Dispersion-Constrained Martingale Schrödinger Bridges: Exact Joint S&P 500/VIX Smile Calibration via Minimum Entropy

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Motivation

- Volatility indices, such as the VIX index, are not only used as market-implied indicators of volatility.
- Futures and options on these indices are also widely used as risk-management tools to hedge the volatility exposure of options portfolios.
- The **very high liquidity of S&P 500 (SPX) and VIX derivatives** requires that financial institutions price, hedge, and risk-manage their SPX and VIX options portfolios using **models that perfectly fit market prices of both SPX and VIX futures and options, jointly**.
- Calibration of stochastic volatility models to liquid hedging instruments: SPX options + VIX futures and options.
- Since VIX options started trading in 2006, many researchers and practitioners have tried to build such a jointly calibrating model, but could only, at best, get approximate fits.
- **"Holy Grail of volatility modeling"**
- **Very challenging problem, especially for short maturities**.

Brief reminder on the VIX index

- VIX = **V**olatility **I**nde**X**.
- Published every 15 seconds by the Chicago Board Options Exchange.
- Indicator of short-term options-implied volatility. Known as "fear factor."
- **Objective of CBOE: VIX is meant to reflect the 30-day implied volatility of SPX options.**
- Problem: implied vol of SPX call/put options depend on the option strike. VIX should be a strike-free measure of SPX implied vol.
- Natural choice: define VIX as the implied volatility of a 30-day variance swap on SPX.
- Problem: Variance swaps are OTC. Not listed on an exchange.
- $\Longrightarrow$ **VIX is defined as the implied volatility of a 30-day log-contract on SPX**:

$$(\mathrm{VIX}_t)^2 := -\frac{2}{\tau}\mathrm{Price}_t\left[\ln\left(\frac{S_{t+\tau}}{F_t^{t+\tau}}\right)\right], \qquad \tau = 30 \text{ days}$$

- The log-contract is not listed on an exchange but it can be replicated at t using OTM call and put options on the SPX with maturity $t+\tau$.

Motivation

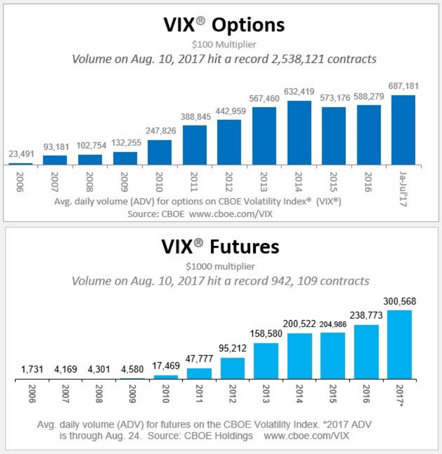


Figure: Average daily volume for VIX options and VIX futures. Source: CBOE

Motivation

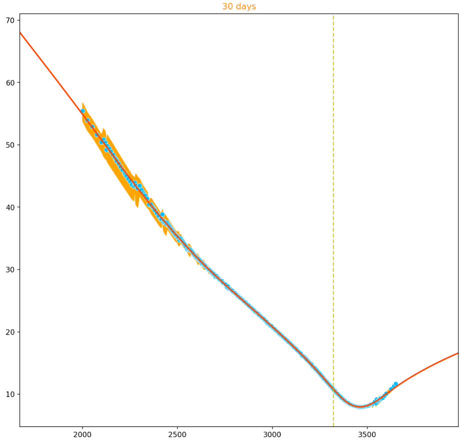


Figure: SPX smile as of January 22, 2020, $T = 30$ days

Motivation

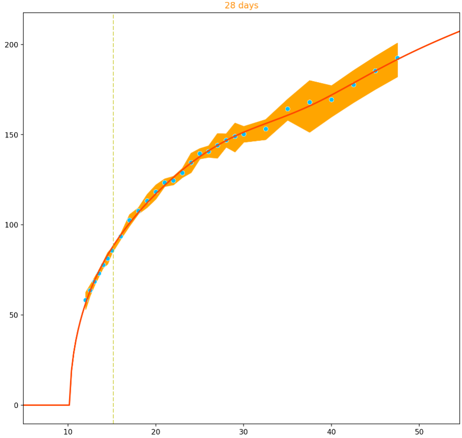


Figure: VIX smile as of January 22, 2020, $T = 28$ days

Motivation

- ATM skew:

$$
\begin{array}{rccl}
\text{Definition:} & \mathcal{S}_T & = & \left.\dfrac{d\sigma_{\mathrm{BS}}(K,T)}{\frac{dK}{K}}\right|_{K=F_T} \\
\text{SPX, small } T\text{:} & \mathcal{S}_T & \approx & -1.5 \\
\text{Classical one-factor SV model:} & \mathcal{S}_T & \underset{T\to 0}{\longrightarrow} & \dfrac{1}{2} \times \text{spot-vol correl} \times \text{vol-of-vol}
\end{array}
$$

- Calibration to short-term ATM SPX skew $\Longrightarrow$

vol-of-vol $\geq 3 = 300\% \gg$ short-term ATM VIX implied vol

The **very large negative skew of short-term SPX options**, which in classical continuous SV models implies a very large volatility of volatility, **seems inconsistent with the comparatively low levels of VIX implied volatilities**.

Gatheral (2008)

Consistent Modeling of SPX and VIX options

Consistent Modeling of SPX and VIX options

Jim Gatheral

Merrill Lynch

The Fifth World Congress of the Bachelier Finance Society
London, July 18, 2008

Double CEV dynamics

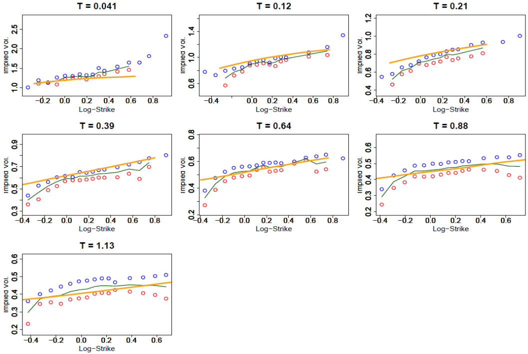
- Buehler's affine variance curve functional is consistent with double mean reverting dynamics of the form:

$$\begin{aligned} \frac{dS}{S} &= \sqrt{v}\, dW \\ dv &= -\kappa\,(v - v')\, dt + \eta_1\, v^{\alpha}\, dZ_1 \\ dv' &= -c\,(v' - z_3)\, dt + \eta_2\, v'^{\beta}\, dZ_2 \end{aligned} \tag{2}$$

  for any choice of $\alpha,\ \beta \in [1/2, 1]$.
  - We will call the case $\alpha = \beta = 1/2$ *Double Heston*,
  - the case $\alpha = \beta = 1$ *Double Lognormal*,
  - and the general case *Double CEV*.
- All such models involve a short term variance level v that reverts to a moving level v' at rate κ. v' reverts to the long-term level z_3 at the slower rate $c < \kappa$.

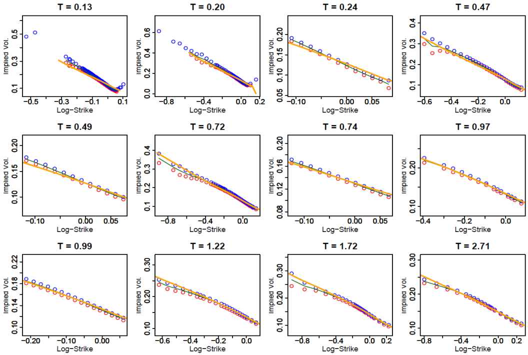
Double CEV fit to VIX options as of 03-Apr-2007

Setting the correlation ρ between volatility factors z_1 and z_2 to its historical average (see later) and iterating on the volatility of volatility parameters ξ_1 and ξ_2 to minimize the differences between model and market VIX option prices, we obtain the following fits (orange lines):

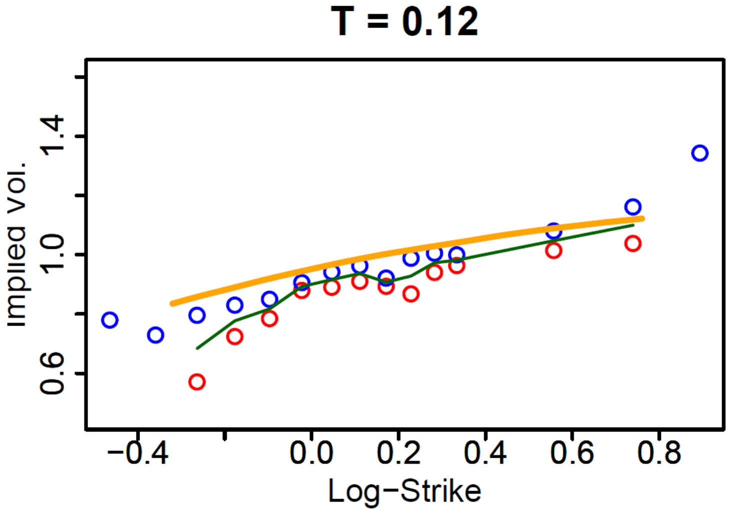


Double CEV fit to SPX options as of 03-Apr-2007

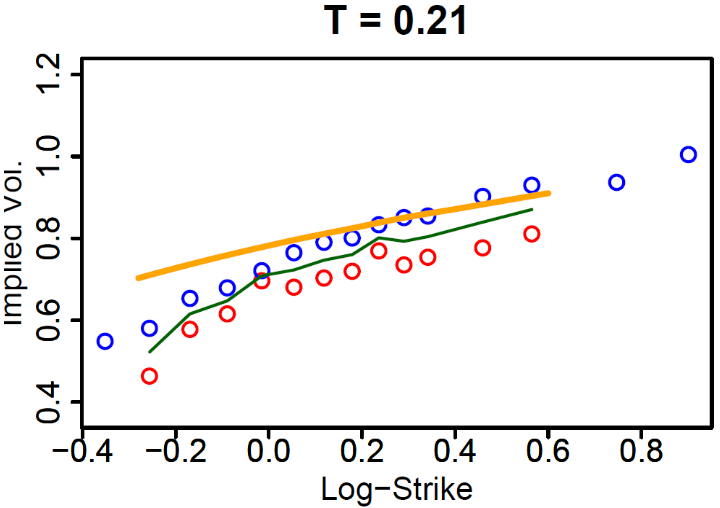
Minimizing the differences between model and market SPX option prices, we find $\rho_1 = -0.9$, $\rho_2 = -0.7$ and obtain the following fits to SPX option prices (orange lines):



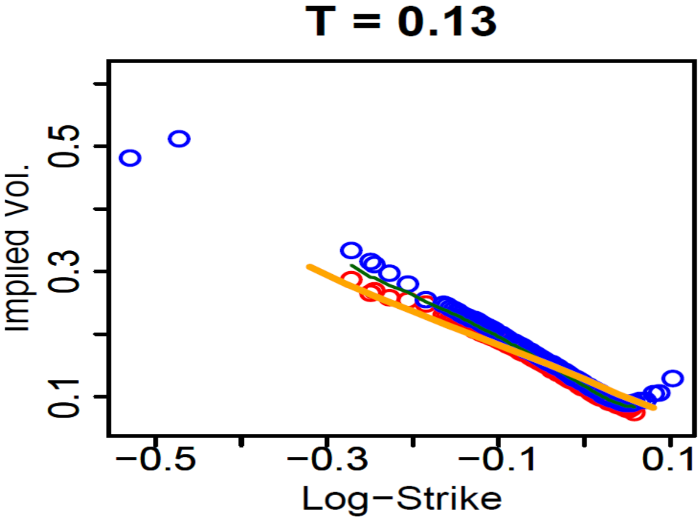
Fit to VIX options



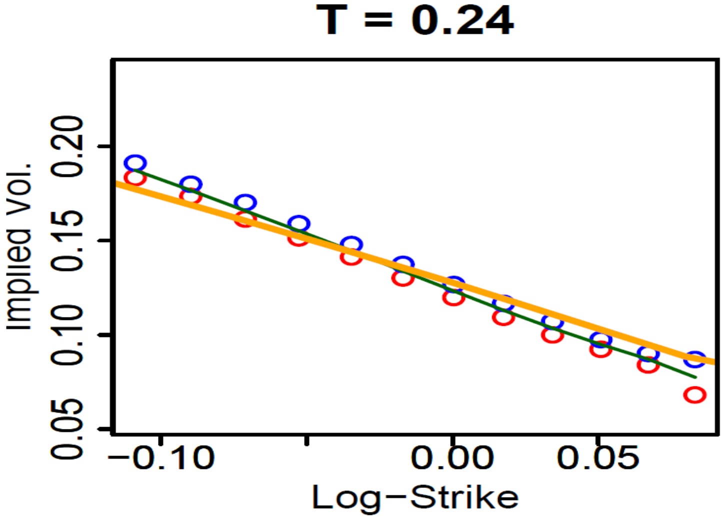
Fit to VIX options



Fit to SPX options



Fit to SPX options



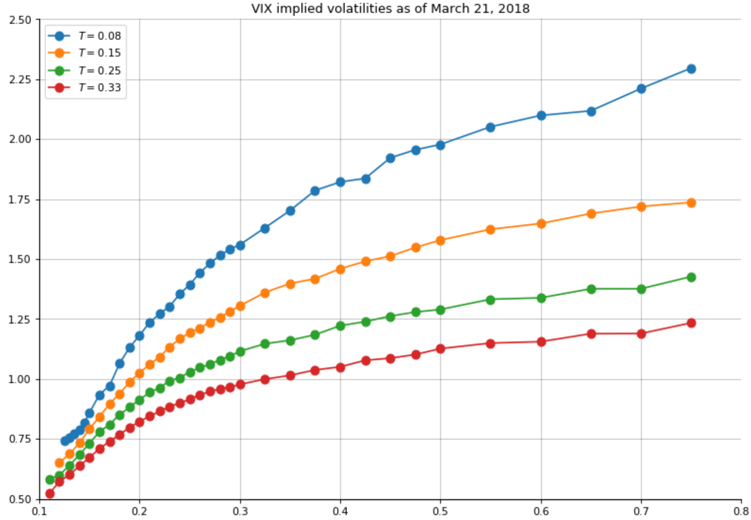
Similar experiments with other models

- Skewed 2-factor Bergomi model (Bergomi 2008)
- Skewed rough Bergomi model (G. 2018, De Marco 2018):

$$\sigma_t^2 = \xi_0^t \left((1-\lambda)\mathcal{E}\left(\nu_0 \int_0^t (t-s)^{H-\frac{1}{2}} dZ_s\right) + \lambda\mathcal{E}\left(\nu_1 \int_0^t (t-s)^{H-1/2} dZ_s\right)\right)$$

  with $\lambda \in [0,1]$.
- Quadratic rough Heston model (Gatheral Jusselin Rosenbaum 2020)
- **VIX smile well calibrated $\Longrightarrow$ not enough SPX ATM skew**

Skewed rough Bergomi: Calibration to VIX futures and options (G. 2018)



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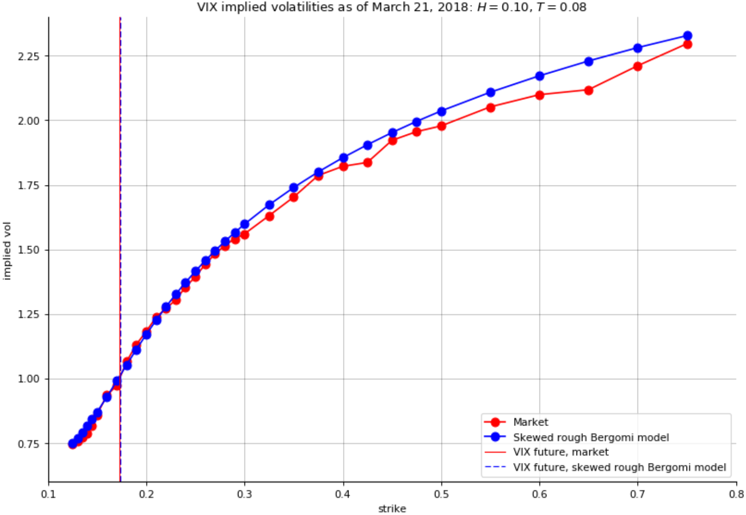

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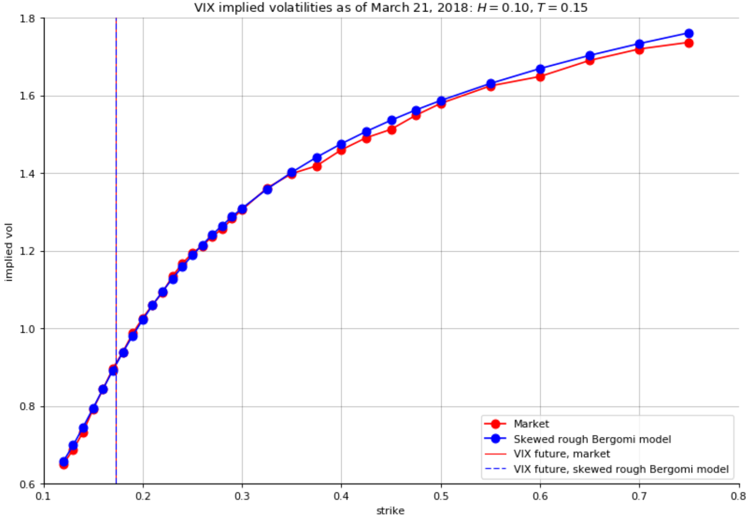

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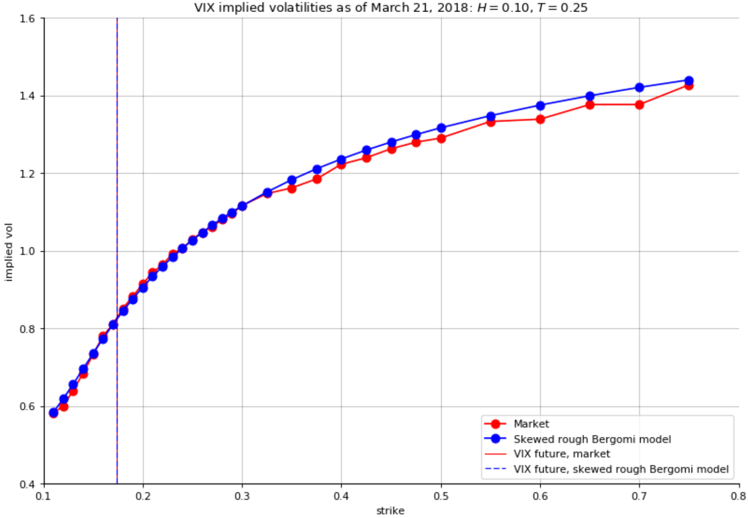
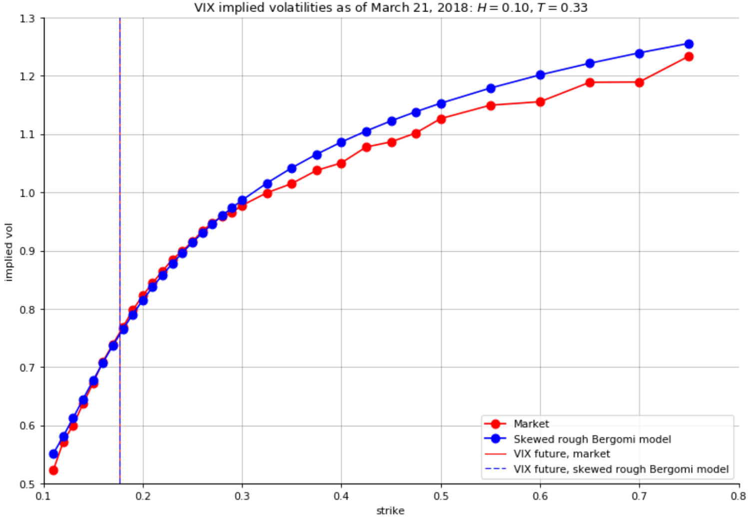
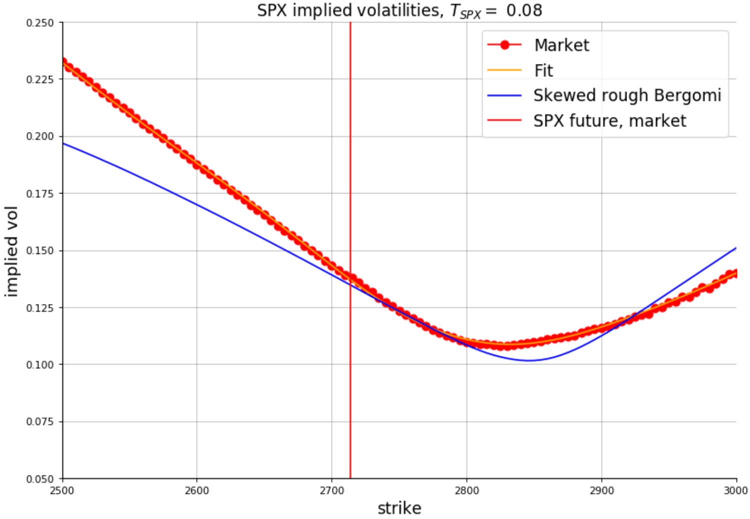


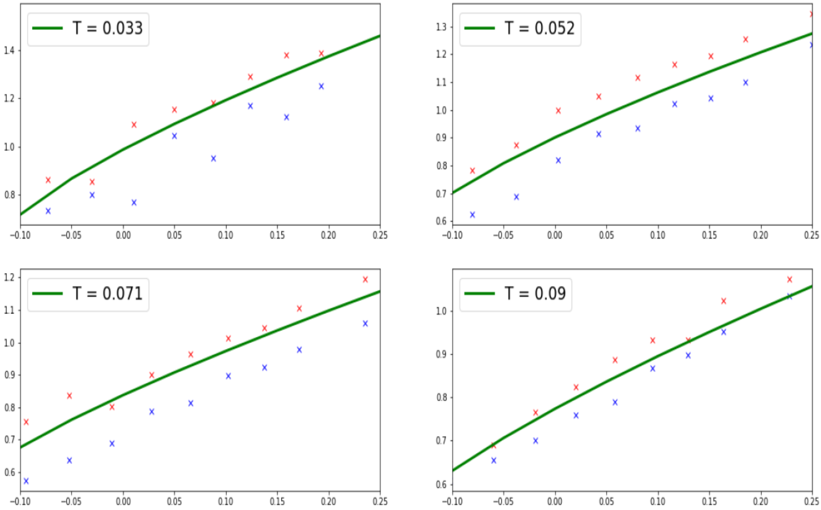
Skewed rough Bergomi: Calibration to VIX future and options (G. 2018)



Skewed rough Bergomi calibrated to VIX: SPX smile (G. 2018)



Quadratic rough Heston model (Gatheral Jusselin Rosenbaum 2020)



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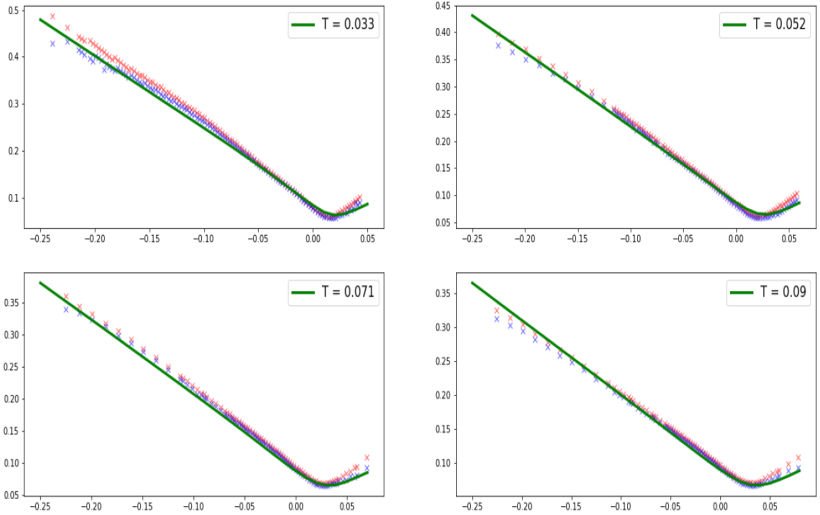

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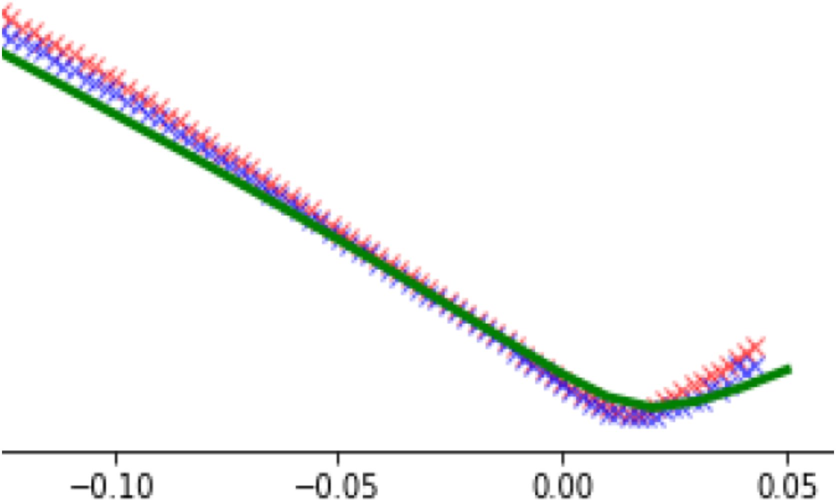

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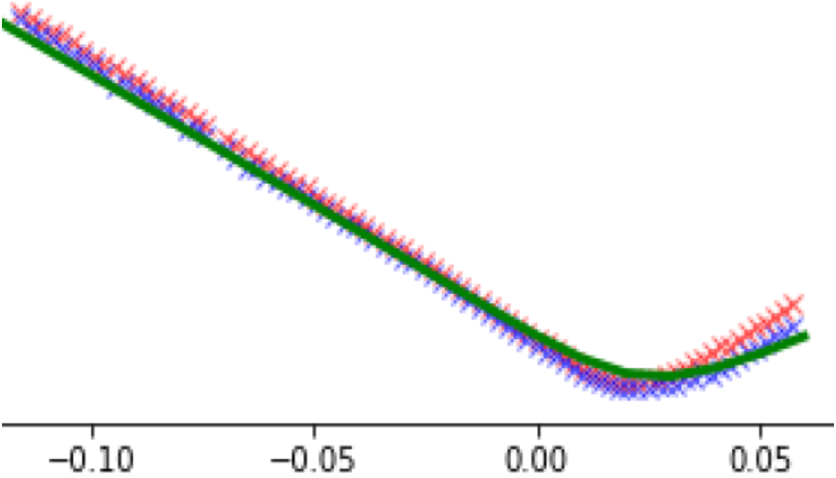


Related works with continuous models on the SPX

- Jacquier-Martini-Muguruza, *On the VIX futures in the rough Bergomi model* (2017):

  "Interestingly, we observe a 20% difference between the [vol-of-vol] parameter obtained through VIX calibration and the one obtained through SPX. This suggests that the volatility of volatility in the SPX market is 20% higher when compared to VIX, revealing potential data inconsistencies (arbitrage?)."

- Goutte-Ismail-Pham (2017), Heston with parameters driven by a Hidden Markov jump process.
- Fouque-Saporito (2018), Heston with stochastic vol-of-vol. See later. **Param**
- Gatheral-Jusselin-Rosenbaum (2020): quadratic rough Heston volatility model. See later. **Param**
- Guo-Loeper-Obłój-Wang (2020): joint calibration via semimartingale nonlinear optimal transport. **Closely related to VIX-contrained martingale Schrödinger bridges**. See later. **Nonparam**
- G. (2020): The VIX Future in Bergomi Models: Analytical Expansions and Joint Calibration with S&P 500 Skew.

Motivation

- To try to jointly fit the SPX and VIX smiles, many authors have incorporated **jumps** in the dynamics of the SPX: Sepp, Cont-Kokholm, Papanicolaou-Sircar, Baldeaux-Badran, Pacati-Pompa-Renò, Kokholm-Stisen, Bardgett-Gourier-Leippold, Forde-Gerhold-Smith...
- Jumps offer extra degrees of freedom to partly decouple the ATM SPX skew and the ATM VIX implied volatility.
- So far all the attempts at solving the joint SPX/VIX smile calibration problem only produced an **approximate fit**.

Exact joint SPX/VIX smile calibration: a dispersion-constrained martingale Schrödinger problem approach

(G. 2019)

Exact joint calibration as a dispersion-constrained martingale Schrödinger problem (G. 2019)

- **A completely different approach**: instead of postulating a parametric continuous-time (jump-)diffusion model on the SPX, we build a **nonparametric discrete-time model**:
  - Help decouple SPX skew and VIX implied vol.
  - Perfectly fits the smiles.
- Given a VIX future maturity T_1, we build a **joint probability measure on** (S_1, V, S_2) which is **perfectly calibrated** to the SPX smiles at T_1 and $T_2 = T_1 + 30$ days, and the VIX future and VIX smile at T_1.
- S_1: SPX at T_1, V: VIX at T_1, S_2: SPX at T_2.
- Our model satisfies:
  - **Martingality constraint** on the SPX;
  - **Consistency condition**: the VIX at T_1 is the implied volatility of the 30-day log-contract on the SPX.
- Our model is cast as the solution of a **dispersion-constrained martingale transport problem** which is solved using the **Sinkhorn algorithm**, in the spirit of De March and Henry-Labordère (2019).

Risk, April 2020

The joint S&P 500/Vix smile calibration puzzle solved

Since Vix options started trading in 2006, many researchers have tried to build a model that jointly and exactly calibrates to the prices of Standard & Poor's 500 options, Vix futures and Vix options. In this article, Julien Guyon solves this long-standing puzzle by casting it as a discrete-time dispersion-constrained martingale transport problem, which he solves in a non-parametric way using Sinkhorn's algorithm

Volatility indexes, such as the Vix index, do not just serve as market-implied indicators of volatility. Futures and options on these indexes are also widely used as risk management tools to hedge the volatility exposure of options portfolios. The existence of a liquid market for these futures and options has led to the need for models that jointly calibrate to the prices of options on the underlying asset and the prices of volatility derivatives. Without such models, financial institutions could possibly arbitrage each other: even market-making desks within the same institution could do so, eg, the Vix desk could arbitrage the S&P 500 (SPX) desk. By using models that fail to correctly incorporate the prices of the hedging instruments, such as SPX options, Vix futures and Vix options, exotic desks may misprice options, especially (but not only) those with payoffs that involve both the underlying and its volatility index.

For this reason, since Vix options began trading in 2006, many researchers and practitioners have tried to build a model that jointly and exactly calibrates to the prices of SPX futures, SPX options, Vix futures and Vix options. This is known to be a very challenging problem, especially for short maturities. In particular, the very large negative skew of short-term SPX options, and Vix smiles: that the distribution of the Dupire market local variance be smaller than the distribution of the (instantaneous) Vix squared in the convex order, at all times. He also reported that for short maturities the distribution of the true Vix squared in the market local volatility model is actually larger than the market-implied distribution of the true Vix squared in the convex order. Guyon showed numerically that when the (typically negative) spot-vol correlation is large enough in absolute value, both (a) traditional stochastic volatility models with large mean reversion and (b) rough volatility models with a small Hurst exponent can reproduce this inversion of convex ordering. Acciaio & Guyon (2020) provide a mathematical proof that the inversion of convex ordering can be produced by continuous models. However, the inversion of convex ordering is only a necessary condition for the joint SPX/Vix calibration of continuous models; it is not sufficient.

Since it looks to be very difficult to jointly calibrate the SPX and Vix smiles with continuous models, many authors have incorporated jumps in the dynamics of the SPX: see references in Guyon (2019a). Jumps offer extra degrees of freedom to partly decouple the ATM SPX skew and the ATM Vix implied volatility. However, the attempts so far have only produced

Preprint, 2021

DISPERSION-CONSTRAINED MARTINGALE SCHRÖDINGER PROBLEMS AND THE EXACT JOINT S&P 500/VIX SMILE CALIBRATION PUZZLE*

JULIEN GUYON
QUANTITATIVE RESEARCH, BLOOMBERG L.P.

ABSTRACT. We solve for the first time* a longstanding puzzle of quantitative finance that has often been described as the Holy Grail of volatility modeling: build a model that jointly and exactly calibrates to the prices of S&P 500 (SPX) options, VIX futures, and VIX options. So far the best attempts, which used parametric continuous-time (jump-)diffusion models on the SPX, only produced approximate fits. We use a very different, nonparametric and discrete-time approach. Given a VIX future maturity T_1, we consider the set $\mathcal{P}$ of all joint probability measures on the SPX at T_1, the VIX at T_1, and the SPX at $T_2 = T_1 + 30$ days which are perfectly calibrated to the full SPX smiles at T_1 and T_2, and the full VIX smile at T_1, and which also satisfy the martingality constraint on the SPX as well as the requirement that the VIX is the implied volatility of the 30-day log-contract on the SPX.

We first consider robust hedging in this setting. By casting the superreplication problem as what we call a *dispersion-constrained martingale optimal transport problem*, we establish a strong duality theorem and, as a result, prove that the absence of joint SPX/VIX arbitrage is equivalent to the set $\mathcal{P}$ of jointly calibrating models being nonempty. Should they arise, joint arbitrages are identified using classical linear programming. In the absence of joint arbitrage, we then provide a solution to the joint calibration puzzle by solving what we call a *dispersion-constrained martingale Schrödinger problem*: we choose a reference measure and build the unique jointly calibrating model that minimizes the relative entropy. We establish several dual versions of the problem, one of which has a natural financial interpretation in terms of exponential utility indifference pricing, and prove absence of duality gaps. The minimum entropy jointly calibrating model is explicit in terms of what we call the dual *Schrödinger portfolio*, i.e., the maximizer of the dual problems, should it exist. We numerically compute this Schrödinger portfolio using an extension of the Sinkhorn algorithm, in the spirit of De March and Henry-Labordère (2019). Our numerical experiments show that the algorithm performs very well in both low and high volatility regimes.

Along the way, we provide new variants, as well as a new proof, of strong duality theorems for the classical Schrödinger problem and for a mixed *Schrödinger-Monge-Kantorovich problem* (also known as *entropic optimal transport problem*) that has recently attracted a lot of attention in the optimal transport community, which are interesting in themselves. Our methodology applies not only to the VIX, but also to any index computed as a function of the price of an option on some underlying asset.

1. INTRODUCTION

Implied volatility indices, such as the VIX index [17], do not only serve as market-implied indicators of volatility. Futures and options on these indices are also widely used as risk-management tools to hedge the

Setting and notation

- For simplicity: zero interest rates, repos, and dividends.
- μ_1 = risk-neutral distribution of $S_1 \longleftrightarrow$ market smile of SPX at T_1.
- μ_V = risk-neutral distribution of $V \longleftrightarrow$ market smile of VIX at T_1.
- μ_2 = risk-neutral distribution of $S_2 \longleftrightarrow$ market smile of SPX at T_2.
- F_V: value at time 0 of VIX future maturing at T_1.
- We denote $\mathbb{E}^i := \mathbb{E}^{\mu_i}$, $\mathbb{E}^V := \mathbb{E}^{\mu_V}$ and assume

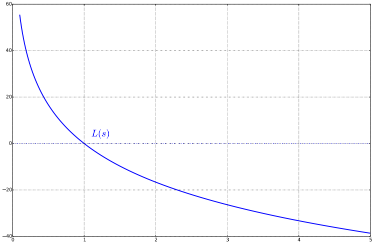
$$\mathbb{E}^i[S_i] = S_0, \quad \mathbb{E}^i[|\ln S_i|] < \infty, \quad i \in \{1,2\}; \qquad \mathbb{E}^V[V] = F_V, \quad \mathbb{E}^V[V^2] < \infty.$$

- No calendar arbitrage $\Longleftrightarrow \mu_1 \leq_c \mu_2$ (convex order)

Setting and notation

$$V^2 := (\mathrm{VIX}_{T_1})^2 := -\frac{2}{\tau}\mathrm{Price}_{T_1}\left[\ln\left(\frac{S_2}{S_1}\right)\right] = \mathrm{Price}_{T_1}\left[L\left(\frac{S_2}{S_1}\right)\right]$$

- $\tau := 30$ days.
- $L(x) := -\frac{2}{\tau}\ln x$: convex, decreasing.



Build a jointly calibrating model

- Let $\mathcal{P}(\mu_1, \mu_V, \mu_2)$:= probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ s.t.

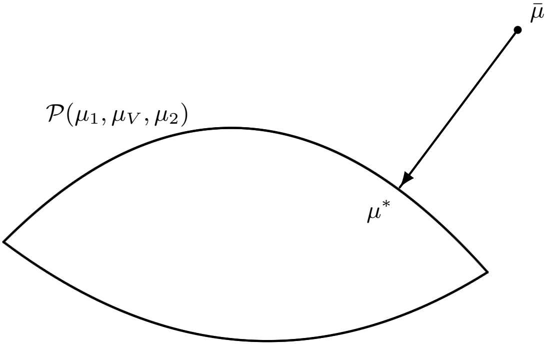
$$S_1 \sim \mu_1, \quad V \sim \mu_V, \quad S_2 \sim \mu_2, \quad \mathbb{E}^{\mu}\left[S_2 | S_1, V\right] = S_1, \quad \mathbb{E}^{\mu}\left[L\left(\frac{S_2}{S_1}\right) \middle| S_1, V\right] = V^2.$$

- **Strong duality** for **dispersion-constrained martingale optimal transport** (G., 2019): Absence of joint SPX/VIX arbitrage $\Longleftrightarrow \mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$.
- **Build a model $\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)$ = solve the joint calibration puzzle**.
- Our strategy is inspired by Avellaneda (1998, 2001) and De March and Henry-Labordère (2019).
- Choose a **reference probability measure** $\bar{\mu}$ on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ and look for the measure $\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)$ that **minimizes the relative entropy** $H(\mu|\bar{\mu})$ of μ w.r.t. $\bar{\mu}$, also known as the Kullback-Leibler divergence:

$$D_{\bar{\mu}} := \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu|\bar{\mu}), \quad H(\mu|\bar{\mu}) := \begin{cases} \mathbb{E}^{\mu}\left[\ln \frac{d\mu}{d\bar{\mu}}\right] = \mathbb{E}^{\bar{\mu}}\left[\frac{d\mu}{d\bar{\mu}} \ln \frac{d\mu}{d\bar{\mu}}\right] & \text{if } \mu \ll \bar{\mu}, \\ +\infty & \text{otherwise.} \end{cases}$$

- This is a **strictly convex problem** that can **efficiently be solved after dualization using an extension of Sinkhorn's algorithm** (Sinkhorn, 1967).

Build a jointly calibrating model



Reminder on Lagrange multipliers

$$\inf_{g(x,y)=c} f(x,y) = \inf_{x,y} \sup_{\lambda\in\mathbb{R}} \{f(x,y) - \lambda(g(x,y)-c)\}$$
$$= \sup_{\lambda\in\mathbb{R}} \inf_{x,y} \{f(x,y) - \lambda(g(x,y)-c)\}$$

- To compute the inner inf over x, y unconstrained, simply solve $\nabla f(x,y) = \lambda \nabla g(x,y)$: easy!
- Then maximize the result over λ unconstrained: easy!
- Constraint $g(x,y) = c \iff \frac{\partial}{\partial\lambda}\{f(x,y) - \lambda(g(x,y)-c)\} = 0$.

$$\inf_{\mu \text{ s.t. } S_1\sim\mu_1} H(\mu|\bar{\mu}) = \inf_{\mu} \sup_{u_1(\cdot)} \left\{ H(\mu|\bar{\mu}) + \mathbb{E}^1[u_1(S_1)] - \mathbb{E}^\mu[u_1(S_1)] \right\}$$

$$\inf_{\mu \text{ s.t. } \mathbb{E}^\mu[S_2|S_1,V]=S_1} H(\mu|\bar{\mu}) = \inf_{\mu} \sup_{\Delta_S(\cdot,\cdot)} \left\{ H(\mu|\bar{\mu}) - \mathbb{E}^\mu\left[\Delta_S(S_1,V)(S_2-S_1)\right] \right\}$$

$$\inf_{\mu \text{ s.t. } \mathbb{E}^\mu\left[L\left(\frac{S_2}{S_1}\right)\middle|S_1,V\right]=V^2} H(\mu|\bar{\mu}) = \inf_{\mu} \sup_{\Delta_L(\cdot,\cdot)} \left\{ H(\mu|\bar{\mu}) - \mathbb{E}^\mu\left[\Delta_L(S_1,V)\left(L\left(\frac{S_2}{S_1}\right) - V^2\right)\right] \right\}$$

Reminder on Lagrange multipliers

$$\begin{aligned}\inf_{g(x,y)=c} f(x,y) &= \inf_{x,y} \sup_{\lambda\in\mathbb{R}} \{f(x,y)-\lambda(g(x,y)-c)\} \\ &= \sup_{\lambda\in\mathbb{R}} \inf_{x,y} \{f(x,y)-\lambda(g(x,y)-c)\}\end{aligned}$$

- To compute the inner inf over x, y unconstrained, simply solve $\nabla f(x,y) = \lambda \nabla g(x,y)$: easy!
- Then maximize the result over λ unconstrained: easy!
- Constraint $g(x,y)=c \iff \frac{\partial}{\partial\lambda}\{f(x,y)-\lambda(g(x,y)-c)\} = 0$.

$$\inf_{\mu \text{ s.t. } S_1\sim\mu_1} H(\mu|\bar{\mu}) = \inf_{\mu} \sup_{u_1(\cdot)} \left\{H(\mu|\bar{\mu}) + \mathbb{E}^1[u_1(S_1)] - \mathbb{E}^{\mu}\left[u_1(S_1)\right]\right\}$$

$$\inf_{\mu \text{ s.t. } \mathbb{E}^{\mu}[S_2|S_1,V]=S_1} H(\mu|\bar{\mu}) = \inf_{\mu} \sup_{\Delta_S(\cdot,\cdot)} \left\{H(\mu|\bar{\mu}) - \mathbb{E}^{\mu}\left[\Delta_S(S_1,V)(S_2-S_1)\right]\right\}$$

$$\inf_{\mu \text{ s.t. } \mathbb{E}^{\mu}\left[L\left(\frac{S_2}{S_1}\right)\middle|S_1,V\right]=V^2} H(\mu|\bar{\mu}) = \inf_{\mu} \sup_{\Delta_L(\cdot,\cdot)} \left\{H(\mu|\bar{\mu}) - \mathbb{E}^{\mu}\left[\Delta_L(S_1,V)\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)\right]\right\}$$

Reminder on Lagrange multipliers

$$\begin{aligned}\inf_{g(x,y)=c} f(x,y) &= \inf_{x,y} \sup_{\lambda\in\mathbb{R}} \{f(x,y)-\lambda(g(x,y)-c)\} \\ &= \sup_{\lambda\in\mathbb{R}} \inf_{x,y} \{f(x,y)-\lambda(g(x,y)-c)\}\end{aligned}$$

- To compute the inner inf over x, y unconstrained, simply solve $\nabla f(x,y) = \lambda \nabla g(x,y)$: easy!
- Then maximize the result over λ unconstrained: easy!
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Build a jointly calibrating model

- $\mathcal{M}_1$: set of probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$: **unconstrained**
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$$
\begin{aligned}
D_{\bar{\mu}} &:= \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu|\bar{\mu}) \\
&= \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}} \Big\{ H(\mu|\bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
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\end{aligned}
$$

Build a jointly calibrating model

$$D_{\bar{\mu}} = \sup_{u\in\mathcal{U}} \Big\{\mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)]$$
$$+ \inf_{\mu\in\mathcal{M}_1} \Big\{H(\mu|\bar{\mu}) - \mathbb{E}^{\mu}\Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Big(\Delta_S^{(S)} + \Delta_L^{(L)}\Big)(S_1, V, S_2)\Big]\Big\}\Big\}$$

- **Remarkable fact: The inner infimum can be exactly computed:**

$$\inf_{\mu\in\mathcal{M}_1} \{H(\mu|\bar{\mu}) - \mathbb{E}^{\mu}[X]\} = -\ln \mathbb{E}^{\bar{\mu}}\left[e^X\right]$$

and **the infimum is attained at $\mu = \bar{\mu}_X$ defined by** (Gibbs type)

$$\frac{d\bar{\mu}_X}{d\bar{\mu}} = \frac{e^X}{\mathbb{E}^{\bar{\mu}}[e^X]}.$$

- That is why we like (and chose) the “distance” $H(\mu|\bar{\mu})$!

Build a jointly calibrating model

$$D_{\bar\mu} := \inf_{\mu\in\mathcal{P}(\mu_1,\mu_V,\mu_2)} H(\mu|\bar\mu) \;=\; \sup_{u\in\mathcal{U}} J^{\ln}_{\bar\mu}(u) =: P_{\bar\mu}$$

$$J^{\ln}_{\bar\mu}(u) := \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] - \ln \mathbb{E}^{\bar\mu}\left[e^{u_1(S_1)+u_V(V)+u_2(S_2)+\Delta_S^{(S)}(S_1,V,S_2)+\Delta_L^{(L)}(S_1,V,S_2)}\right].$$

- $\inf_{\mu\in\mathcal{P}(\mu_1,\mu_V,\mu_2)}$: **constrained** optimization, **difficult**.
- $\sup_{u\in\mathcal{U}}$: **unconstrained** optimization, **easy!** If sup is attained, to find the optimum $u^* = (u_1^*, u_V^*, u_2^*, \Delta_S^*, \Delta_L^*)$, simply cancel the gradient of $J^{\ln}_{\bar\mu}$.
- Most important, $\inf_{\mu\in\mathcal{P}(\mu_1,\mu_V,\mu_2)} H(\mu|\bar\mu)$ is reached at

$$\mu^*(ds_1,dv,ds_2) = \bar\mu(ds_1,dv,ds_2)\frac{e^{u_1^*(s_1)+u_V^*(v)+u_2^*(s_2)+\Delta_S^{*(S)}(s_1,v,s_2)+\Delta_L^{*(L)}(s_1,v,s_2)}}{\mathbb{E}^{\bar\mu}\left[e^{u_1^*(S_1)+u_V^*(V)+u_2^*(S_2)+\Delta_S^{*(S)}(S_1,V,S_2)+\Delta_L^{*(L)}(S_1,V,S_2)}\right]}.$$

- **Problem solved:** $\mu^* \in \mathcal{P}(\mu_1,\mu_V,\mu_2)$**!**

Strong duality for the VIX-constrained martingale Schrödinger problem

- Notation $(u_1 \oplus u_V \oplus u_2)(s_1, v, s_2) := u_1(s_1) + u_V(v) + u_2(s_2)$
- Recall

$$D_{\bar\mu} := \inf_{\mu\in\mathcal{P}(\mu_1,\mu_V,\mu_2)} H(\mu|\bar\mu) \;=\; \sup_{u\in\mathcal{U}} J_{\bar\mu}^{\ln}(u) =: P_{\bar\mu}$$

$$J_{\bar\mu}^{\ln}(u) := \mathbb{E}^1[u_1(S_1)]+\mathbb{E}^V[u_V(V)]+\mathbb{E}^2[u_2(S_2)]-\ln \mathbb{E}^{\bar\mu}\left[e^{\left(u_1\oplus u_V\oplus u_2+\Delta_S^{(S)}+\Delta_L^{(L)}\right)(S_1,V,S_2)}\right]$$

- Generalization:

$$\mathcal{G} := \{g : (0,+\infty] \to (-\infty,+\infty] \mid \forall x \in (0,+\infty],\ g(x) \geq \ln x \quad \text{and} \quad g(1) = 0\}$$

For $g \in \mathcal{G}$ we define $J_{\bar\mu}^{g} : \mathcal{U} = L^1 B \to [-\infty,+\infty)$ by

$$J_{\bar\mu}^{g}(u) := \mathbb{E}^1[u_1(S_1)]+\mathbb{E}^V[u_V(V)]+\mathbb{E}^2[u_2(S_2)]-g\left(\mathbb{E}^{\bar\mu}\left[e^{\left(u_1\oplus u_V\oplus u_2+\Delta_S^{(S)}+\Delta_L^{(L)}\right)(S_1,V,S_2)}\right]\right)$$

$$Z_{\bar\mu}(u) := \mathbb{E}^{\bar\mu}\left[e^{\left(u_1\oplus u_V\oplus u_2+\Delta_S^{(S)}+\Delta_L^{(L)}\right)(S_1,V,S_2)}\right]$$

For $E \in \{L^1B, CC_b, C_bC_b\}$, we denote $E_{\exp} := \{u \in E \,|\, Z_{\bar\mu}(u) < +\infty\}$.

Strong duality for the VIX-constrained martingale Schrödinger problem

Theorem (G. 2020)

Let $\bar\mu \in \mathcal{M}_1$ *and* $g \in \mathcal{G}$. *The following equality holds in* $[0, +\infty]$:

$$\sup_{u \in CC_b} J^g_{\bar\mu}(u) = \sup_{u \in L^1 B} J^g_{\bar\mu}(u) = \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu|\bar\mu).$$

Moreover:

1. *The stronger* $C_b C_b$*-duality* $\sup_{u \in C_b C_b} J^g_{\bar\mu}(u) = \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu|\bar\mu)$ *holds if for all* $(\Delta_S, \Delta_L) \in C_b^2$, $\mathbb{E}^{\bar\mu}\left[e^{\left(\Delta_S^{(S)} + \Delta_L^{(L)}\right)(S_1, V, S_2)}\right] < +\infty$.
2. *If* $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$, *then the infimum is attained.*
3. *If the problem is finite, then* $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$ *and the infimum is uniquely attained. We then denote by* $\mu^* := \arg\min_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu|\bar\mu)$ *the minimum entropy jointly calibrating model.*

Strong duality for the VIX-constrained martingale Schrödinger problem

Theorem (G. 2020, cont'd)

4 *Let* $E \in \{L^1B, CC_b\}$ *and* $u^* \in E$. *The following assertions are equiv.:*
  - (i) $J_{\bar{\mu}}^{\ln}(u^*) = \sup_{u \in E} J_{\bar{\mu}}^{\ln}(u)$.
  - (ii) *The problem is finite,* $u^* \in E_{\exp}$, *and*

$$\frac{d\mu^*}{d\bar{\mu}} = Z_{\bar{\mu}}(u^*)^{-1} \, e^{\left(u_1^* \oplus u_V^* \oplus u_2^* + \Delta_S^{*(S)} + \Delta_L^{*(L)}\right)(S_1, V, S_2)} \qquad \bar{\mu}\text{−a.s.}$$

In this case, let $u^\dagger := (u_1^* - \ln Z_{\bar{\mu}}(u^*), u_V^*, u_2^*, \Delta_S^*, \Delta_L^*)$. *Then* $u^\dagger \in E$ *and* $u^\dagger$ *satisfies the three equivalent assertions below.*

5 *Let* $E \in \{L^1B, CC_b\}$ *and* $u^\dagger \in E$. *The following assertions are equivalent:*
  - (i) $J_{\bar{\mu}}^{\mathrm{id}-1}(u^\dagger) = \sup_{u \in E} J_{\bar{\mu}}^{\mathrm{id}-1}(u)$.
  - (ii) *For all* $g \in \mathcal{G}$, $J_{\bar{\mu}}^{g}(u^\dagger) = \sup_{u \in E} J_{\bar{\mu}}^{g}(u)$.
  - (iii) *The problem is finite and*

$$\frac{d\mu^*}{d\bar{\mu}} = e^{\left(u_1^\dagger \oplus u_V^\dagger \oplus u_2^\dagger + \Delta_S^{\dagger(S)} + \Delta_L^{\dagger(L)}\right)(S_1, V, S_2)} \qquad \bar{\mu}\text{−a.s.}$$

In this case, $Z_{\bar{\mu}}(u^\dagger) = 1$.

Strong duality for the VIX-constrained martingale Schrödinger problem

$$\frac{d\mu^*}{d\bar{\mu}} = Z_{\bar{\mu}}(u^*)^{-1} e^{\left(u_1^* \oplus u_V^* \oplus u_2^* + \Delta_S^{*(S)} + \Delta_L^{*(L)}\right)(S_1,V,S_2)} \quad \bar{\mu}-\text{a.s.}$$

$$= e^{\left(u_1^\dagger \oplus u_V^\dagger \oplus u_2^\dagger + \Delta_S^{\dagger(S)} + \Delta_L^{\dagger(L)}\right)(S_1,V,S_2)} \quad \bar{\mu}-\text{a.s.}$$

- We call maximizers $(u_1^*, u_V^*, u_2^*, \Delta_S^*, \Delta_L^*)$ and $(u_1^\dagger, u_V^\dagger, u_2^\dagger, \Delta_S^\dagger, \Delta_L^\dagger)$ **Schrödinger potentials** (if they exist).
- We call the corresponding portfolios
  $\pi_{u^*} := u_1^* \oplus u_V^* \oplus u_2^* + \Delta_S^{*(S)} + \Delta_L^{*(L)}$ and
  $\pi_{u^\dagger} := u_1^\dagger \oplus u_V^\dagger \oplus u_2^\dagger + \Delta_S^{\dagger(S)} + \Delta_L^{\dagger(L)}$ **Schrödinger portfolios**.
- The Schrödinger portfolio is essentially unique: two Schrödinger portfolios π_{u^*} and $\pi_{u^\dagger}$ are $\bar{\mu}$-a.e. equal up to an additive constant.
- We call $\pi_{u^\dagger}$ the **standard Schrödinger portfolio**.

Strong duality for the VIX-constrained martingale Schrödinger problem

Sketch of the proof of strong duality:

1. Prove strong duality for the classical Schrödinger problem (marginal constraints only, $E = L^1$ or C or C_b)

$$\inf_{\mu\in\Pi(\mu_1,\mu_V,\mu_2)} H(\mu|\bar\mu) = \sup_{(u_1,u_V,u_2)\in E} J^{\mathsf{S},g}_{\bar\mu}(u_1,u_V,u_2)$$

using Fenchel-Rockafellar convex duality theorem. Be careful: in general $\mathcal{M}(\mathcal{X}) \subsetneq C_b(\mathcal{X})^*$ when $\mathcal{X}$ is not compact!

2. Extend to the mixed Schrödinger-Monge-Kantorovich problem (or entropy-regularized optimal transport problem, $E = L^1$ or C)

$$\inf_{\mu\in\Pi(\mu_1,\mu_V,\mu_2)} \{H(\mu|\bar\mu) - \mathbb{E}^\mu[f(S_1,V,S_2)]\} = \sup_{(u_1,u_V,u_2)\in E} J^{\mathsf{SMK},g}_{\bar\mu,f}(u_1,u_V,u_2).$$

$$J^{\mathsf{S},g}_{\bar\mu}(u_1,u_V,u_2) := \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] - g\left(\mathbb{E}^{\bar\mu}\left[e^{(u_1\oplus u_V\oplus u_2)(S_1,V,S_2)}\right]\right)$$

$$J^{\mathsf{SMK},g}_{\bar\mu,f}(u_1,u_V,u_2) := \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] - g\left(\mathbb{E}^{\bar\mu}\left[e^{(u_1\oplus u_V\oplus u_2+f)(S_1,V,S_2)}\right]\right)$$

Strong duality for the VIX-constrained martingale Schrödinger problem

Sketch of the proof:

1. Prove strong duality for the classical Schrödinger problem (marginal constraints only, $E = L^1$ or C or C_b)

$$\inf_{\mu\in\Pi(\mu_1,\mu_V,\mu_2)} H(\mu|\bar\mu) = \sup_{(u_1,u_V,u_2)\in E} J^{\mathsf{S},g}_{\bar\mu}(u_1,u_V,u_2)$$

using Fenchel-Rockafellar convex duality theorem. Be careful: in general $\mathcal{M}(\mathcal{X}) \subsetneq C_b(\mathcal{X})^*$ when $\mathcal{X}$ is not compact!

2. Extend to the mixed Schrödinger-Monge-Kantorovich problem (or entropy-regularized optimal transport problem, $E = L^1$ or C)

$$\inf_{\mu\in\Pi(\mu_1,\mu_V,\mu_2)} \{H(\mu|\bar\mu) - \mathbb{E}^\mu[f(S_1,V,S_2)]\} = \sup_{(u_1,u_V,u_2)\in E} J^{\mathsf{SMK},g}_{\bar\mu,f}(u_1,u_V,u_2).$$

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Strong duality for the VIX-constrained martingale Schrödinger problem

Sketch of proof:

1. Prove strong duality for the classical Schrödinger problem

$$\inf_{\mu\in\Pi(\mu_1,\mu_V,\mu_2)} H(\mu|\bar{\mu}) = \sup_{(u_1,u_V,u_2)\in E} J^{\mathsf{S},g}_{\bar{\mu}}(u_1,u_V,u_2)$$

2. Extend to the mixed Schrödinger-Monge-Kantorovich problem

$$\inf_{\mu\in\Pi(\mu_1,\mu_V,\mu_2)} \{H(\mu|\bar{\mu}) - \mathbb{E}^\mu[f(S_1,V,S_2)]\} = \sup_{(u_1,u_V,u_2)\in E} J^{\mathsf{SMK},g}_{\bar{\mu},f}(u_1,u_V,u_2).$$

3. Back to the dispersion-constrained martingale Schrödinger problem:
   1. Dualize the martingality and dispersion constraints:

$$\inf_{\mu\in\mathcal{P}(\mu_1,\mu_V,\mu_2)} H(\mu|\bar{\mu}) = \inf_{\mu\in\Pi(\mu_1,\mu_V,\mu_2)} \sup_{(\Delta_S,\Delta_L)\in C_b} \left\{H(\mu|\bar{\mu}) - \mathbb{E}^\mu\left[\left(\Delta_S^{(S)} + \Delta_L^{(L)}\right)(S_1,V,S_2)\right]\right\}$$

   2. Use the weak compactness of $\Pi(\mu_1,\mu_V,\mu_2)$ and Sion's minimax theorem to swap inf and sup.
   3. Apply the SMK strong duality.

The Schrödinger equations (a.k.a. Schrödinger system)

$$
\begin{aligned}
\frac{\partial J^{\mathrm{id}-1}_{\bar\mu}}{\partial u_1(s_1)}=0: &\quad \forall s_1>0, & u_1(s_1) &= \Phi_1(s_1;u_V,u_2,\Delta_S,\Delta_L)\\
\frac{\partial J^{\mathrm{id}-1}_{\bar\mu}}{\partial u_V(v)}=0: &\quad \forall v\geq 0, & u_V(v) &= \Phi_V(v;u_1,u_2,\Delta_S,\Delta_L)\\
\frac{\partial J^{\mathrm{id}-1}_{\bar\mu}}{\partial u_2(s_2)}=0: &\quad \forall s_2>0, & u_2(s_2) &= \Phi_2(s_2;u_1,u_V,\Delta_S,\Delta_L)\\
\frac{\partial J^{\mathrm{id}-1}_{\bar\mu}}{\partial \Delta_S(s_1,v)}=0: &\quad \forall s_1>0,\ \forall v\geq 0, & 0 &= \Phi_{\Delta_S}(s_1,v;u_2,\Delta_S(s_1,v),\Delta_L(s_1,v))\\
\frac{\partial J^{\mathrm{id}-1}_{\bar\mu}}{\partial \Delta_L(s_1,v)}=0: &\quad \forall s_1>0,\ \forall v\geq 0, & 0 &= \Phi_{\Delta_L}(s_1,v;u_2,\Delta_S(s_1,v),\Delta_L(s_1,v))
\end{aligned}
$$

- We could have simply postulated a model of the form

$$\mu(ds_1,dv,ds_2)=\bar\mu(ds_1,dv,ds_2)\,e^{u_1(s_1)+u_V(v)+u_2(s_2)+\Delta_S^{(S)}(s_1,v,s_2)+\Delta_L^{(L)}(s_1,v,s_2)}.$$

- Then the 5 conditions defining $\mathcal{P}(\mu_1,\mu_V,\mu_2)$ translate into the 5 above equations.
- The system of equations is solved using **Sinkhorn's algorithm**.

Sinkhorn's algorithm

- Sinkhorn's algorithm (1967) is a coordinate ascent method which was first used in the context of optimal transport by Cuturi (2013). It performs **alternating projections**.
- Extension: **Fixed point method that alternates maximizations in the different directions (one per Lagrange multiplier) to approximate the maximizer $u^\dagger$.**
- Start from initial guess $u^{(0)} = (u_1^{(0)}, u_V^{(0)}, u_2^{(0)}, \Delta_S^{(0)}, \Delta_L^{(0)})$, recursively define $u^{(n+1)}$ knowing $u^{(n)}$ by

$$
\begin{aligned}
\forall s_1 > 0, \qquad u_1^{(n+1)}(s_1) &= \Phi_1(s_1; u_V^{(n)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
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\forall s_2 > 0, \qquad u_2^{(n+1)}(s_2) &= \Phi_2(s_2; u_1^{(n+1)}, u_V^{(n+1)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_1 > 0,\ \forall v \geq 0, \qquad 0 &= \Phi_{\Delta_S}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n)}(s_1, v)) \\
\forall s_1 > 0,\ \forall v \geq 0, \qquad 0 &= \Phi_{\Delta_L}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n+1)}(s_1, v))
\end{aligned}
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  until convergence.
- Each of the above 5 lines corresponds to a Bregman projection in the space of measures.

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\forall s_2 > 0, \quad & u_2^{(n+1)}(s_2) &=& \ \Phi_2(s_2; u_1^{(n+1)}, u_V^{(n+1)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_1 > 0, \ \forall v \geq 0, \quad & 0 &=& \ \Phi_{\Delta_S}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n)}(s_1, v)) \\
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\end{aligned}$$

  until convergence.
- Each of the above 5 lines corresponds to a **Bregman projection** in the space of measures.

Sinkhorn's algorithm

- **If the algorithm diverges**, then $P_{\bar{\mu}} = +\infty$, so $D_{\bar{\mu}} = +\infty$, i.e.,

$$\mathcal{P}(\mu_1, \mu_V, \mu_2) \cap \{\mu \in \mathcal{M}_1 | H(\mu|\bar{\mu}) < +\infty\} = \emptyset.$$

- In practice, when $\bar{\mu}$ has full support, this is a sign that **there likely exists a joint SPX/VIX arbitrage**.
- One should directly check if $\mathcal{P}(\mu_1, \mu_V, \mu_2) = \emptyset$ (linear program).
- We have never experienced this situation in our numerical tests, which covered both low and high volatility regimes.

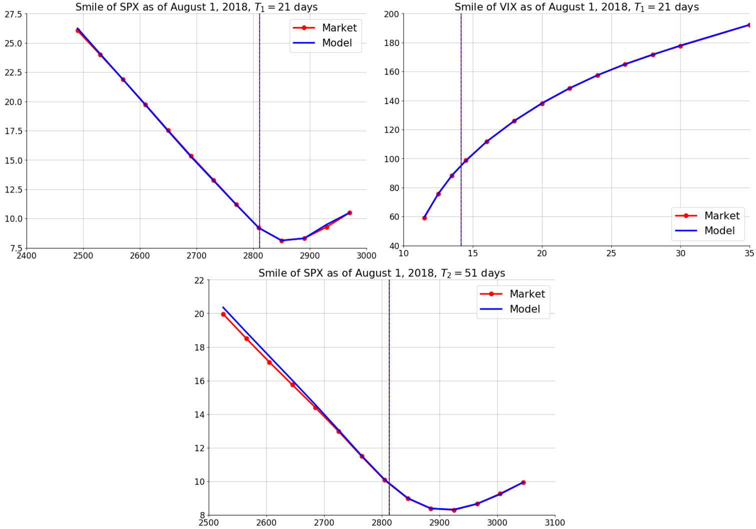
Numerical experiments

Implementation details

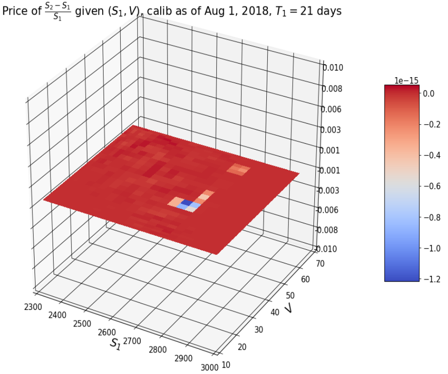
- Choice of $\bar{\mu}$:
  - $S_1 \sim \mu_1$ and $V \sim \mu_V$ independent;
  - Conditional on (S_1, V), S_2 lognormal with mean S_1 and variance V.

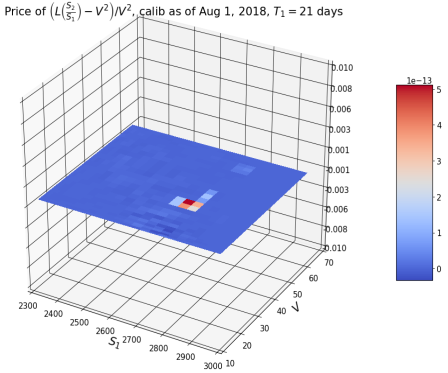
  Under $\bar{\mu}$, $S_2 \not\sim \mu_2$.
- Instead of abstract payoffs u_1, u_V, u_2, we work with market strikes and market prices of vanilla options on S_1, V, and S_2.
- Canceling the gradient of $J_{\bar{\mu}}^{\ln} \to$ system of equations solved using Sinkhorn's algorithm.
- Enough accuracy is typically reached after ≈ 100 iterations.

August 1, 2018, $T_1 = 21$ days



August 1, 2018, $T_1 = 21$ days





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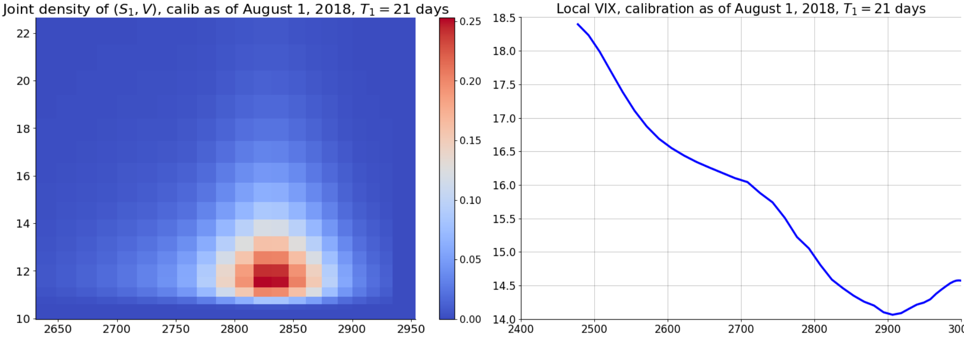


Figure: Joint distribution of (S_1, V) and local VIX function $\mathrm{VIX}_{\mathrm{loc}}(s_1)$

$$\mathrm{VIX}^2_{\mathrm{loc}}(S_1) := \mathbb{E}^{\mu^*}\left[V^2 \middle| S_1\right]$$

August 1, 2018, $T_1 = 21$ days

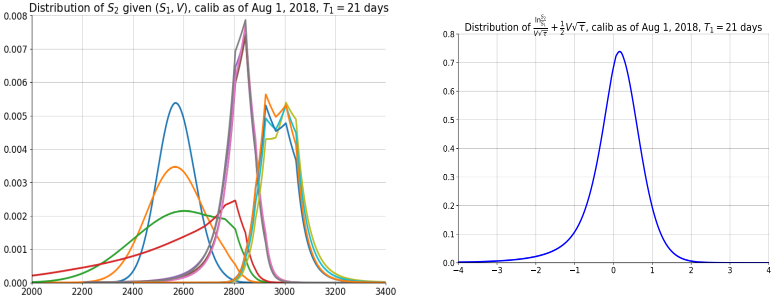
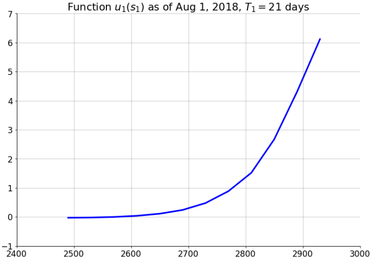
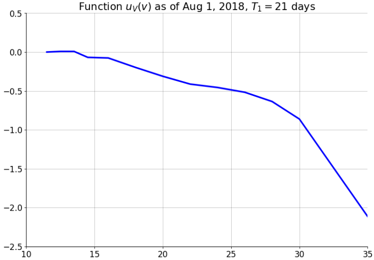
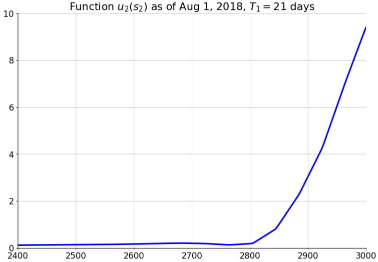


Figure: Conditional distribution of S_2 given (s_1, v) under μ^* for different vales of (s_1, v): $s_1 \in \{2571, 2808, 3000\}$, $v \in \{10.10, 15.30, 23.20, 35.72\}\%$, and distribution of the normalized return $R := \frac{\ln(S_2/S_1)}{V\sqrt{\tau}} + \frac{1}{2}V\sqrt{\tau}$

August 1, 2018, $T_1 = 21$ days







August 1, 2018, $T_1 = 21$ days

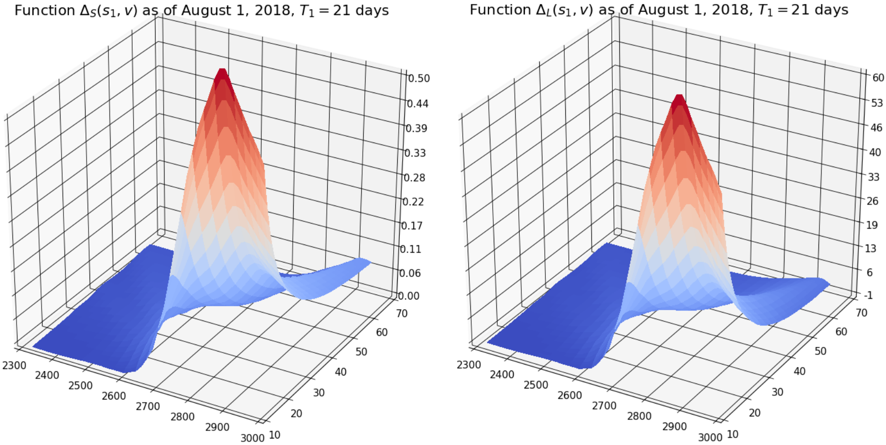
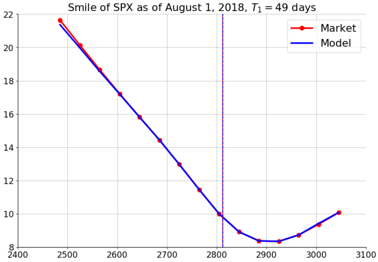
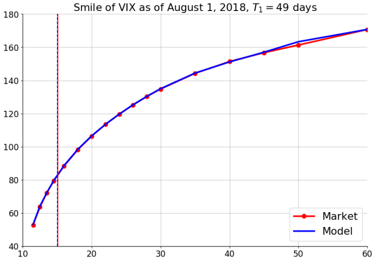
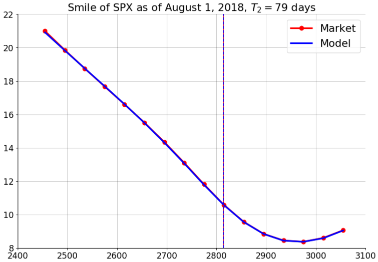


Figure: Optimal functions $\Delta_S^*(s_1, v)$ and $\Delta_L^*(s_1, v)$ for (s_1, v) in the quadrature grid

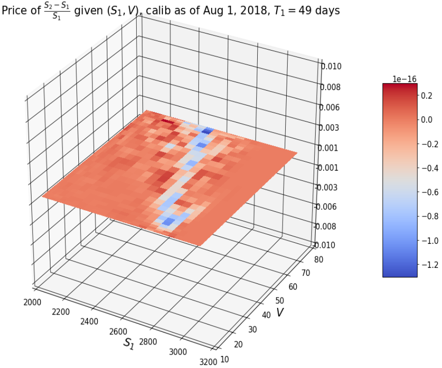
August 1, 2018, $T_1 = 49$ days

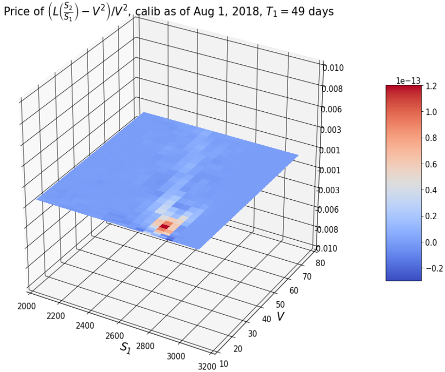




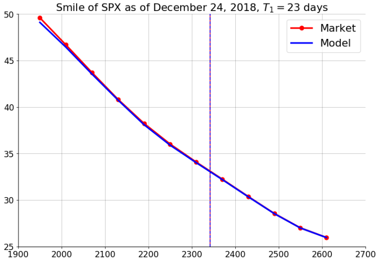


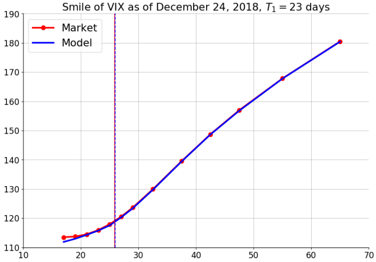
August 1, 2018, $T_1 = 49$ days

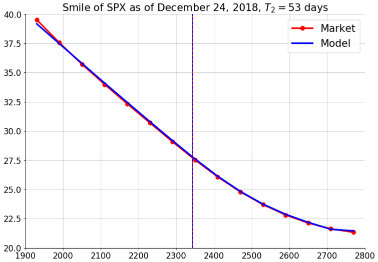




December 24, 2018, $T_1 = 23$ days: large VIX, $F_V \approx 26\%$







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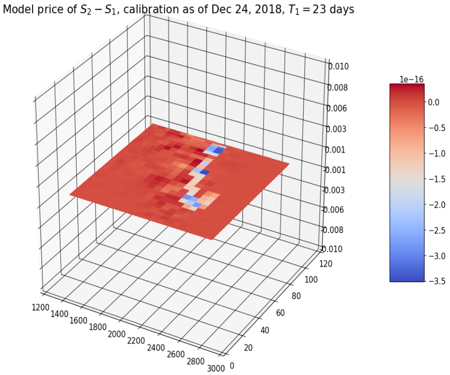
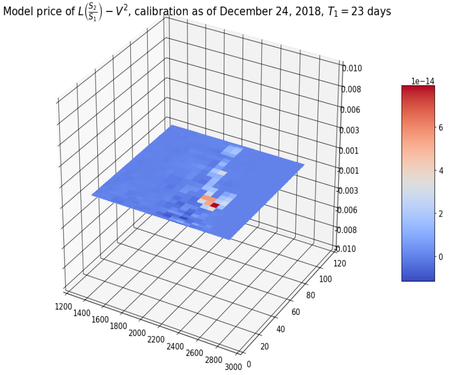




Strong duality for the VIX-constrained martingale optimal transport (Monge-Kantorovich) problem

$\mathcal{U}_f$: set of superreplicating portfolios, i.e., the set of all functions $(u_1, u_V, u_2, \Delta_S, \Delta_L)$ that satisfy the superreplication constraint:

$$u_1(s_1) + u_V(v) + u_2(s_2) + \Delta_S^{(S)}(s_1, v, s_2) + \Delta_L^{(L)}(s_1, v, s_2) \geq f(s_1, v, s_2).$$

Theorem (G. 2020)

Let $f : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}$ *be upper semicontinuous and satisfy*

$$|f(s_1, v, s_2)| \leq C\big(1 + s_1 + s_2 + |L(s_1)| + |L(s_2)| + v^2\big)$$

for some constant $C > 0$. *Then*

$$P_f := \inf_{\mathcal{U}_f} \Big\{ \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \Big\}$$
$$= \sup_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} \mathbb{E}^\mu[f(S_1, V, S_2)] =: D_f.$$

Moreover, $D_f \neq -\infty$ *if and only if* $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$, *and in that case the supremum is attained.*

Characterization of joint SPX/VIX arbitrage

Definition: A joint SPX/VIX arbitrage, or (S_1, S_2, V)-arbitrage, is a portfolio that superreplicates $f \equiv 0$ with negative price.

Theorem (G. 2020)

The following assertions are equivalent:

(i) *The market is free of joint SPX/VIX arbitrage.*

(ii) $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$.

(iii) *There exists a coupling* ν *of* μ_1 *and* μ_V *such that* $\mathrm{Law}_\nu(S_1, L(S_1) + V^2)$ *and* $\mathrm{Law}_{\mu_2}(S_2, L(S_2))$ *are in convex order, i.e., for any convex function* $f : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}^\nu[f(S_1, L(S_1) + V^2)] \leq \mathbb{E}^2[f(S_2, L(S_2))].$$

Continuous time: Exact joint calibration via dispersion-constrained martingale Schrödinger bridges

(G. 2020)

Martingale optimal transport approach in continuous time

- Same point of view as the discrete-time model: Pick a reference measure $\mathbb{P}_0 \longleftrightarrow$ a particular SV model:

$$\begin{aligned} \frac{dS_t}{S_t} &= a_t\, dW_t^0 \\ da_t &= b(a_t)\, dt + \sigma(a_t) \left(\rho\, dW_t^0 + \sqrt{1-\rho^2} dW_t^{0,\perp}\right) \end{aligned}$$

- We want to prove that $\mathcal{P} \neq \emptyset$ and build $\mathbb{P} \in \mathcal{P}$, where

$$\mathcal{P} := \{\mathbb{P} \in \mathcal{M}_1 \,|\, S_1 \sim \mu_1, S_2 \sim \mu_2, \sqrt{\mathbb{E}^{\mathbb{P}}[L(S_2/S_1)|\mathcal{F}_1]} \sim \mu_V, S \text{ is a } \mathbb{P}\text{-martingale}\}.$$

- No need to introduce a new r.v. for the VIX: $\text{VIX} = \sqrt{\mathbb{E}^{\mathbb{P}}[L(S_2/S_1)|\mathcal{F}_1]}$.
- We look for $\mathbb{P} \in \mathcal{P}$ that minimizes the relative entropy w.r.t. $\mathbb{P}_0$:

$$D := \inf_{\mathbb{P} \in \mathcal{P}} H(\mathbb{P}|\mathbb{P}_0)$$

- Inspired by Henry-Labordère 2019: *From (Martingale) Schrödinger Bridges to a New Class of Stochastic Volatility Models* (calib to SPX smiles)
- Follows closely the construction of **Schrödinger bridges**

Simple Schrödinger bridge (à la Follmer, Saint-Flour 1988)

$$\begin{aligned} dX_t &= dW_t^0, \qquad X_0 = x_0 \\ \mathcal{P} &:= \{\mathbb{P} \in \mathcal{M}_1 \,|\, X_1 \sim \mu_1\} \end{aligned}$$

$$\begin{aligned} D &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P}|\mathbb{P}_0) \\ &= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1)} \left\{ H(\mathbb{P}|\mathbb{P}_0) + \mathbb{E}^{\mu_1}\left[u_1(X_1)\right] - \mathbb{E}^{\mathbb{P}}\left[u_1(X_1)\right] \right\} \\ &= \sup_{u_1\in L^1(\mu_1)} \inf_{\mathbb{P}\in\mathcal{M}_1} \left\{ H(\mathbb{P}|\mathbb{P}_0) + \mathbb{E}^{\mu_1}\left[u_1(X_1)\right] - \mathbb{E}^{\mathbb{P}}\left[u_1(X_1)\right] \right\} \end{aligned}$$

Recall the remarkable fact about the inner infimum:

$$\inf_{\mathbb{P}\in\mathcal{M}_1} \left\{ H(\mathbb{P}|\mathbb{P}_0) - \mathbb{E}^{\mathbb{P}}\left[u_1(X_1)\right] \right\} = -\ln \mathbb{E}^{\mathbb{P}_0}\left[e^{u_1(X_1)}\right]$$

and the infimum is reached at $\mathbb{P}^*$ defined by $\dfrac{d\mathbb{P}^*}{d\mathbb{P}_0} = \dfrac{e^{u_1(X_1)}}{\mathbb{E}^{\mathbb{P}_0}\left[e^{u_1(X_1)}\right]}$.

Simple Schrödinger bridge (à la Follmer, Saint-Flour 1988)

$$
\begin{aligned}
dX_t &= dW_t^0, \qquad X_0 = x_0 \\
\mathcal{P} &:= \{\mathbb{P} \in \mathcal{M}_1 \,|\, X_1 \sim \mu_1\}
\end{aligned}
$$

$$
\begin{aligned}
D &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P}|\mathbb{P}_0) \\
&= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1)} \left\{ H(\mathbb{P}|\mathbb{P}_0) + \mathbb{E}^{\mu_1}[u_1(X_1)] - \mathbb{E}^{\mathbb{P}}[u_1(X_1)] \right\} \\
&= \sup_{u_1\in L^1(\mu_1)} \inf_{\mathbb{P}\in\mathcal{M}_1} \left\{ H(\mathbb{P}|\mathbb{P}_0) + \mathbb{E}^{\mu_1}[u_1(X_1)] - \mathbb{E}^{\mathbb{P}}[u_1(X_1)] \right\}
\end{aligned}
$$

Recall the remarkable fact about the inner infimum:

$$
\inf_{\mathbb{P}\in\mathcal{M}_1} \left\{ H(\mathbb{P}|\mathbb{P}_0) - \mathbb{E}^{\mathbb{P}}[u_1(X_1)] \right\} = -\ln \mathbb{E}^{\mathbb{P}_0}\left[e^{u_1(X_1)}\right]
$$

and the infimum is reached at $\mathbb{P}^*$ defined by $\dfrac{d\mathbb{P}^*}{d\mathbb{P}_0} = \dfrac{e^{u_1(X_1)}}{\mathbb{E}^{\mathbb{P}_0}\left[e^{u_1(X_1)}\right]}$.

Simple Schrödinger bridge (à la Follmer, Saint-Flour 1988)

$$D := \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P}|\mathbb{P}_0) = \sup_{u_1\in L^1(\mu_1)} \left\{\mathbb{E}^{\mu_1}\left[u_1(X_1)\right] - \ln\mathbb{E}^{\mathbb{P}_0}\left[e^{u_1(X_1)}\right]\right\} =: P$$

- Assume $P < +\infty$ and the sup is reached at u_1^*. Then

$$M_{T_1} := \frac{d\mathbb{P}^*}{d\mathbb{P}_0} = e^{u_1^*(X_1)} \qquad (Z = 1 \text{ by cash adjustment of } u_1^*)$$

- Let $M_t := \mathbb{E}^{\mathbb{P}_0}[M_{T_1}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}_0}[e^{u_1^*(X_1)}|\mathcal{F}_t]$. Then $M_t = U^*(t, X_t)$ where

$$\partial_t U^* + \frac{1}{2}\partial_x^2 U^* = 0, \qquad U^*(T_1, x) = e^{u_1^*(x)}.$$

- By Girsanov, $W_t^* := W_t^0 - \int_0^t \partial_x \ln U^*(s, X_s)\, ds$ is a $\mathbb{P}^*$-Brownian motion,

$$dX_t = \partial_x \ln U^*(t, X_t)\, dt + dW_t^* = \partial_x \ln \mathbb{E}^{\mathbb{P}_0}[e^{u_1^*(X_1)}|X_t = x]_{|X_t}\, dt + dW_t^*$$

- Brownian motion with drift, which is explicitly known.
- In practice, $u_1(X_1)$ is replaced by $\sum_{K\in\mathcal{K}} \alpha_K (X_1 - K)_+$. The gradient of

$$\mathbb{E}^{\mu_1}\left[\sum_{K\in\mathcal{K}} \alpha_K (X_1 - K)_+\right] - \ln\mathbb{E}^{\mathbb{P}_0}\left[e^{\sum_{K\in\mathcal{K}} \alpha_K (X_1-K)_+}\right]$$

is simply the vector of differences between model and market call prices.

VIX-constrained martingale Schrödinger bridge

$$\frac{dS_t}{S_t} = a_t\, dW_t^*$$
$$da_t = \left(b(a_t) + (1-\rho^2)\sigma(a_t)^2 \partial_a u^*(t, S_t, a_t)\right) dt + \sigma(a_t)\left(\rho\, dW_t^* + \sqrt{1-\rho^2} dW_t^{*,\perp}\right)$$

- Let $P := \sup_{u_1, u_V, u_2} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) - u(0, S_0, a_0) \right\}$ where u is solution to a nonlinear Hamilton-Jacobi-Bellman PDE:

$$\begin{aligned} u(T_2, s, a; \delta^L) &= u_2(s) + \delta^L L(s), \\ \partial_t u + \mathcal{L}^0 u + \frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 &= 0, \quad t \in (T_1, T_2), \\ \Phi(s,a) := \sup_{v\geq 0} \inf_{\delta^L \in \mathbb{R}} \Big\{ u_V(v) &- \delta^L(L(s) + v^2) + u(T_1, s, a; \delta^L) \Big\}, \\ u(T_1, s, a) &= u_1(s) + \Phi(s,a), \\ \partial_t u + \mathcal{L}^0 u + \frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 &= 0, \quad t \in [0, T_1). \end{aligned}$$

- Assume $P < +\infty$ and (u_1^*, u_V^*, u_2^*) maximizes $P \longrightarrow u^*$

VIX-constrained martingale Schrödinger bridge

$$\frac{dS_t}{S_t} = a_t\, dW_t^*$$

$$da_t = \left(b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a u^*(t,S_t,a_t)\right) dt + \sigma(a_t)\left(\rho\, dW_t^* + \sqrt{1-\rho^2}dW_t^{*,\perp}\right)$$

- Optimal deltas:

$$\Delta_t^* = -\partial_s u^*(t,S_t,a_t) - \rho\frac{\sigma(a_t)}{a_t S_t}\partial_a u^*(t,S_t,a_t); \quad \Delta^{*,L} = \delta^{*,L}(S_1,a_1)$$

- The drift of (a_t) under $\mathbb{P}^*$ also reads as

$$b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a \ln \mathbb{E}^0[e^{u_1^*(S_1)+\int_t^{T_1}\Delta^*(r,S_r,a_r)dS_r+\Phi^*(S_1,a_1)}|S_t,a_t], \quad t\in[0,T_1],$$

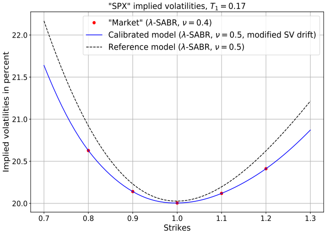
$$b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a \ln \mathbb{E}^0[e^{u_2^*(S_2)+\int_t^{T_2}\Delta^*(r,S_r,a_r)dS_r+\delta^{*,L}(S_1,a_1)L(S_2)}|S_t,a_t], \quad t\in[T_1,T_2].$$

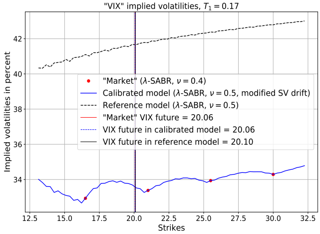
- It is path-dependent on $[T_1,T_2]$.
- If $P=+\infty$, then

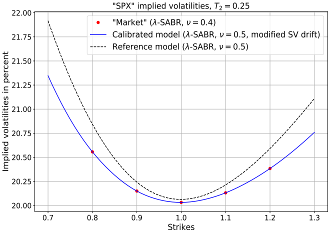
$$\mathcal{P}\cap\{\mathbb{P}\in\mathcal{M}_1\,|\,H(\mathbb{P}|\mathbb{P}_0)<+\infty\}=\emptyset.$$

$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 0.4$, $\mathbb{P}_0 : \nu = 0.5$

$$k = 1.5, \quad a_0 = m = 0.2, \quad \rho = 0$$







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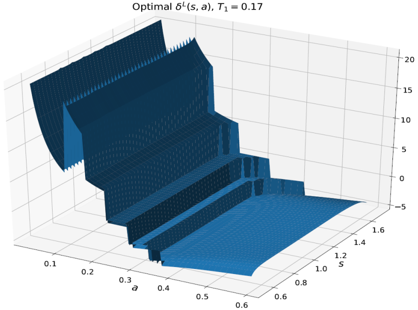

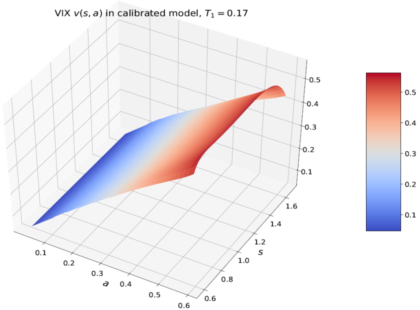

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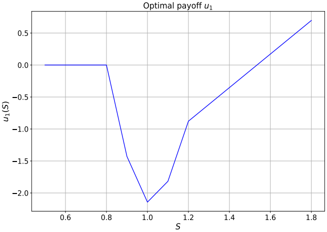
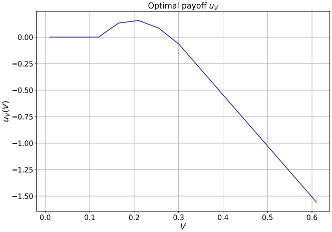
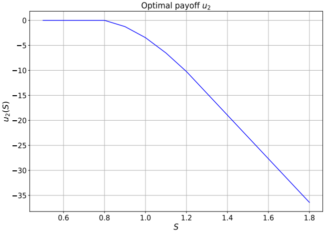
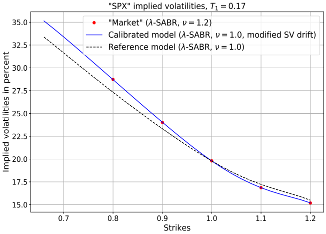


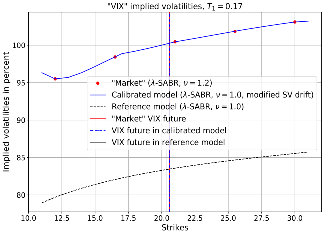


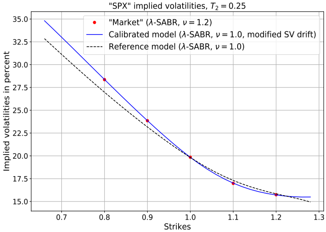


$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 1.2$, $\mathbb{P}_0 : \nu = 1$

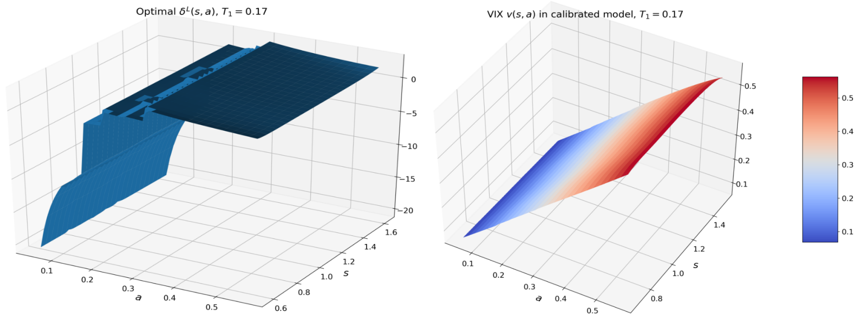
$$k = 1.5, \quad a_0 = m = 0.2, \quad \rho = -0.7$$



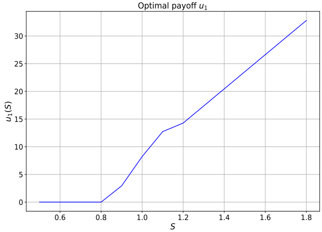


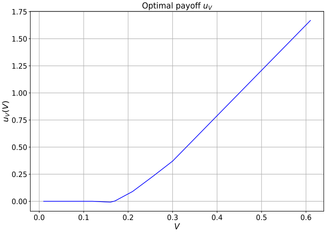


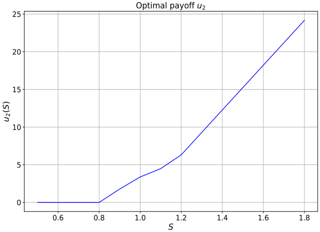
$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 1.2$, $\mathbb{P}_0 : \nu = 1$



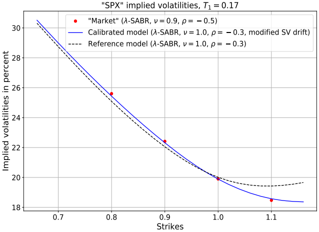
$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 1.2$, $\mathbb{P}_0 : \nu = 1$

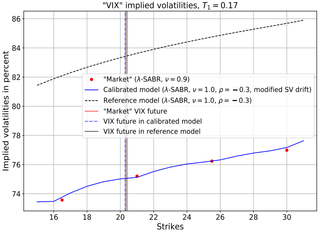


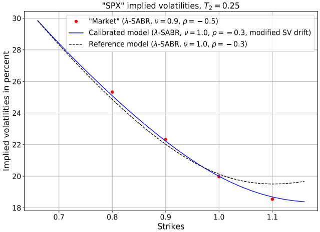




$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 0.9$, $\rho = -0.5$, $\mathbb{P}_0 : \nu = 1$, $\rho = -0.3$







Thanks!

Equations for $u^* = (u_1^*, u_V^*, u_2^*, \Delta_S^*, \Delta_L^*)$

$$
\begin{aligned}
\frac{\partial J_{\bar\mu}^{\ln}}{\partial u_1(s_1)}=0: \quad & \forall s_1 > 0, & u_1(s_1) &= \Phi_1(s_1; u_V, u_2, \Delta_S, \Delta_L) \\
\frac{\partial J_{\bar\mu}^{\ln}}{\partial u_V(v)}=0: \quad & \forall v \geq 0, & u_V(v) &= \Phi_V(v; u_1, u_2, \Delta_S, \Delta_L) \\
\frac{\partial J_{\bar\mu}^{\ln}}{\partial u_2(s_2)}=0: \quad & \forall s_2 > 0, & u_2(s_2) &= \Phi_2(s_2; u_1, u_V, \Delta_S, \Delta_L) \\
\frac{\partial J_{\bar\mu}^{\ln}}{\partial \Delta_S(s_1,v)}=0: \quad & \forall s_1 > 0,\ \forall v \geq 0, & 0 &= \Phi_{\Delta_S}(s_1, v; \Delta_S(s_1,v), \Delta_L(s_1,v)) \\
\frac{\partial J_{\bar\mu}^{\ln}}{\partial \Delta_L(s_1,v)}=0: \quad & \forall s_1 > 0,\ \forall v \geq 0, & 0 &= \Phi_{\Delta_L}(s_1, v; \Delta_S(s_1,v), \Delta_L(s_1,v))
\end{aligned}
$$

$$
\begin{aligned}
\Phi_1(s_1; u_V, \Delta_S, \Delta_L) &:= \ln \mu_1(s_1) - \ln\left(\int \bar\mu(s_1, dv, ds_2) e^{u_V(v)+u_2(s_2)+\Delta_S^{(S)}(s_1,v,s_2)+\Delta_L^{(L)}(s_1,v,s_2)}\right) \\
\Phi_V(v; u_1, \Delta_S, \Delta_L) &:= \ln \mu_V(v) - \ln\left(\int \bar\mu(ds_1, v, ds_2) e^{u_1(s_1)+u_2(s_2)+\Delta_S^{(S)}(s_1,v,s_2)+\Delta_L^{(L)}(s_1,v,s_2)}\right) \\
\Phi_2(s_2; u_1, u_V, \Delta_S, \Delta_L) &:= \ln \mu_2(s_2) - \ln\left(\int \bar\mu(ds_1, dv, s_2) e^{u_1(s_1)+u_V(v)+\Delta_S^{(S)}(s_1,v,s_2)+\Delta_L^{(L)}(s_1,v,s_2)}\right) \\
\Phi_{\Delta_S}(s_1, v; u_2, \delta_S, \delta_L) &:= \int \bar\mu(s_1, v, ds_2)(s_2 - s_1) e^{u_2(s_2)+\delta_S(s_2-s_1)+\delta_L\left(L\left(\frac{s_2}{s_1}\right)-v^2\right)} \\
\Phi_{\Delta_L}(s_1, v; u_2, \delta_S, \delta_L) &:= \int \bar\mu(s_1, v, ds_2)\left(L\left(\frac{s_2}{s_1}\right) - v^2\right) e^{u_2(s_2)+\delta_S(s_2-s_1)+\delta_L\left(L\left(\frac{s_2}{s_1}\right)-v^2\right)}.
\end{aligned}
$$

Implementation details

Practically, we consider market strikes $\mathcal{K} := (\mathcal{K}_1, \mathcal{K}_V, \mathcal{K}_2)$ and market prices (C^1_K, C^V_K, C^2_K) of vanilla options on S_1, V, and S_2, and we build the model

$$\mu^*_{\mathcal{K}}(ds_1, dv, ds_2) = \bar{\mu}(ds_1, dv, ds_2) e^{c^* + \Delta^{0*}_S s_1 + \Delta^{0*}_V v + \sum_{K\in\mathcal{K}_1} a^{1*}_K (s_1 - K)_+}$$
$$e^{\sum_{K\in\mathcal{K}_V} a^{V*}_K (v-K)_+ + \sum_{K\in\mathcal{K}_2} a^{2*}_K (s_2 - K)_+ + \Delta^{*(S)}_S (s_1, v, s_2) + \Delta^{*(L)}_L (s_1, v, s_2)}$$

where $\theta^* := (c^*, \Delta^{0*}_S, \Delta^{0*}_V, a^{1*}, a^{V*}, a^{2*}, \Delta^*_S, \Delta^*_L)$ maximizes

$$J^{\mathrm{id}-1}_{\bar{\mu},\mathcal{K}}(\theta) := c + \Delta^0_S S_0 + \Delta^0_V F_V + \sum_{K\in\mathcal{K}_1} a^1_K C^1_K + \sum_{K\in\mathcal{K}_V} a^V_K C^V_K + \sum_{K\in\mathcal{K}_2} a^2_K C^2_K$$
$$-\mathbb{E}^{\bar{\mu}}\left[e^{c + \Delta^0_S S_1 + \Delta^0_V V + \sum_{\mathcal{K}_1} a^1_K (S_1 - K)_+ + \sum_{\mathcal{K}_V} a^V_K (V-K)_+ + \sum_{\mathcal{K}_2} a^2_K (S_2 - K)_+ + \Delta^{(S)}_S (\dots) + \Delta^{(L)}_L (\dots)}\right]$$

over the set Θ of portfolios $\theta := (c, \Delta^0_S, \Delta^0_V, a^1, a^V, a^2, \Delta_S, \Delta_L)$ such that $c, \Delta^0_S, \Delta^0_V \in \mathbb{R}$, $a^1 \in \mathbb{R}^{\mathcal{K}_1}$, $a^V \in \mathbb{R}^{\mathcal{K}_V}$, $a^2 \in \mathbb{R}^{\mathcal{K}_2}$, and $\Delta_S, \Delta_L : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ are measurable functions of (s_1, v).

Implementation details

- This corresponds to solving the entropy minimization problem

$$P_{\bar\mu,\mathcal{K}} := \inf_{\mu\in\mathcal{P}(\mathcal{K})} H(\mu|\bar\mu) = \sup_{\theta\in\Theta} J^{\mathrm{id}-1}_{\bar\mu,\mathcal{K}}(\theta) =: D_{\bar\mu,\mathcal{K}}$$

where $\mathcal{P}(\mathcal{K})$ denotes the set of probability measures μ on $\mathbb{R}_{>0}\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0}$ such that

$$\mathbb{E}^\mu[S_1] = S_0, \quad \mathbb{E}^\mu[V] = F_V, \quad \forall K\in\mathcal{K}_1,\ \mathbb{E}^\mu\left[(S_1-K)_+\right] = C^1_K,$$
$$\forall K\in\mathcal{K}_V,\ \mathbb{E}^\mu\left[(V-K)_+\right] = C^V_K, \quad \forall K\in\mathcal{K}_2,\ \mathbb{E}^\mu\left[(S_2-K)_+\right] = C^2_K,$$
$$\mathbb{E}^\mu\left[S_2|S_1,V\right] = S_1, \quad \mathbb{E}^\mu\left[L\left(\frac{S_2}{S_1}\right)\middle|S_1,V\right] = V^2.$$

- One can directly check that model $\mu^*_{\mathcal{K}}$ is an arbitrage-free model that jointly calibrates the prices of SPX futures, options, VIX future, and VIX options. Indeed, if $J^{\mathrm{id}-1}_{\bar\mu,\mathcal{K}}$ reaches its maximum at θ^*, then θ^* is solution to $\frac{\partial J^{\mathrm{id}-1}_{\bar\mu,\mathcal{K}}}{\partial\theta_i}(\theta) = 0$:

Implementation details

$$J^{\text{id}-1}_{\bar{\mu},\mathcal{K}}(\theta) := c + \Delta^0_S S_0 + \Delta^0_V F_V + \sum_{K\in\mathcal{K}_1} a^1_K C^1_K + \sum_{K\in\mathcal{K}_V} a^V_K C^V_K + \sum_{K\in\mathcal{K}_2} a^2_K C^2_K$$

$$-\mathbb{E}^{\bar{\mu}}\left[e^{c+\Delta^0_S S_1+\Delta^0_V V+\sum_{\mathcal{K}_1} a^1_K (S_1-K)_+ + \sum_{\mathcal{K}_V} a^V_K (V-K)_+ + \sum_{\mathcal{K}_2} a^2_K (S_2-K)_+ + \Delta^{(S)}_S(\dots)+\Delta^{(L)}_L(\dots)}\right]$$

$$\frac{\partial J^{\text{id}-1}_{\bar{\mu},\mathcal{K}}}{\partial c} = 0 : \mathbb{E}^{\bar{\mu}}\left[\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = 1 \qquad \frac{\partial J^{\text{id}-1}_{\bar{\mu},\mathcal{K}}}{\partial \Delta^0_S} = 0 : \mathbb{E}^{\bar{\mu}}\left[S_1\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = S_0$$

$$\frac{\partial J^{\text{id}-1}_{\bar{\mu},\mathcal{K}}}{\partial \Delta^0_V} = 0 : \mathbb{E}^{\bar{\mu}}\left[V\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = F_V \qquad \frac{\partial J^{\text{id}-1}_{\bar{\mu},\mathcal{K}}}{\partial a^1_K} = 0 : \mathbb{E}^{\bar{\mu}}\left[(S_1-K)_+\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = C^1_K$$

$$\frac{\partial J^{\text{id}-1}_{\bar{\mu},\mathcal{K}}}{\partial a^V_K} = 0 : \mathbb{E}^{\bar{\mu}}\left[(V-K)_+\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = C^V_K \qquad \frac{\partial J^{\text{id}-1}_{\bar{\mu},\mathcal{K}}}{\partial a^2_K} = 0 : \mathbb{E}^{\bar{\mu}}\left[(S_2-K)_+\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = C^2_K$$

$$\frac{\partial J^{\text{id}-1}_{\bar{\mu},\mathcal{K}}}{\partial \Delta_S(s_1,v)} = 0 : \mathbb{E}^{\bar{\mu}}\left[(S_2-S_1)\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\middle| S_1=s_1, V=v\right] = 0, \quad \forall s_1\geq 0, v>0$$

$$\frac{\partial J^{\text{id}-1}_{\bar{\mu},\mathcal{K}}}{\partial \Delta_L(s_1,v)} = 0 : \mathbb{E}^{\bar{\mu}}\left[\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\middle| S_1=s_1, V=v\right] = 0, \quad \forall s_1\geq 0, v>0$$