 5$, $\beta = 6$, $\gamma = 1$

VIX and VVIX in pure PDV model with EWMA

$$
\begin{aligned}
\text{VIX}_t^2 &= \frac{1}{\tau}\int_t^{t+\tau} \mathbb{E}[\sigma(X_u)^2|\mathcal{F}_t]\,du \\
&= \frac{1}{\tau}\int_t^{t+\tau} \mathbb{E}[\sigma(X_u)^2|X_t]\,du \;=\; \text{VIX}^2(X_t)
\end{aligned}
$$

- $h \gg \tau = 30$ days $\Longrightarrow \text{VIX}(X_t) \approx \sigma(X_t)$
- $h \ll \tau = 30$ days $\Longrightarrow \text{VIX}(X_t) \approx \sqrt{\int \sigma^2(x)\,d\mu(x)}$, μ stationary distribution of X, due to mean-reversion, ergodicity

$$
\begin{aligned}
\text{VVIX}_t^2 &= -\frac{2}{\tau}\mathbb{E}[\ln \text{VIX}_{t+\tau} - \ln \mathbb{E}[\text{VIX}_{t+\tau}|\mathcal{F}_t]|\mathcal{F}_t] \\
&= -\frac{2}{\tau}\mathbb{E}[\ln \text{VIX}(X_{t+\tau}) - \ln \mathbb{E}[\text{VIX}(X_{t+\tau})|X_t]|X_t] \;=\; \text{VVIX}^2(X_t)
\end{aligned}
$$

- $h \gg \tau \Longrightarrow \text{VVIX}(X_t) \approx \alpha + \beta\sigma(X_t)$, inst. vol of $\sigma(X_t)$, i.e., $\text{VVIX}(X_t) \approx \alpha + \beta\text{VIX}(X_t)$
- $h \ll \tau \Longrightarrow \text{VVIX}(X_t) \approx 0$

Calibration to SPX smile

$$\frac{dS_t}{S_t} = \sigma(X_t) l_\sigma(t, S_t)\, dW_t, \qquad X_t = \frac{S_t}{\bar{S}_t^h}$$

- Model calibrated to SPX smile $\iff \mathbb{E}[\sigma(X_t)^2 l_\sigma(t, S_t)^2 | S_t] = \sigma_{\mathrm{loc}}(t, S_t)^2$

$$\frac{dS_t}{S_t} = \frac{\sigma(X_t)}{\sqrt{\mathbb{E}[\sigma(X_t)^2 | S_t]}} \sigma_{\mathrm{loc}}(t, S_t)\, dW_t$$

- **Particle method** = estimate conditional expectation within MC (kernel regression)
- EWMA: (X_t, S_t) Markov, X_t not Markov

$$\frac{dX_t}{X_t} = -\frac{1}{h}(X_t - 1)\, dt + \frac{dS_t}{S_t} = -\frac{1}{h}(X_t - 1)\, dt + \sigma(X_t) l_\sigma(t, S_t)\, dW_t$$

Spot-starting smile and forward-starting smile

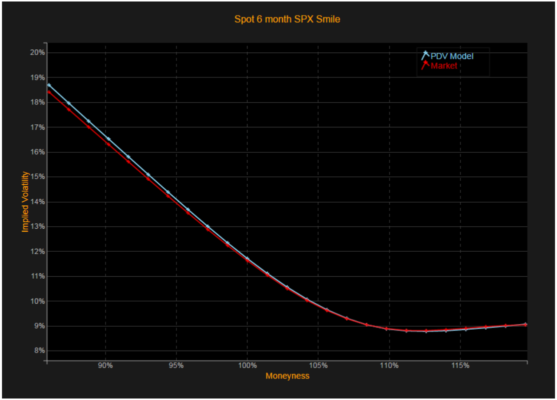


Figure: Spot-starting smile on May 8, 2017, $T = 6$ months, $h = 30$ days, $\alpha = 2.5$, $\beta = 1.2$, $\gamma = 2.3$

Spot-starting smile and forward-starting smile

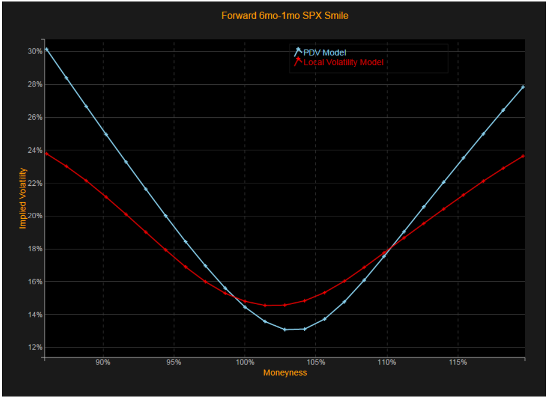


Figure: Forward-starting smile 1 month in 6 months, $h = 30$ days, $\alpha = 2.5$, $\beta = 1.2$, $\gamma = 2.3$

Spot-starting smile and forward-starting smile (flat spot smile)

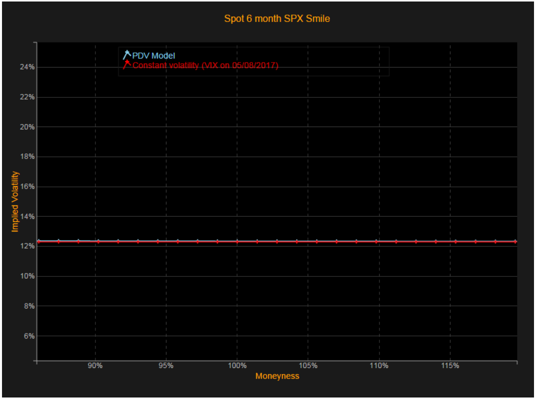


Figure: Spot-starting flat smile 6 months, $h = 30$ days, $\alpha = 2.5$, $\beta = 1.2$, $\gamma = 2.3$

Spot-starting smile and forward-starting smile (flat spot smile)

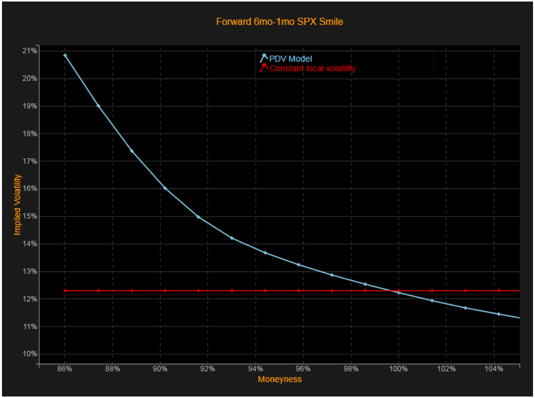


Figure: Forward-starting smile 1 month in 6 months, $h = 30$ days, $\alpha = 2.5$, $\beta = 1.2$, $\gamma = 2.3$, with a flat spot smile

VIX vs recent trend of SPX: $h \gg 30$ days

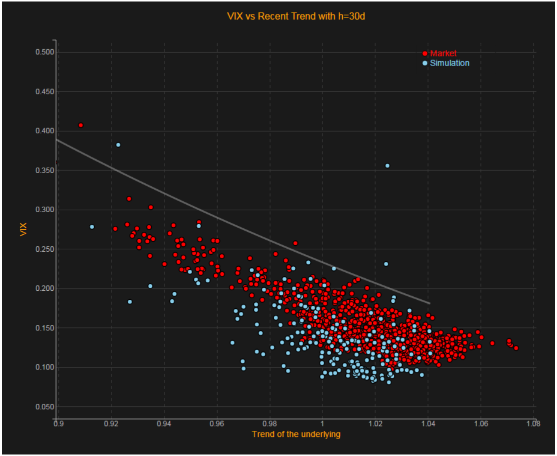


Figure: Leveraged Case with $\alpha = 1$, $\beta = 1.5$, $\gamma = 0.9$

VIX vs recent trend of SPX: $h \approx 30$ days

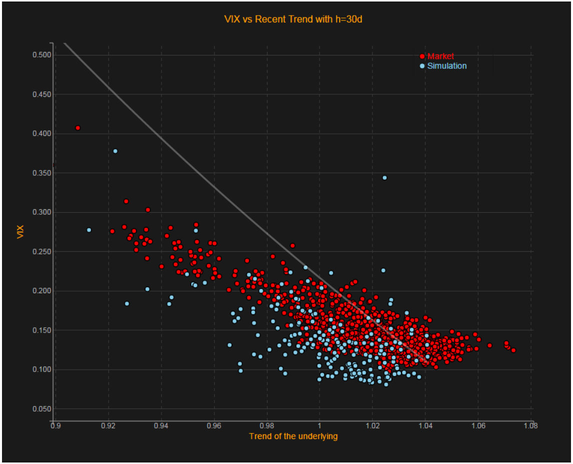


Figure: Leveraged Case with $\alpha = 2.5$, $\beta = 1.2$, $\gamma = 2.3$

VIX vs recent trend of SPX: $h \ll 30$ days

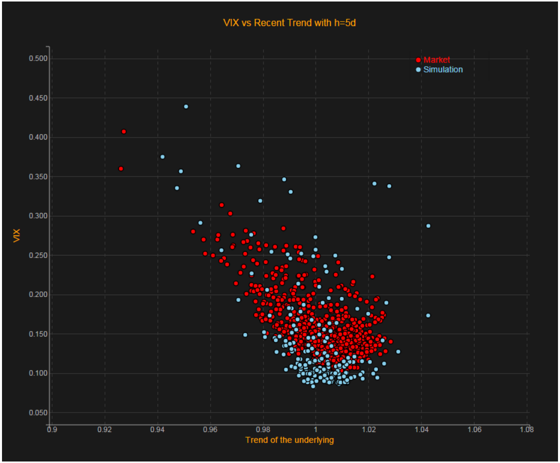


Figure: Leveraged Case with $\alpha = 5$, $\beta = 6$, $\gamma = 1$

Calibration to VIX futures and VIX smiles

- $\text{VIX}_t^{T_i}$ = monthly VIX future expiring at T_i; $\text{VIX}_{T_i}^{T_i} = \text{VIX}_{T_i}$
- We assume $T_{i+1} - T_i = 30$ days (28 or 35 days, in fact)
- VIX options pay at T_i $(\text{VIX}_{T_i} - K)_+ = (\text{VIX}_{T_i}^{T_i} - K)_+$ or $(K - \text{VIX}_{T_i})_+ = (K - \text{VIX}_{T_i}^{T_i})_+$
- Goal: for each T_i, calibration to $\text{VIX}_0^{T_i}$ and all T_i-VIX options, **on top of calibration to SPX options**

$$\frac{dS_t}{S_t} = \frac{\sigma_i(X_t)}{\sqrt{\mathbb{E}[\sigma_i(X_t)^2|S_t]}}\sigma_{\text{loc}}(t, S_t)\, dW_t, \qquad t \in [T_i.T_{i+1})$$

- Find σ_i s.t. this model is calibrated to $\text{VIX}_0^{T_i}$ and all T_i-VIX options

The choice of X matters

$$\mathrm{VIX}_{T_i}^2 = \frac{1}{\tau}\int_{T_i}^{T_{i+1}} \mathbb{E}\left[\frac{\sigma_i(X_t)^2}{\mathbb{E}[\sigma_i(X_t)^2|S_t]}\sigma_{\mathrm{loc}}(t,S_t)^2\middle|\mathcal{F}_{T_i}\right] dt$$

- If $X_t = f(t, S_t)$ then local volatility (LV) model:

$$\frac{\sigma_i(X_t)}{\sqrt{\mathbb{E}[\sigma_i(X_t)^2|S_t]}} = 1, \qquad \frac{dS_t}{S_t} = \sigma_{\mathrm{loc}}(t,S_t)\,dW_t, \qquad t \in [T_i.T_{i+1})$$

- If X is the recent SPX trend with time window h:
  - If h large, then calibrated model is close to LV model $\Longrightarrow$ hard to calibrate to VIX futures and VIX options
  - If $h \ll \tau = 30$ days, then fast mean-reversion $\Longrightarrow$ $\mathrm{VIX}_{T_i}^2$ has small variance $\Longrightarrow$ hard to calibrate to VIX options
- X could be driven by another Brownian motion, e.g.:

$$dX_t = -k(X_t - m)\,dt + \nu\,dZ_t, \qquad d\langle W, Z\rangle_t = \rho\,dt$$

Stochastic local volatility model

Optimization problem

$$\mathsf{VIX}^2_{T_i} = \frac{1}{\tau}\int_{T_i}^{T_{i+1}} \mathbb{E}\left[\frac{\sigma_i(X_t)^2}{\mathbb{E}[\sigma_i(X_t)^2|S_t]}\sigma_{\text{loc}}(t,S_t)^2 \middle| \mathcal{F}_{T_i}\right] dt$$

Find σ_i s.t.

$$\begin{aligned} \mathbb{E}\left[\mathsf{VIX}_{T_i}\right] &= \mathsf{VIX}_0^{T_i,\text{mkt}} \\ \forall K, \quad \mathbb{E}\left[(\mathsf{VIX}_{T_i} - K)_+\right] &= C^{\text{mkt}}_{\mathsf{VIX}}(T_i,K) \end{aligned}$$

In the case where SPX smile is flat at $\bar{\sigma}_{\mathsf{SPX}}$:

$$\begin{aligned} \mathbb{E}\left[\bar{\sigma}_{\mathsf{SPX}}\sqrt{\frac{1}{\tau}\int_{T_i}^{T_{i+1}} \mathbb{E}\left[\frac{\sigma_i(X_t)^2}{\mathbb{E}[\sigma_i(X_t)^2|S_t]} \middle| \mathcal{F}_{T_i}\right] dt}\right] &= \mathsf{VIX}_0^{T_i,\text{mkt}} \\ \forall K, \quad \mathbb{E}\left[\left(\bar{\sigma}_{\mathsf{SPX}}\sqrt{\frac{1}{\tau}\int_{T_i}^{T_{i+1}} \mathbb{E}\left[\frac{\sigma_i(X_t)^2}{\mathbb{E}[\sigma_i(X_t)^2|S_t]} \middle| \mathcal{F}_{T_i}\right] dt} - K\right)_+\right] &= C^{\text{mkt}}_{\mathsf{VIX}}(T_i,K) \end{aligned}$$

In the case where SPX smile is flat at $\bar{\sigma}_{\mathsf{SPX}}$:

$$\begin{aligned} \mathbb{E}\left[\bar{\sigma}_{\mathsf{SPX}}\sqrt{\frac{1}{\tau}\int_{T_i}^{T_{i+1}}\mathbb{E}\left[\frac{\sigma_i(X_t)^2}{\mathbb{E}[\sigma_i(X_t)^2|S_t]}\middle|\mathcal{F}_{T_i}\right]dt}\right] &= \mathsf{VIX}_0^{T_i,\mathrm{mkt}} \\ \forall K, \quad \mathbb{E}\left[\left(\bar{\sigma}_{\mathsf{SPX}}\sqrt{\frac{1}{\tau}\int_{T_i}^{T_{i+1}}\mathbb{E}\left[\frac{\sigma_i(X_t)^2}{\mathbb{E}[\sigma_i(X_t)^2|S_t]}\middle|\mathcal{F}_{T_i}\right]dt} - K\right)_+\right] &= C_{\mathsf{VIX}}^{\mathrm{mkt}}(T_i,K) \end{aligned}$$

- In order to calibrate to a nonzero VIX smile, we need $\frac{1}{\tau}\int_{T_i}^{T_{i+1}}\mathbb{E}\left[\frac{\sigma_i(X_t)^2}{\mathbb{E}[\sigma_i(X_t)^2|S_t]}\middle|\mathcal{F}_{T_i}\right]dt$ to have enough variance. Moreover, VIX smiles are typically increasing so we need the distribution of VIX_{T_i} to have positive skew.

- EWMA: (S_t, X_t) is Markov, so

$$\mathrm{VIX}_{T_i}^2 = \frac{1}{\tau}\mathbb{E}\left[\int_{T_i}^{T_{i+1}} \frac{\sigma_i(X_t)^2}{\mathbb{E}[\sigma_i(X_t)^2|S_t]}\,dt\middle|S_{T_i}, X_{T_i}\right] = \mathrm{VIX}_{T_i}^2(S_{T_i}, X_{T_i})$$

$\Longrightarrow$ Nonparametric regression in dimension 2

- Approximation

$$\frac{1}{\tau}\int_{T_i}^{T_{i+1}} \mathbb{E}\left[\frac{\sigma_i(X_t)^2}{\mathbb{E}[\sigma_i(X_t)^2|S_t]}\middle|\mathcal{F}_{T_i}\right]\,dt \approx \frac{\sigma_i(X_{T_i})^2}{\mathbb{E}[\sigma_i(X_{T_i})^2|S_{T_i}]}.$$

is good only if $h \gg 30$ days, but in that case X_t is very correlated with S_t so this quantity has little variance...

- We can also work with another X process that has mean-reversion time $h \gg 30$ days but is not necessarily very correlated with S_t, e.g., an Ornstein-Uhlenbeck process:

$$\begin{aligned}\frac{dS_t}{S_t} &= \frac{\sigma_i(X_t)}{\sqrt{\mathbb{E}[\sigma_i(X_t)^2|S_t]}}\sigma_{\mathrm{loc}}(t, S_t)\,dW_t, \qquad t \in [T_i.T_{i+1}) \\ dX_t &= -k(X_t - 1)\,dt + \nu dZ_t, \qquad d\langle W, Z\rangle_t = \rho\,dt\end{aligned}$$

Results (work in progress): calibration to VIX futures

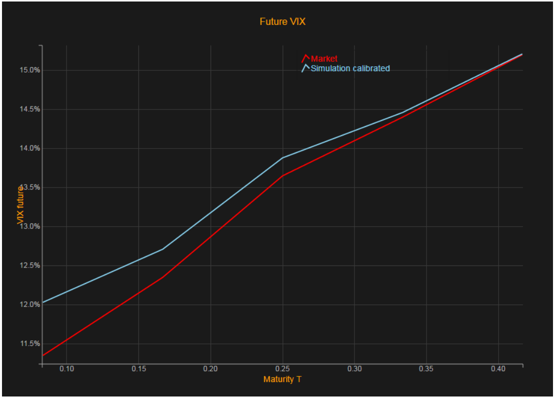


Figure: X is an O-U process, mean-reversion $1/k = 120$ days, vol of vol $\nu = 1$, $\rho = 0$. Flat (time-dependent) SPX smile

Results (work in progress): calibration to VIX smiles

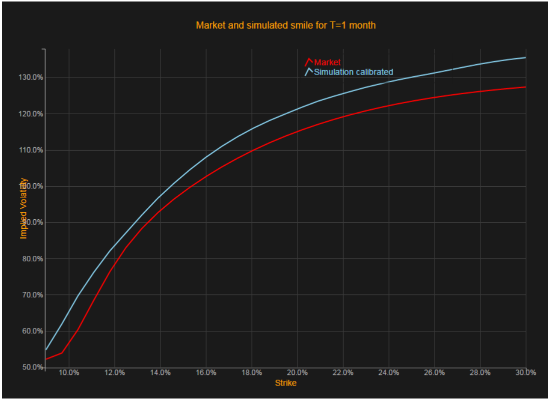


Figure: $T = 1$ month; X is an O-U process, mean-reversion $1/k = 120$ days, vol of vol $\nu = 1$, $\rho = 0$. Flat (time-dependent) SPX smile

Results (work in progress): calibration to VIX smiles

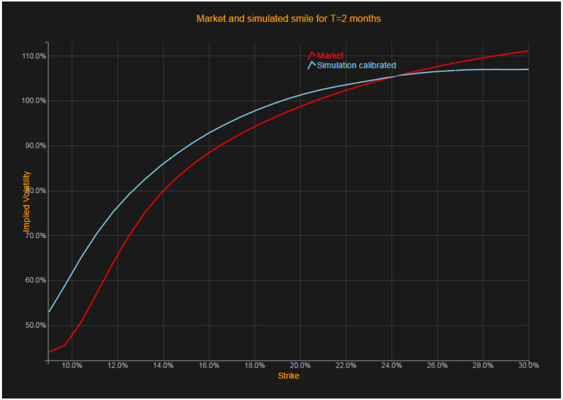


Figure: $T = 2$ months; X is an O-U process, mean-reversion $1/k = 120$ days, vol of vol $\nu = 1$, $\rho = 0$. Flat (time-dependent) SPX smile

Results (work in progress): calibration to VIX smiles

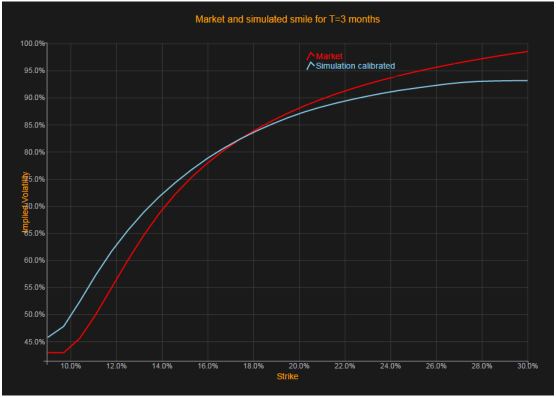


Figure: $T = 3$ months; X is an O-U process, mean-reversion $1/k = 120$ days, vol of vol $\nu = 1$, $\rho = 0$. Flat (time-dependent) SPX smile

Results (work in progress): calibration to VIX smiles

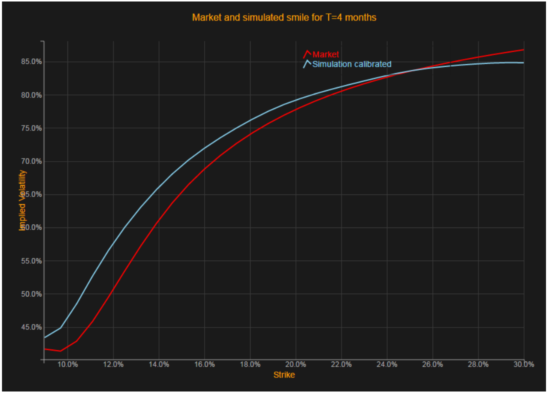


Figure: $T = 4$ months; X is an O-U process, mean-reversion $1/k = 120$ days, vol of vol $\nu = 1$, $\rho = 0$. Flat (time-dependent) SPX smile

Results (work in progress): calibration to VIX smiles

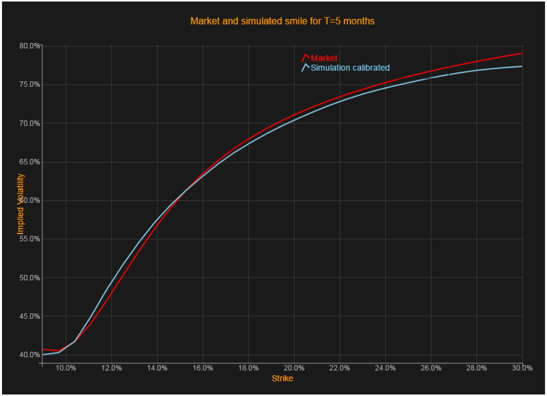


Figure: $T = 5$ months; X is an O-U process, mean-reversion $1/k = 120$ days, vol of vol $\nu = 1$, $\rho = 0$. Flat (time-dependent) SPX smile

Results (work in progress): optimal σ_i

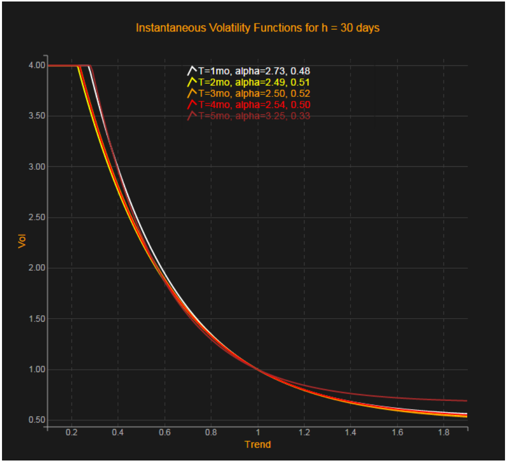


Figure: Corresponding optimal functions $\sigma_i(x) = 1 + \alpha_1(\exp(-\alpha_0(x-1)) - 1)$

Comparison with stochastic volatility models

- Closest SV model:

$$\begin{aligned}
\frac{dS_t}{S_t} &= \sigma(Y_t)\,dW_t \\
\frac{dY_t}{Y_t} &= -k(Y_t-1)\,dt + \sigma(Y_t)\,dZ_t, \qquad d\langle W,Z\rangle_t = \rho\,dt \\
\sigma(y) &= -\frac{\alpha}{\beta} + \gamma y^{-\beta}, \qquad -y\sigma'(y) = \alpha + \beta\sigma(y)
\end{aligned}$$

for which

$$\frac{d\sigma(Y_t)}{\sigma(Y_t)} = \{\text{mean-reversion}\}\,dt + (\alpha + \beta\sigma(Y_t))\,dZ_t$$

- Another close SV model:

$$\begin{aligned}
\frac{dS_t}{S_t} &= a_t\,dW_t \\
\frac{da_t}{a_t} &= \{\text{mean-reversion}\}\,dt + (\alpha + \beta a_t)\,dZ_t, \qquad d\langle W,Z\rangle_t = \rho\,dt
\end{aligned}$$

Complete variance curve models (after Dupire, Buehler, Bergomi)

$$\xi_t^T = \xi_0^T F^T(t,(S_u, u \le t)), \qquad \frac{dS_t}{S_t} = \sqrt{\xi_t^t}\, dW_t$$

- Instantaneous forward variances $(\xi_t^T, 0 \le t \le T)$ are martingales. Functional Itô calculus (Dupire, 2009) $\Longrightarrow$

$$\Delta_t F^T + \frac{1}{2} S_t^2 \xi_0^t F^t(t,(S_u, u \le t)) \Delta_x^2 F^T = 0$$

- Particular case $\xi_t^T = \xi_0^T F^T(t, X_t)$ where $X_t = \frac{S_t}{\bar{S}_t^h}$, $\bar{S}_t^h$ EWMA:

$$\frac{dX_t}{X_t} = -\frac{1}{h}(X_t - 1)\, dt + \sqrt{\xi_t^t}\, dW_t$$

The ξ^T are martingales iff

$$\partial_t F^T(t,x) - \frac{1}{h} x(x-1)\partial_x F^T(t,x) + \frac{1}{2} x^2 \xi_0^t F^t(t,x) \partial_x^2 F^T(t,x) = 0$$

Complete variance curve models

$$\partial_t F^T(t,x) - \frac{1}{h} x(x-1)\partial_x F^T(t,x) + \frac{1}{2} x^2 \xi_0^t F^t(t,x) \partial_x^2 F^T(t,x) = 0$$

- For each given $F^T(T,\cdot)$, $F^T(t,x)$, $t < T$ is the solution to a 1d PDE. E.g., choose $F^T(T,x) = \sigma(x)^2$, i.e., $\sqrt{\xi_t^t} = \sqrt{\xi_0^t}\sigma(X_t)$:

$$\begin{aligned} \partial_t F^T(t,x) - \frac{1}{h} x(x-1)\partial_x F^T(t,x) + \frac{1}{2} x^2 \xi_0^t \sigma(x)^2 \partial_x^2 F^T(t,x) &= 0 \\ F^T(T,x) &= \sigma(x)^2 \end{aligned}$$

- If $(\xi_0^T, T \geq 0)$ is flat, then $F^T(t,x) = F(T-t,x)$.
- The whole variance curve is diffused.
- Any initial VS curve $(\xi_0^T, T \geq 0)$ is calibrated, without resorting to a leverage function.
- SPX smile is *not* calibrated.

Comparison of model families

	inst. vol model	variance curve model
Equation	$\frac{dS_t}{S_t} = \sigma(X_t) l_\sigma(t, S_t)\, dW_t$	$\begin{cases} \xi_t^T = \xi_0^T F^T(t, X_t) \\ \frac{dS_t}{S_t} = \sqrt{\xi_t^t}\, dW_t \\ \qquad = F^t(t, X_t)\sqrt{\xi_0^t}\, dW_t \end{cases}$
Calib. to VS curve at time 0	Yes $\Uparrow$	Yes
Calib. to SPX smile at time 0	Yes	No
Simul. of future inst. fwd variance ξ_t^T	Difficult $\mathbb{E}[\sigma(X_t)^2 l_\sigma(t, S_t)^2 \mid S_t, X_t]$	Easy $\xi_0^T F^T(t, X_t)$
Calib. to VIX smile at time 0	Depends on choice of X	Depends on choice of X

Examples of choices of X:

- $X_t = \frac{S_t}{\bar{S}_t^h}$, $\bar{S}_t^h$ EWMA
- $X_t =$ Ornstein-Uhlenbeck process driven by extra Brownian motion Z

Conclusion

We have introduced a simple path-dependent volatility model that:

- calibrates to SPX smile
- produces rich joint dynamics of spot, implied vol, and implied vol of vol
- captures historical joint behavior of SPX trend, VIX, and VVIX
- naturally generates mean reverting volatility
- jointly calibrates to SPX and VIX smiles? (work in progress)

This joint calibration approach to SPX and VIX smiles can use any X process, not necessarily path-dependent. Further results hopefully coming soon...

Thanks

I would like to thank Maxime Cauchois for producing the graphs.

A few selected references

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