The VIX Future in Bergomi Models

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QuantMinds 2019
Vienna, May 14, 2019

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Motivation

- Volatility indices, such as the VIX index, are not only used as market-implied indicators of volatility.
- Futures and options on these indices are also widely used as risk-management tools to hedge the volatility exposure of options portfolios.
- Existence of a liquid market for these futures and options $\Longrightarrow$ need for models that jointly calibrate to the prices of options the underlying asset and prices of volatility derivatives.
- Since VIX options started trading in 2006, many researchers and practitioners have tried to build a model that jointly and exactly calibrates to the prices of S&P 500 (SPX) options, VIX futures and VIX options.
- Very challenging problem, especially for short maturities.

Motivation

- The **very large negative skew of short-term SPX options**, which in continuous models implies a very large volatility of volatility, **seems inconsistent with the comparatively low levels of VIX implied volatilities**.
- One should decrease the volatility of volatility to decrease the latter, but this would also decrease the former, which is already too small. See G. (2017, 2018).
- **Objective: quantitatively describe the structural constraints that continuous stochastic volatility models jointly put on SPX and VIX derivatives**.

Motivation

- In particular, we focus on Bergomi models: one factor, two factors (+ skewed versions that calibrate to VIX smile) .
- Popular variance curve models that can be used to price SPX and VIX derivatives.
- Bergomi-G. (2012) have already derived a general expansion of the smile in variance curve models at order two in vol-of-vol
- Objective: derive an **expansion of the price of VIX futures in Bergomi models**. Order 6.
- Precisely pinpoint the roles of **vol-of-vol** and **mean-reversion**.
- **Understand the structural constraints that flexible continuous stochastic volatility models like Bergomi models jointly put on SPX and VIX derivatives.**

One-factor Bergomi model

One-factor Bergomi model

- ξ_t^u: instantaneous lognormal variance of the SPX S at time $u > t$ seen from t.
- Forward instantaneous variances are driftless (Dupire, Bergomi).
- Second generation stochastic volatility models directly model the dynamics of $(\xi_t^u, t \in [0,u])$ under a risk-neutral measure. Only requirement: that these processes, indexed by u, be **nonnegative** and **driftless** (in t).
- One-factor Bergomi model: the simplest model on $(\xi_t^u, t \in [0,u])$. First suggested by Dupire (1993). Assumes that forward instantaneous variances are lognormal and all driven by a single standard one-dimensional Brownian motion Z, correlated with the Brownian motion W that drives the SPX dynamics:

$$\frac{d\xi_t^u}{\xi_t^u} = \omega e^{-k(u-t)} dZ_t, \qquad \frac{dS_t}{S_t} = (r_t - q_t)\,dt + \sqrt{\xi_t^t}\,dW_t, \qquad \omega, k > 0$$

- r_t, q_t: instantaneous interest rate and dividend yield, inclusive of repo.

One-factor Bergomi model

$$\frac{d\xi_t^u}{\xi_t^u} = \omega e^{-k(u-t)} dZ_t, \qquad \frac{dS_t}{S_t} = (r_t - q_t)\, dt + \sqrt{\xi_t^t}\, dW_t, \qquad \omega, k > 0$$

- Time-homogeneous exponential kernel $K(u-t) = \omega e^{-k(u-t)}$ motivated by two objectives: (1) K **decreasing** function; (2) ξ_t^u admits a **one-dimensional Markov representation**:

$$\xi_t^u = \xi_0^u f^u(t, X_t) \tag{1.1}$$

  with a Markov process X which does not depend on u.
- Indeed, (1.1) holds with

$$X_t := \int_0^t e^{-k(t-s)} dZ_s, \quad f^u(t,x) := \exp\left(\omega e^{-k(u-t)} x - \frac{\omega^2}{2} e^{-2k(u-t)} v_t\right),$$

$$v_t := \mathrm{Var}(X_t) = \frac{1 - e^{-2kt}}{2k}$$

  where the Ornstein-Uhlenbeck process X follows the Markov dynamics:

$$dX_t = -kX_t\, dt + dZ_t, \qquad X_0 = 0 \tag{1.2}$$

One-factor Bergomi model

$$\xi_t^u = \xi_0^u f^u(t, X_t)$$
$$dX_t = -kX_t\,dt + dZ_t, \qquad X_0 = 0$$
$$\sigma_t^2 := \xi_t^t = \xi_0^t \exp\left(\omega X_t - \frac{\omega^2}{2}\mathrm{Var}(X_t)\right)$$

- k: parameter of **mean-reversion** of the instantaneous volatility.
- ω: instantaneous (lognormal) volatility of the instantaneous variance; $\omega/2$: instantaneous (lognormal) volatility of the instantaneous volatility σ_t. ω referred to as **vol-of-vol**.
- Initial condition ξ_0^u computed from market prices $\mathrm{VS}(T)$ of variances swaps on the SPX: $\xi_0^u = \frac{d}{du}(u\mathrm{VS}(u))$. Assumed strictly positive and bounded.

- Markov representation is very convenient. Will be instrumental in our derivation of an expansion of the price of VIX futures in small vol-of-vol.
- In particular, our technique of proof does not apply to the rough Bergomi models (see pricing methods in Jacquier, Martini, Muguruza, *On VIX Futures in the Rough Bergomi Model*, 2017).

Expansion of the price of VIX futures in small vol-of-vol

- Let $T \geq 0$. By definition, the (idealized) VIX at time T is the implied volatility of a 30-day log-contract on the SPX index starting at T.
- For continuous models on the SPX such as the one-factor Bergomi model, this translates into

$$\mathrm{VIX}_T^2 = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_u^2\, du \middle| \mathcal{F}_T\right] = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}\left[\sigma_u^2 \middle| \mathcal{F}_T\right] du = \frac{1}{\tau}\int_T^{T+\tau} \xi_T^u\, du$$

- $\tau = \frac{30}{365}$ (30 days)
- $\mathcal{F}_t$: information available at time t, in this case the filtration generated by the Brownian motions W and Z

For any continuous model on the SPX:

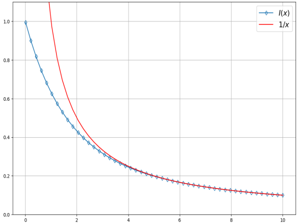
$$\mathrm{VIX}_T^2 = \frac{1}{\tau}\int_T^{T+\tau} \xi_T^u\, du$$

Expansion of the price of VIX futures in small vol-of-vol

$$\Xi_n := \frac{1}{\tau}\int_T^{T+\tau} \xi_0^u e^{-nk(u-T)} du > 0, \qquad I_n := \frac{\Xi_n}{\Xi_0}, \qquad n \in \mathbb{N}$$

$$I(x) := \frac{1-e^{-x}}{x}, \quad x > 0, \qquad I(0) := 1.$$

- When the initial term-structure of forward instantaneous variances $u \mapsto \xi_0^u$ is flat at level ξ, $\Xi_n = \xi I(nk\tau)$ and $I_n = I(nk\tau)$ are known in closed form.
- Otherwise, the computation of I_n requires a one-dimensional quadrature. Note: $v_t = tI(2kt)$.



Expansion of the price of VIX futures in small vol-of-vol

Proposition

In the one-factor Bergomi model, the price of a VIX future satisfies

$$\mathbb{E}[\mathrm{VIX}_T] = \sqrt{\Xi_0}\left\{1 + \alpha_2\omega^2 v_T + \alpha_4(\omega^2 v_T)^2 + \alpha_6(\omega^2 v_T)^3\right\} + O(\omega^7)$$

where $v_T = \frac{1-e^{-2kT}}{2k}$ *and*

$$\begin{aligned}
\alpha_2 &= -\frac{1}{8}I_1^2, \\
\alpha_4 &= -\frac{1}{16}I_2^2 + \frac{3}{16}I_1^2 I_2 - \frac{15}{128}I_1^4, \\
\alpha_6 &= -\frac{1}{48}I_3^2 + \frac{1}{16}I_2^3 + \frac{3}{16}I_1 I_2 I_3 - \frac{75}{128}I_1^2 I_2^2 - \frac{5}{32}I_1^3 I_3 + \frac{105}{128}I_1^4 I_2 - \frac{315}{1024}I_1^6.
\end{aligned}$$

Expansion of the price of VIX futures in small vol-of-vol

Proposition (cont'd)

In particular, this expansion provides a closed form expression of the prices of VIX futures in the one-factor Bergomi model at order 6 in small vol-of-vol when $u \mapsto \xi_0^u$ is flat at level ξ:

$$\mathbb{E}[\mathrm{VIX}_T] = \sqrt{\xi}\left\{1 + \alpha_2(k\tau)\omega^2 v_T + \alpha_4(k\tau)(\omega^2 v_T)^2 + \alpha_6(k\tau)(\omega^2 v_T)^3\right\} + O(\omega^7)$$

where the functions $\alpha_i(\cdot)$ are defined by:

$$\begin{aligned}
\alpha_2(x) &= -\frac{1}{8}I(x)^2, \\
\alpha_4(x) &= -\frac{1}{16}I(2x)^2 + \frac{3}{16}I(x)^2 I(2x) - \frac{15}{128}I(x)^4, \\
\alpha_6(x) &= -\frac{1}{48}I(3x)^2 + \frac{1}{16}I(2x)^3 + \frac{3}{16}I(x)I(2x)I(3x) \\
&\quad - \frac{75}{128}I(x)^2 I(2x)^2 - \frac{5}{32}I(x)^3 I(3x) + \frac{105}{128}I(x)^4 I(2x) - \frac{315}{1024}I(x)^6.
\end{aligned}$$

Expansion of the price of VIX futures in small vol-of-vol

Remark

In the case where ω_u is maturity-dependent, we can still expand in small vol-of-vol by multiplying ω_u by a dimensionless parameter ε (that can later be taken equal to one) and expand in powers of ε. Then the expansion still holds by replacing ω by ε and with

$$\Xi_n := \frac{1}{\tau}\int_T^{T+\tau} \xi_0^u \omega_u^n e^{-nk(u-T)} du, \qquad I_n := \frac{\Xi_n}{\Xi_0}, \qquad n \in \mathbb{N}.$$

Expansion of the price of VIX futures in small vol-of-vol

Remark

In the case where the mean reversion $k(t)$ is time-dependent,

$$dX_t = -k(t) X_t \, dt + dZ_t,$$

the expansion still holds with

$$\Xi_n := \frac{1}{\tau} \int_T^{T+\tau} \xi_0^u e^{-n(K(u)-K(T))} du, \qquad I_n := \frac{\Xi_n}{\Xi_0},$$

$$v_t := e^{-2K(t)} \int_0^T e^{2K(s)} ds$$

where

$$K(t) := \int_0^t k(s)\, ds.$$

Of course one can mix maturity-dependent ω_u with time-dependent $k(t)$.

Expansion of the price of VIX futures in small vol-of-vol

Remark

In the case where both $k(t)$ and $\omega(t)$ are time-dependent,

$$dX_t = -k(t)X_t\,dt + \omega(t)\,dZ_t,$$

we can still expand in small vol-of-vol by multiplying $\omega(t)$ by a dimensionless parameter ε (that can later be taken equal to one) and expand in powers of ε. Then the expansion still holds by replacing ω by ε and with

$$\begin{aligned} \Xi_n &:= \frac{1}{\tau}\int_T^{T+\tau} \xi_0^u e^{-n(K(u)-K(T))}du, \qquad I_n := \frac{\Xi_n}{\Xi_0}, \\ v_t &:= e^{-2K(t)}\int_0^T \omega(s)^2 e^{2K(s)}ds. \end{aligned}$$

Expansion of the price of VIX futures in small vol-of-vol

Remark

One can easily mix maturity-dependent ω_u with time-dependent $\omega(t)$ (and time-dependent $k(t)$) if they are in **product form:** $\omega_u(t) = \omega_u \omega(t)$. Then, as above, we can expand in powers of ε and the expansion still holds by replacing ω by ε and with

$$\Xi_n := \frac{1}{\tau}\int_T^{T+\tau} \xi_0^u \omega_u^n e^{-n(K(u)-K(T))} du, \qquad I_n := \frac{\Xi_n}{\Xi_0},$$

$$v_t := e^{-2K(t)} \int_0^T \omega(s)^2 e^{2K(s)} ds.$$

Numerical inspection of the formula

$$\mathbb{E}[\mathrm{VIX}_T] = \sqrt{\Xi_0}\left\{1 + \alpha_2\omega^2 v_T + \alpha_4(\omega^2 v_T)^2 + \alpha_6(\omega^2 v_T)^3\right\} + O(\omega^7)$$

- The formula is essentially an **expansion in powers of $\omega^2 v_T$**, suggesting that the expansion is accurate not only for small ω, but also for small $\omega^2 v_T$.
- Let us define $\nu := \frac{\omega}{\sqrt{2k}}$. As $\mathrm{Var}(\omega X_t) = \omega^2 v_t = \nu^2(1 - e^{-2kt})$, ν is the long term standard deviation of ωX_t.
- Since $\omega^2 v_T \le \nu^2$ and $\omega^2 v_T = \omega^2 T I(2kT) \le \omega^2 T$, we have

$$0 \le \omega^2 v_T \le \min(\nu^2, \omega^2 T).$$

- In particular we expect the expansion to be accurate when ν is small enough or when $\omega\sqrt{T}$ is small enough.
- ν small enough: mean-reversion large enough to mitigate vol-of-vol.
- Both ν and $\omega\sqrt{T}$ are dimensionless quantities, while ω has the dimension of a volatility, i.e., time$^{-1/2}$.
- **We expect the expansion to be accurate when the vol ω is small enough compared to the vols $\sqrt{2k}$ or $1/\sqrt{T}$.**

Numerical inspection of the formula

$$\mathbb{E}[\mathrm{VIX}_T] = \sqrt{\xi}\left\{1 + \alpha_2(k\tau)\omega^2 v_T + \alpha_4(k\tau)(\omega^2 v_T)^2 + \alpha_6(k\tau)(\omega^2 v_T)^3\right\} + O(\omega^7)$$

- How small should ω be, compared to $\sqrt{2k}$ or $1/\sqrt{T}$?
- Dependence of the formula on ξ is trivial: simply proportional to $\sqrt{\xi}$.
- After dividing by $\sqrt{\xi}$, each term in the expansion is of the form $\alpha_{2i}(k\tau)(\omega^2 v_T)^i$, where $\alpha_{2i}(k\tau)$ depends only on k, not on ω or T.
- $\alpha_{2i}(x)$ is small and decreases quickly with i. In particular $\alpha_2(0) = -\frac{1}{8}$, $\alpha_4(0) = \frac{1}{128}$, and $\alpha_6(0) = -\frac{1}{3072}$.
- $\alpha_2(x)$ and both ratios $\alpha_4(x)/\alpha_2(x)$ and $\alpha_6(x)/\alpha_4(x)$ take values around -5% for reasonable values of $x = k\tau$, e.g., $x \in [0, 2]$.
- Suggests that the **expansion should be accurate for $\omega^2 v_T$ up to ≈ 7**: if $\omega^2 v_T = 7$, then the order i term in the expansion $\alpha_{2i}(k\tau)(\omega^2 v_T)^i$ is only about a third, in absolute value, of the order $i-1$ term.
- However, if $\omega^2 v_T \geq 20$, then the order i term can be larger than the order $i-1$ term, suggesting divergence of the series.

Numerical inspection of the formula

$$\mathbb{E}[\mathrm{VIX}_T] = \sqrt{\xi}\left\{1 + \alpha_2(k\tau)\omega^2 v_T + \alpha_4(k\tau)(\omega^2 v_T)^2 + \alpha_6(k\tau)(\omega^2 v_T)^3\right\} + O(\omega^7)$$

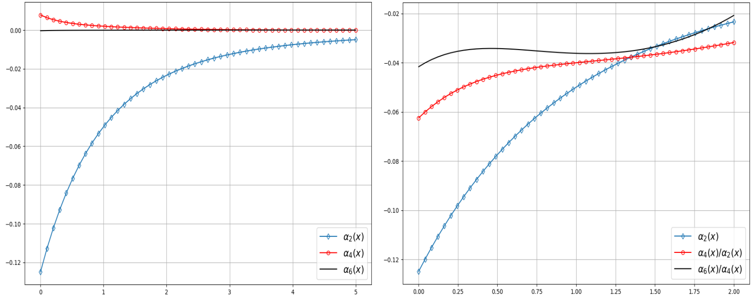


Figure: Left: Graph of functions α_2, α_4, and α_6 of Formula (1.3). Right: Graph of α_2 and of ratios α_4/α_2 and α_6/α_4

Numerical experiments: $k = 0.25$

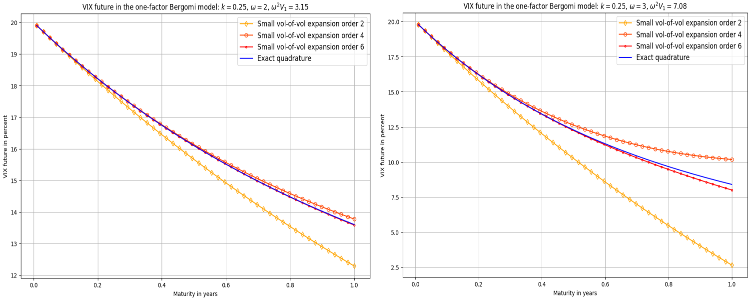


Figure: VIX future in the one-factor Bergomi model as a function of maturity (in years). Comparison of small vol-of-vol expansion at orders 2, 4, and 6 with the exact quadrature for several sets of parameters. Left: $\omega^2 v_1 \approx 3$. Right: $\omega^2 v_1 \approx 7$

Numerical experiments: $k = 2$

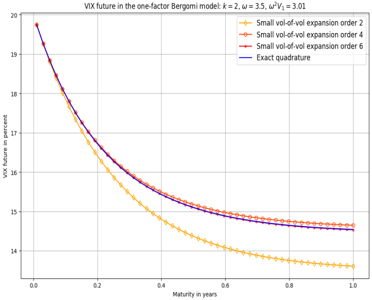
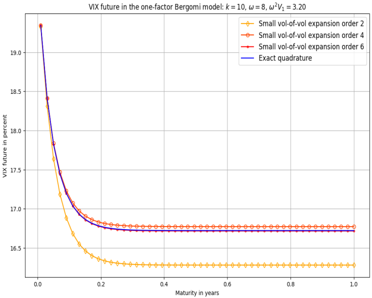


Figure: VIX future in the one-factor Bergomi model as a function of maturity (in years). Comparison of small vol-of-vol expansion at orders 2, 4, and 6 with the exact quadrature for several sets of parameters. Left: $\omega^2 v_1 \approx 3$. Right: $\omega^2 v_1 \approx 7$

Numerical experiments: $k = 10$



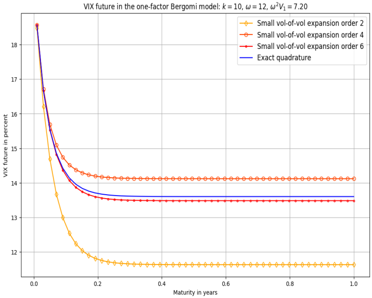


Figure: VIX future in the one-factor Bergomi model as a function of maturity (in years). Comparison of small vol-of-vol expansion at orders 2, 4, and 6 with the exact quadrature for several sets of parameters. Left: $\omega^2 v_1 \approx 3$. Right: $\omega^2 v_1 \approx 7$

Numerical experiments: $\omega^2 v_1 \approx 15$

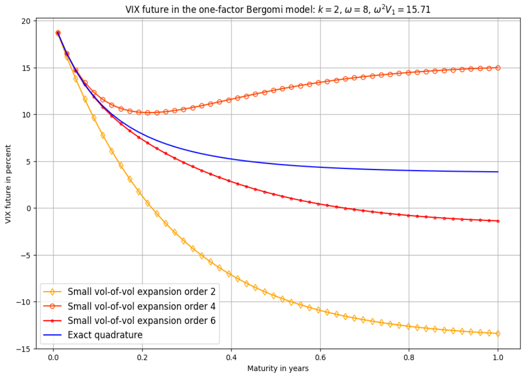


Figure: VIX future in the one-factor Bergomi model as a function of maturity (in years). Comparison of small vol-of-vol expansion at orders 2, 4, and 6 with the exact quadrature when $\omega = 8$ and $k = 2$ ($\omega^2 v_1 \approx 15$; $\omega^2 v_{2/12} \approx 7$)

Contango vs backwardation

- Since we used a flat initial term-structure of forward instantaneous variances $u \mapsto \xi_0^u$, the model generates a decreasing term-structure of VIX futures (backwardation).
- To recover an increasing term-structure (contango), as usually observed in the market, we should use an increasing term-structure of forward instantaneous variances.
- The term-structure implied from the market prices of variances swaps on the SPX $\xi_0^u = \frac{d}{du}(u\mathrm{VS}(u))$ is typically increasing, except during those periods when the VIX index blows up.

Expansion of the volatility of the squared VIX implied by VIX future prices

- It is natural to quote the price of a VIX future in terms of the **implied (lognormal) volatility of the squared VIX**.
- The (undiscounted) time 0 price of the payoff VIX_T^2 is known from the market prices of variance swaps on the SPX:

$$\mathsf{Price}[\mathsf{VIX}_T^2] = \frac{(T+\tau)\mathrm{VS}(T+\tau) - T\mathrm{VS}(T)}{\tau} = \frac{1}{\tau}\int_T^{T+\tau} \xi_0^u \, du = \Xi_0.$$

- $\Longrightarrow$ The volatility of the squared VIX implied by the VIX future price for maturity T is the value $\sigma_{\mathsf{VIX}_T^2}$ such that

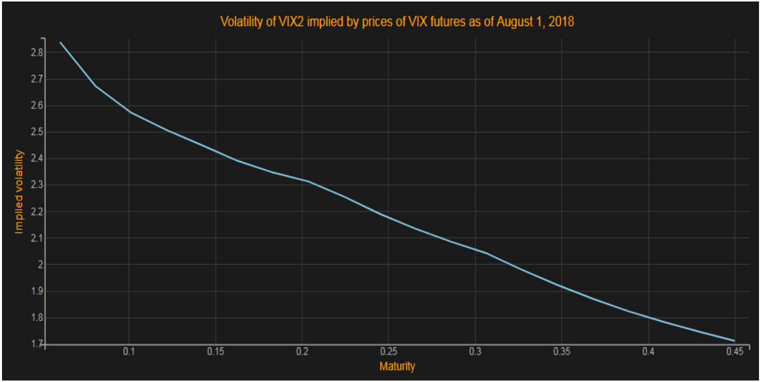
$$\mathsf{Price}[\mathsf{VIX}_T] = \sqrt{\Xi_0}\exp\left(-\frac{1}{8}\sigma_{\mathsf{VIX}_T^2}^2 T\right).$$

- R.h.s. is the (undiscounted) time 0 price of the payoff VIX_T in the model where VIX_T^2 is lognormal with mean Ξ_0 and volatility $\sigma_{\mathsf{VIX}_T^2}$.

$$\sigma_{\mathsf{VIX}_T^2} = \sqrt{-\frac{8}{T}\ln\frac{\mathsf{Price}[\mathsf{VIX}_T]}{\sqrt{\Xi_0}}} = \sqrt{-\frac{8}{T}\ln\frac{\mathsf{Price}[\mathsf{VIX}_T]}{\sqrt{\mathsf{Price}[\mathsf{VIX}_T^2]}}}.$$

- No arbitrage $\Longrightarrow$ $\mathsf{Price}[\mathsf{VIX}_T] \le \sqrt{\mathsf{Price}[\mathsf{VIX}_T^2]}$ so $\sigma_{\mathsf{VIX}_T^2}$ is well defined.

Volatility of VIX^2 implied by VIX future, as of August 1, 2018



Expansion of the volatility of the squared VIX implied by VIX future prices

Proposition

In the one-factor Bergomi model, the volatility $\sigma_{\mathrm{VIX}_T^2}$ of the squared VIX implied by the VIX future price for maturity T satisfies

$$\sigma_{\mathrm{VIX}_T^2} = \omega I_1 \sqrt{I(2kT)} \left\{ 1 + \beta_2 \omega^2 v_T + \beta_4 (\omega^2 v_T)^2 \right\} + O(\omega^6)$$

where $v_T = \frac{1-e^{-2kT}}{2k} = TI(2kT)$ *and*

$$\beta_2 = \frac{1}{2}\left(\frac{\alpha_4}{\alpha_2} - \frac{\alpha_2}{2}\right), \tag{1.3}$$

$$\beta_4 = \frac{1}{2}\left(\frac{\alpha_6}{\alpha_2} - \alpha_4 + \frac{\alpha_2^2}{3}\right) - \frac{1}{8}\left(\frac{\alpha_4}{\alpha_2} - \frac{\alpha_2}{2}\right)^2. \tag{1.4}$$

Expansion of the volatility of the squared VIX implied by VIX future prices

Proposition (cont'd)

In particular, this formula provides a closed form expression of the implied volatility $\sigma_{\mathrm{VIX}_T^2}$ in the one-factor Bergomi model at order 5 in small vol-of-vol when $u \mapsto \xi_0^u$ is flat:

$$\sigma_{\mathrm{VIX}_T^2} = \omega I(k\tau)\sqrt{I(2kT)}\left\{1 + \beta_2(k\tau)\omega^2 v_T + \beta_4(k\tau)(\omega^2 v_T)^2\right\} + O(\omega^6)$$

where the functions $\beta_2(\cdot)$ and $\beta_4(\cdot)$ are defined from the functions $\alpha_i(\cdot)$ by (1.3)-(1.4). In particular, at first order in vol-of-vol ω,

$$\sigma_{\mathrm{VIX}_T^2} = \omega \frac{1-e^{-k\tau}}{k\tau}\sqrt{\frac{1-e^{-2kT}}{2kT}} + O(\omega^3).$$

Numerical inspection of the formula

$$\sigma_{\mathrm{VIX}_T^2} = \omega I(k\tau)\sqrt{I(2kT)}\left\{1 + \beta_2(k\tau)\omega^2 v_T + \beta_4(k\tau)(\omega^2 v_T)^2\right\} + O(\omega^6)$$

- Formula is essentially expansion in powers of $\omega^2 v_T \Longrightarrow$ Accurate for $\omega^2 v_T$ small enough, in particular when ν or $\omega\sqrt{T}$ are small enough.
- **The domain of accuracy of the implied volatility expansion is actually much larger than that of the price expansion**.
- Indeed, both $\beta_2(x)$ and the ratio $\beta_4(x)/\beta_2(x)$ take very small values, around -1%, for $x = k\tau \in [0, 2]$, suggesting that the implied vol expansion should be accurate even for $\omega^2 v_T \approx$ 20–30.
- Moreover, contrary to the ratios $\alpha_{2i}(x)/\alpha_{2i-2}(x)$, both $\beta_2(x)$ and the ratio $\beta_4(x)/\beta_2(x)$ tend to zero, together with their first order derivatives, when x tends to zero.
- $\Longrightarrow$ Even when v_T becomes extremely large ($k \to 0$, $T \to \infty$), the first two ratios of consecutive terms in the expansion (with $\beta_0(x) := 1$)

$$\left|\frac{\beta_{2i}(k\tau)}{\beta_{2i-2}(k\tau)}\omega^2 v_T\right| \leq \frac{1}{2k}\left|\frac{\beta_{2i}(k\tau)}{\beta_{2i-2}(k\tau)}\right|\omega^2$$

stay bounded (they tend to zero when v_T tends to infinity).

Numerical inspection of the formula

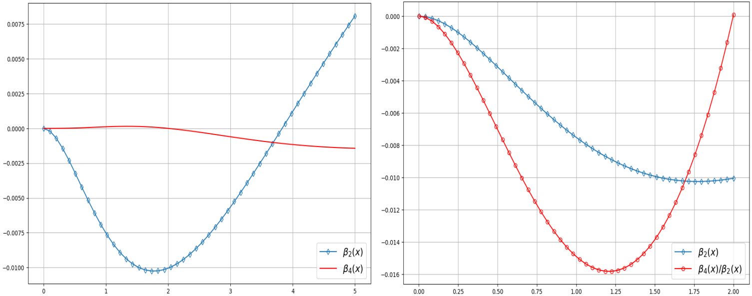


Figure: Left: Graph of functions β_2 and β_4. Right: Graph of β_2 and of ratio β_4/β_2

Numerical inspection of the formula

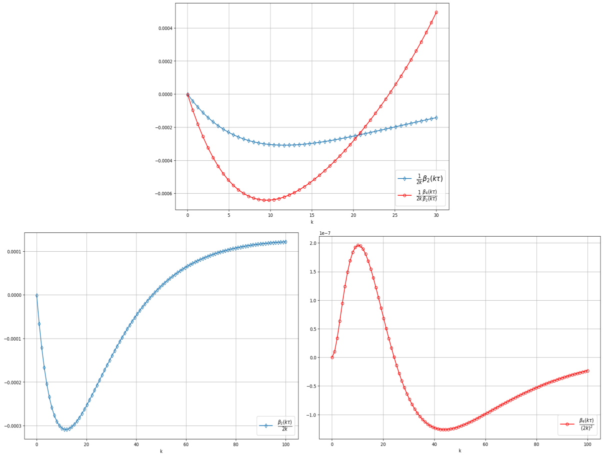


Figure: Top: Graph of $k \mapsto \frac{\beta_2(k\tau)}{2k}$ and $k \mapsto \frac{1}{2k}\frac{\beta_4(k\tau)}{\beta_2(k\tau)}$ for $0 \le k \le 30$. Bottom: Graph of $k \mapsto \frac{\beta_2(k\tau)}{2k}$ (left) and $k \mapsto \frac{\beta_4(k\tau)}{(2k)^2} \times 10^{-7}$ (right) for $0 \le k \le 100$

Numerical inspection of the formula

$$\sigma_{\mathrm{VIX}^2_T} = \omega I(k\tau)\sqrt{I(2kT)}\left\{1 + \beta_2(k\tau)\omega^2 v_T + \beta_4(k\tau)(\omega^2 v_T)^2\right\} + O(\omega^6)$$

$$\left|\frac{\beta_{2i}(k\tau)}{\beta_{2i-2}(k\tau)}\omega^2 v_T\right| \leq \frac{1}{2k}\left|\frac{\beta_{2i}(k\tau)}{\beta_{2i-2}(k\tau)}\right|\omega^2$$

- The r.h.s. are bounded above by $7 \times 10^{-4}\omega^2$ for all $k \leq 30$. This suggests that, for all T and a very wide range of values of k, the above expansion should be very accurate even for extremely large ω, say, $\omega = 10$.
- Even for this unreasonably large value of ω, the first two correcting terms in the expansion are small **whatever the value of k and T**:

$$\beta_2(k\tau)\omega^2 v_T \leq 3.1 \times 10^{-2}, \qquad \beta_4(k\tau)(\omega^2 v_T)^2 \leq 2 \times 10^{-3}.$$

Numerical experiment

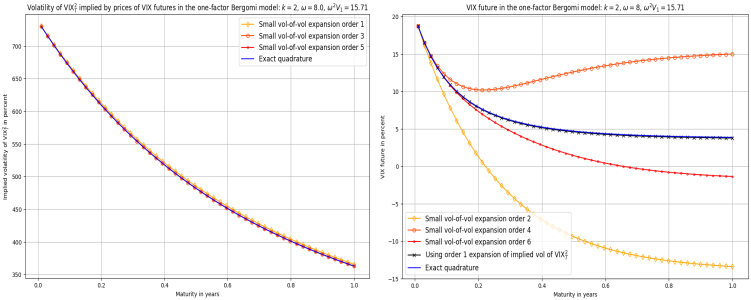


Figure: Implied vol of VIX squared (left) and price of VIX future (right) in the one-factor Bergomi model as a function of maturity (in years). Comparison of small vol-of-vol expansion with the exact quadrature when $\omega = 8$ and $k = 2$ ($\omega^2 v_1 \approx 15$; $\omega^2 v_{2/12} \approx 7$)

Inspection of the first order formula

$$\sigma_{\mathsf{VIX}^2_T} = \omega \frac{1-e^{-k\tau}}{k\tau}\sqrt{\frac{1-e^{-2kT}}{2kT}} + O(\omega^3).$$

- In fact, for practical purposes, **the first order formula can be considered exact**.
- The implied volatility of a very short VIX^2_T is the volatility ω of the instantaneous variance ξ^t_t, dampened by the factor $I(k\tau) = \frac{1-e^{-k\tau}}{k\tau}$ which accounts for the mean-reversion of volatility over 30 days.
- For non-zero maturities T, this is multiplied by $\sqrt{I(2kT)} = \sqrt{\frac{1-e^{-2kT}}{2kT}}$.
- For large T, the term-structure of the implied volatility of the squared VIX decays as the power law $T^{-1/2}$.
- Interpretation: Mean-reversion causes the price of the VIX future to converge when T increases, as the Ornstein-Uhlenbeck process X reaches its stationary distribution. $\text{Price}[\mathsf{VIX}_T] = \sqrt{\Xi_0}\exp\left(-\frac{1}{8}\sigma^2_{\mathsf{VIX}^2_T}T\right)$
$\Longrightarrow \sigma^2_{\mathsf{VIX}^2_T}T$ must converge, so $\sigma_{\mathsf{VIX}^2_T}$ behaves like $T^{-1/2}$.
- **For large k, $\sigma_{\mathsf{VIX}^2_T} \sim \frac{\omega}{k^{3/2}}$.**

Two-factor Bergomi model

Two-factor Bergomi model

- In the one-factor Bergomi model, all forward variances are driven by a single Brownian motion Z.
- A positive move of the short end of the variance curve ($dZ_t > 0$) implies a positive move of the long end of the curve.
- To allow for more flexibility for the dynamics of forward variances, at least 2 factors are needed.
- 2 factors actually enough to mimic power-law-like decay of term-structure of vols of variance swap rates (Bergomi) as well as power-law-like decay of term-structure of ATM implied vols of equity indices.
- In the two-factor Bergomi model (Bergomi 2005), the curve $\xi_t^{\cdot}$ is driven by two Brownian motions Z^1 and Z^2 whose constant correlation is denoted by ρ:

$$\frac{d\xi_t^u}{\xi_t^u} = \omega\alpha_\theta \left\{\theta_1 e^{-k_1(u-t)} dZ_t^1 + \theta_2 e^{-k_2(u-t)} dZ_t^2\right\},$$

$$\alpha_\theta = \left(\theta_1^2 + 2\rho\theta_1\theta_2 + \theta_2^2\right)^{-\frac{1}{2}}, \qquad k_1, k_2 > 0, \quad \theta_1, \theta_2 \in [0,1], \quad \theta_1 + \theta_2 = 1$$

Two-factor Bergomi model

$$\frac{d\xi_t^u}{\xi_t^u} = \omega\alpha_\theta \left\{\theta_1 e^{-k_1(u-t)} dZ_t^1 + \theta_2 e^{-k_2(u-t)} dZ_t^2\right\},$$

$$\alpha_\theta = \left(\theta_1^2 + 2\rho\theta_1\theta_2 + \theta_2^2\right)^{-\frac{1}{2}}, \qquad k_1, k_2 > 0, \quad \theta_1, \theta_2 \in [0,1], \quad \theta_1 + \theta_2 = 1$$

- Normalizing factor α_θ s.t. ω is the inst vol of the inst variance ξ_t^t.
- For identification purposes, we assume that $k_1 > k_2$:
  Z^1 drives short end of variance curve only (up to $u - t \approx 1/k_1$)
  Z^2 drives both its short and long end (up to $u - t \approx 1/k_2 > 1/k_1$).
- In the two-factor model, ξ_t^u admits a **two-dimensional Markov representation** in terms of two Ornstein-Uhlenbeck processes X^1 and X^2

Two-factor Bergomi model

$$
\begin{aligned}
\xi_t^u &= \xi_0^u f^u(t, x_t^u) = \xi_0^u g^u(t, X_t^1, X_t^2) \\
dX_t^i &= -k_i X_t^i \, dt + dZ_t^i, \qquad X_0^i = 0, \qquad i \in \{1, 2\} \\
x_t^u &:= \alpha_\theta \left\{ \theta_1 e^{-k_1(u-t)} X_t^1 + \theta_2 e^{-k_2(u-t)} X_t^2 \right\} \\
f^u(t, x) &:= \exp\left(\omega x - \frac{\omega^2}{2} v_t(u) \right) \\
v_t(u) &:= \mathrm{Var}(x_t^u) = \alpha_\theta^2 \Big\{ \theta_1^2 e^{-2k_1(u-t)} v_t^1 + \theta_2^2 e^{-2k_2(u-t)} v_t^2 \\
&\qquad\qquad + 2\theta_1\theta_2 e^{-(k_1+k_2)(u-t)} v_t^{1,2} \Big\} \\
v_t^i &:= \frac{1 - e^{-2k_i t}}{2k_i}, \qquad v_t^{1,2} := \rho \frac{1 - e^{-(k_1+k_2)t}}{k_1 + k_2} \\
(X_t^1, X_t^2) &\sim \mathcal{N}\left(0, \begin{pmatrix} v_t^1 & v_t^{1,2} \\ v_t^{1,2} & v_t^2 \end{pmatrix} \right)
\end{aligned}
$$

Expansion of the price of VIX futures in small vol-of-vol

- For $(m,n) \in \mathbb{N}^2$, we define

$$\Xi_{m,n} := \frac{1}{\tau}\int_T^{T+\tau} \xi_0^u e^{-(mk_1+nk_2)(u-T)} du > 0, \qquad I_{m,n} := \frac{\Xi_{m,n}}{\Xi_{0,0}}.$$

- When $u \mapsto \xi_0^u$ is flat at level ξ, $\Xi_{m,n} = \xi I((mk_1+nk_2)\tau)$ and $I_{m,n} = I((mk_1+nk_2)\tau)$ are known in closed form.
- For clarity, T being fixed, we use the notations $v_1 := v_T^1$, $v_2 := v_T^2$, and $v_{1,2} := v_T^{1,2}$.

Expansion of the price of VIX futures in small vol-of-vol

Proposition

In the two-factor Bergomi model, the price of a VIX future satisfies

$$\mathbb{E}[\mathrm{VIX}_T] = \sqrt{\Xi_{0,0}}\left\{1 + \gamma_2(\omega\alpha_\theta)^2 + \gamma_4(\omega\alpha_\theta)^4 + \gamma_6(\omega\alpha_\theta)^6\right\} + O(\omega^7) \quad (2.1)$$

where

$$\begin{aligned}
\gamma_2 &= -\frac{1}{8}\left(\theta_1^2 I_{10}^2 v_1 + 2\theta_1\theta_2 I_{10} I_{01} v_{1,2} + \theta_2^2 I_{01}^2 v_2\right), \\
\gamma_4 &= \left(-\frac{1}{16} I_{20}^2 + \frac{3}{16} I_{10}^2 I_{20} - \frac{15}{128} I_{10}^4\right)\theta_1^4 v_1^2 \\
&+ \left(-\frac{1}{4} I_{20} I_{11} + \frac{3}{8}\left(I_{10}^2 I_{11} + I_{10} I_{20} I_{01}\right) - \frac{15}{32} I_{10}^3 I_{01}\right)\theta_1^3\theta_2 v_1 v_{1,2} \\
&+ \left(-\frac{1}{8} I_{11}^2 + \frac{3}{8} I_{10} I_{11} I_{01} - \frac{15}{64} I_{10}^2 I_{01}^2\right)\theta_1^2\theta_2^2 v_1 v_2 \\
&+ \left(-\frac{1}{8}\left(I_{11}^2 + I_{20} I_{02}\right) + \frac{3}{16}\left(I_{10}^2 I_{02} + I_{20} I_{01}^2\right) + \frac{3}{8} I_{10} I_{11} I_{01} - \frac{15}{32} I_{10}^2 I_{01}^2\right)\theta_1^2\theta_2^2 v_{1,2}^2 \\
&+ \left(-\frac{1}{4} I_{11} I_{02} + \frac{3}{8}\left(I_{11} I_{01}^2 + I_{10} I_{01} I_{02}\right) - \frac{15}{32} I_{10} I_{01}^3\right)\theta_1\theta_2^3 v_{1,2} v_2 \\
&+ \left(-\frac{1}{16} I_{02}^2 + \frac{3}{16} I_{01}^2 I_{02} - \frac{15}{128} I_{01}^4\right)\theta_2^4 v_2^2,
\end{aligned}$$

Expansion of the price of VIX futures in small vol-of-vol

Proposition (cont'd)

and $\gamma_6 = \sum_{p=0}^{6} \gamma_{6-p,p}$ *where*

$$
\begin{aligned}
\gamma_{6,0} &= \left(-\frac{1}{48} I_{30}^2 + \frac{1}{16} I_{20}^3 + \frac{3}{16} I_{10} I_{20} I_{30} - \frac{75}{128} I_{10}^2 I_{20}^2 - \frac{5}{32} I_{10}^3 I_{30} + \frac{105}{128} I_{10}^4 I_{20} - \frac{315}{1024} I_{10}^6\right) \theta_1^6 v_1^3 \\
\gamma_{5,1} &= \left(-\frac{1}{8} I_{30} I_{21} + \frac{3}{8} I_{20}^2 I_{11} + \frac{9}{16} I_{20} I_{10} I_{21} + \frac{3}{8} I_{30} I_{10} I_{11} - \frac{75}{32} I_{10}^2 I_{20} I_{11} - \frac{15}{32} I_{10}^3 I_{21} + \frac{105}{64} I_{10}^4 I_{11} \right. \\
&\quad \left. + \frac{3}{16} I_{20} I_{30} I_{01} - \frac{75}{64} I_{10} I_{20}^2 I_{01} - \frac{15}{32} I_{10}^2 I_{30} I_{01} + \frac{105}{32} I_{10}^2 I_{20} I_{01} - \frac{945}{512} I_{10}^5 I_{01}\right) \theta_1^5 \theta_2 v_1^2 v_{1,2} \\
\gamma_{4,2} &= \left(-\frac{945}{1024} I_{10}^4 I_{01}^2 + \frac{105}{128} I_{20} I_{10}^2 I_{01}^2 - \frac{15}{128} I_{20}^2 I_{01}^2 + \frac{105}{64} I_{10}^3 I_{11} I_{01} - \frac{15}{32} I_{10}^2 I_{21} I_{01} - \frac{15}{16} I_{10} I_{20} I_{11} I_{0} \right. \\
&\quad \left. + \frac{3}{16} I_{20} I_{21} I_{01} - \frac{45}{64} I_{10}^2 I_{11}^2 + \frac{3}{8} I_{10} I_{21} I_{11} + \frac{3}{16} I_{20} I_{11}^2 - \frac{1}{16} I_{21}^2\right) \theta_1^4 \theta_2^2 v_1^2 v_2 \\
&\quad + \left(-\frac{945}{256} I_{10}^4 I_{01}^2 + \frac{525}{128} I_{20} I_{01}^2 I_{10}^2 - \frac{15}{32} I_{10} I_{01}^2 I_{30} - \frac{15}{32} I_{20}^2 I_{01}^2 + \frac{315}{64} I_{10}^3 I_{11} I_{01} - \frac{15}{16} I_{10}^2 I_{21} I_{01} \right. \\
&\quad - \frac{15}{4} I_{10} I_{20} I_{11} I_{01} + \frac{3}{8} I_{30} I_{11} I_{01} + \frac{3}{8} I_{20} I_{21} I_{01} + \frac{105}{128} I_{10}^4 I_{02} - \frac{75}{64} I_{10}^2 I_{20} I_{02} + \frac{3}{16} I_{10} I_{30} I_{02} + \\
&\quad \left. - \frac{15}{32} I_{10}^3 I_{12} - \frac{105}{64} I_{10}^2 I_{11}^2 + \frac{3}{4} I_{10} I_{11} I_{21} + \frac{9}{16} I_{10} I_{20} I_{12} + \frac{9}{16} I_{20} I_{11}^2 - \frac{1}{8} I_{30} I_{12} - \frac{1}{8} I_{21}^2\right) \theta_1^4 \theta_2^2 v
\end{aligned}
$$

Expansion of the price of VIX futures in small vol-of-vol

Proposition (cont'd)

$$
\begin{aligned}
\gamma_{3,3} \quad = \quad & +\Big(-\frac{945}{256} I_{10}^3 I_{01}^3 + \frac{105}{64} I_{10} I_{01}^3 I_{20} + \frac{105}{16} I_{10}^2 I_{11} I_{01}^2 - \frac{15}{16} I_{10} I_{21} I_{01}^2 - \frac{45}{32} I_{20} I_{11} I_{01}^2 + \frac{105}{64} I_{10}^3 I_{01} I \\
& - \frac{15}{16} I_{10} I_{20} I_{01} I_{02} - \frac{15}{16} I_{10}^2 I_{01} I_{12} - \frac{105}{32} I_{10} I_{11}^2 I_{01} + \frac{3}{4} I_{21} I_{11} I_{01} + \frac{3}{8} I_{20} I_{01} I_{12} - \frac{45}{32} I_{10}^2 I_{11} I_{0} \\
& + \frac{3}{8} I_{10} I_{21} I_{02} + \frac{3}{8} I_{20} I_{11} I_{02} + \frac{3}{4} I_{10} I_{11} I_{12} + \frac{3}{8} I_{11}^3 - \frac{1}{4} I_{12} I_{21}\Big) \theta_1^3 \theta_2^3 v_1 v_{12} v_2 \\
& \Big(-\frac{315}{128} I_{10}^3 I_{01}^3 + \frac{105}{64} I_{10} I_{20} I_{01}^3 - \frac{5}{32} I_{30} I_{01}^3 + \frac{105}{32} I_{10}^2 I_{11} I_{01}^2 - \frac{15}{32} I_{10} I_{21} I_{01}^2 - \frac{15}{16} I_{20} I_{11} I_{01}^2 \\
& + \frac{105}{64} I_{10}^3 I_{01} I_{02} - \frac{45}{32} I_{10} I_{20} I_{01} I_{02} + \frac{3}{16} I_{30} I_{01} I_{02} - \frac{15}{32} I_{10}^2 I_{12} I_{01} - \frac{45}{32} I_{10} I_{11}^2 I_{01} + \frac{3}{8} I_{11} I_{21} I_{0} \\
& + \frac{3}{16} I_{20} I_{12} I_{01} - \frac{15}{16} I_{10}^2 I_{11} I_{02} + \frac{3}{16} I_{10} I_{21} I_{02} + \frac{3}{8} I_{20} I_{11} I_{02} - \frac{5}{32} I_{10}^3 I_{03} + \frac{3}{16} I_{10} I_{20} I_{03} - \frac{1}{24} \\
& + \frac{3}{8} I_{10} I_{11} I_{12} + \frac{1}{8} I_{11}^3 - \frac{1}{8} I_{21} I_{12}\Big) \theta_1^3 \theta_2^3 v_{12}^3
\end{aligned}
$$

and $\gamma_{p,6-p}$ is built from $\gamma_{6-p,p}$ by swapping θ_1 and θ_2, v_1 and v_2, and $I_{m,n}$ and $I_{n,m}$. In particular, the expansion provides a closed form expression of the prices of VIX futures in the two-factor Bergomi model at order 6 in small vol-of-vol when $u \mapsto \xi_0^u$ is flat.

Expansion of the volatility of the squared VIX implied by VIX future prices

Proposition

In the two-factor Bergomi model, the volatility $\sigma_{\mathrm{VIX}_T^2}$ of the squared VIX implied by the VIX future price for maturity T satisfies

$$\sigma_{\mathrm{VIX}_T^2} = \omega\alpha_\theta\sqrt{\theta_1^2 I_{10}^2 I(2k_1T) + 2\rho\theta_1\theta_2 I_{10}I_{01}I((k_1+k_2)T) + \theta_2^2 I_{01}^2 I(2k_2T)} \times \left\{1 + \delta_2(\omega\alpha_\theta)^2 + \delta_4(\omega\alpha_\theta)^4\right\} + O(\omega^6)$$

$$\delta_2 = \frac{1}{2}\left(\frac{\gamma_4}{\gamma_2} - \frac{\gamma_2}{2}\right), \quad \delta_4 = \frac{1}{2}\left(\frac{\gamma_6}{\gamma_2} - \gamma_4 + \frac{\gamma_2^2}{3}\right) - \frac{1}{8}\left(\frac{\gamma_4}{\gamma_2} - \frac{\gamma_2}{2}\right)^2.$$

In particular, this provides a closed form expression of the implied volatility $\sigma_{\mathrm{VIX}_T^2}$ in the two-factor Bergomi model at order 5 in small vol-of-vol when the initial curve $u \mapsto \xi_0^u$ is flat. At first order, this closed form expression reads

$$\sigma_{\mathrm{VIX}_T^2} = \omega\sqrt{\frac{\theta_1^2 I(k_1\tau)^2 I(2k_1T) + 2\rho\theta_1\theta_2 I(k_1\tau)I(k_2\tau)I((k_1+k_2)T) + \theta_2^2 I(k_2\tau)^2 I(2k_2T)}{\theta_1^2 + 2\rho\theta_1\theta_2 + \theta_2^2}} + O(\omega^3).$$

Numerical experiments: $\rho = 0$

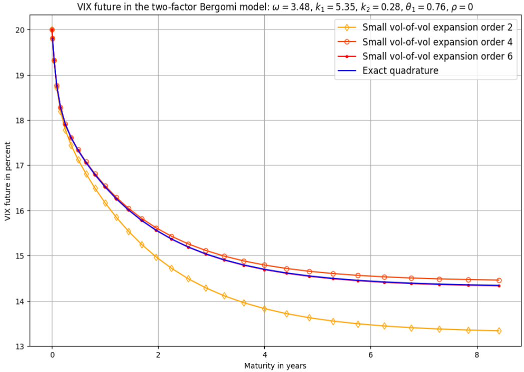


Figure: VIX future in the two-factor Bergomi model as a function of maturity (in years). Comparison of small vol-of-vol expansion at orders 2, 4, and 6 with the exact quadrature for parameter set II of Bergomi (*Stochastic Volatility Modeling*, 2016)

Numerical experiments: $\rho = 0$

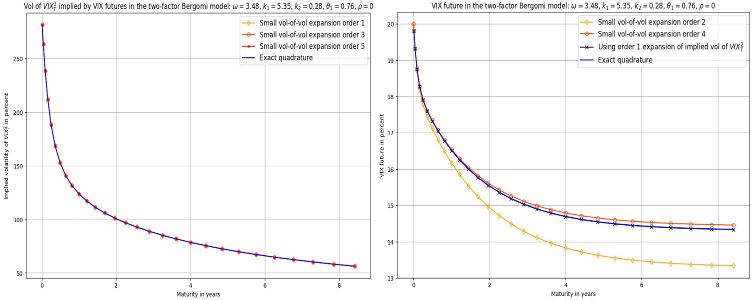


Figure: Left: Implied volatility of the squared VIX. Right: VIX future computed using the implied volatility expansion at order one. Parameter set II of Bergomi (*Stochastic Volatility Modeling*, 2016)

Numerical experiments: $\rho \neq 0$

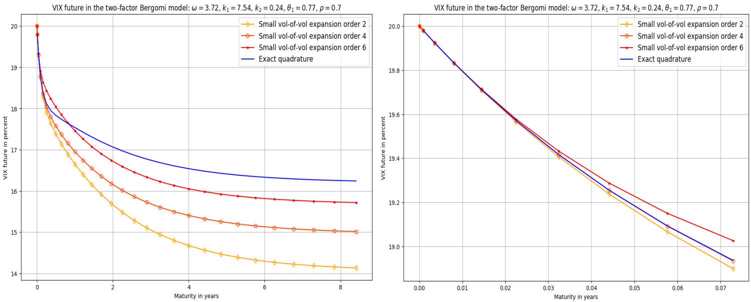


Figure: Left: VIX future in the two-factor Bergomi model as a function of maturity (in years). Comparison of small vol-of-vol expansion at orders 2, 4, and 6 with the exact quadrature for parameter set III of Bergomi (*Stochastic Volatility Modeling*, 2016); $\rho = 0.7$. Right: zoom on small T

Numerical experiments: $\rho \neq 0$

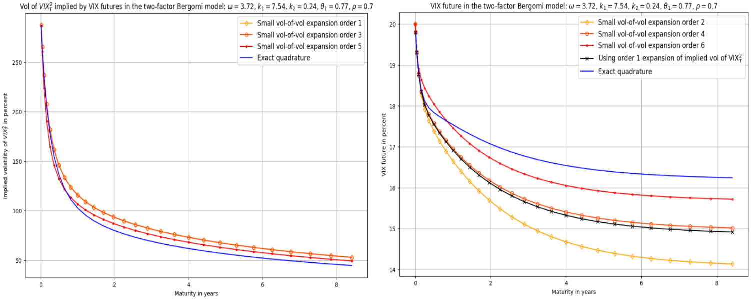


Figure: Left: VIX future in the two-factor Bergomi model as a function of maturity (in years). Comparison of small vol-of-vol expansion at orders 2, 4, and 6 with the exact quadrature for parameter set III of Bergomi (*Stochastic Volatility Modeling*, 2016); $\rho = 0.7$. Right: zoom on small T

Proofs

Proof: one-factor Bergomi model

- $\mathrm{VIX}_T^2 = \frac{1}{\tau}\int_T^{T+\tau} \xi_T^u \, du = f(T, X_T)$ with

$$f(T,x) = \frac{1}{\tau}\int_T^{T+\tau} \xi_0^u \exp\left(\omega e^{-k(u-T)} x - \frac{\omega^2}{2} e^{-2k(u-T)} v_T\right) du.$$

- The **Hermite polynomials** (of unit variance) H_n satisfy for all $(\lambda, z) \in \mathbb{R}^2$

$$e^{\lambda z - \frac{\lambda^2}{2}} = \sum_{n=0}^{\infty} H_n(z)\frac{\lambda^n}{n!}.$$

- As a consequence, for all $(\lambda, z, v) \in \mathbb{R}^3$

$$e^{\lambda z - \frac{\lambda^2}{2} v} = \exp\left(\lambda\sqrt{v}\frac{z}{\sqrt{v}} - \frac{(\lambda\sqrt{v})^2}{2}\right) = \sum_{n=0}^{\infty} H_n\left(\frac{z}{\sqrt{v}}\right)\frac{v^{n/2}\lambda^n}{n!} = \sum_{n=0}^{\infty} H_n(z,v)\frac{\lambda^n}{n!}$$

where

$$H_n(z,v) := v^{n/2} H_n\left(\frac{z}{\sqrt{v}}\right) = \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{(-1)^p n!}{2^p p!(n-2p)!} v^p z^{n-2p} \qquad (3.1)$$

are the Hermite polynomials of variance v.

Proof: one-factor Bergomi model

- The first seven polynomials $H_n(x,v)$ are

$$H_0(x,v) = 1$$
$$H_1(x,v) = x$$
$$H_2(x,v) = x^2 - v$$
$$H_3(x,v) = x^3 - 3vx$$
$$H_4(x,v) = x^4 - 6vx^2 + 3v^2$$
$$H_5(x,v) = x^5 - 10vx^3 + 15v^2x$$
$$H_6(x,v) = x^6 - 15vx^4 + 45v^2x^2 - 15v^3.$$

- In particular,

$$\exp\left(\omega e^{-k(u-T)}x - \frac{\omega^2}{2}e^{-2k(u-T)}v_T\right) = \sum_{n=0}^{\infty} H_n(x, v_T)e^{-nk(u-T)}\frac{\omega^n}{n!}.$$

- We get

$$f(T,x) = \sum_{n=0}^{\infty} \Xi_n H_n(x, v_T)\frac{\omega^n}{n!}.$$

Proof: one-factor Bergomi model

- Denote $I_n := \frac{\Xi_n}{\Xi_0}$, $P_n(x,v) := I_n H_n(x,v)$, and $\varepsilon := \frac{1}{\Xi_0}\sum_{n=1}^{\infty} \Xi_n H_n(x,v_T)\frac{\omega^n}{n!} = \sum_{n=1}^{\infty} P_n(x,v_T)\frac{\omega^n}{n!}$. Then

$$\sqrt{f(T,x)} = \sqrt{\Xi_0}\sqrt{1+\varepsilon} = \sqrt{\Xi_0}\sum_{n=0}^{6} Q_n(x,v_T)\omega^n + O(\omega^7)$$

where the polynomials $Q_n(x,v)$ are expressed in terms of the rescaled Hermite polynomials $P_n(x,v)$:

Lemma

Let $\varepsilon = \sum_{n=1}^{6} P_n \frac{\omega^n}{n!} + O(\omega^7)$. *Then* $\sqrt{1+\varepsilon} = \sum_{n=0}^{6} Q_n\omega^n + O(\omega^7)$ *where*

$$Q_0 = 1$$

$$Q_1 = \frac{1}{2}P_1$$

$$Q_2 = \frac{1}{4}P_2 - \frac{1}{8}P_1^2$$

$$Q_3 = \frac{1}{12}P_3 - \frac{1}{8}P_1P_2 + \frac{1}{16}P_1^3$$

$$Q_4 = \frac{1}{48}P_4 - \frac{1}{24}P_1P_3 - \frac{1}{32}P_2^2 + \frac{3}{32}P_1^2P_2 - \frac{5}{128}P_1^4$$

$$Q_5 = \frac{P_5}{240} - \frac{P_1P_4}{96} - \frac{P_2P_3}{48} + \frac{P_1^2P_3}{32} + \frac{3}{64}P_1P_2^2 - \frac{5}{64}P_1^3P_2 + \frac{7}{256}P_1^5$$

$$Q_6 = \frac{1}{1440}P_6 - \frac{1}{480}P_1P_5 - \frac{1}{192}P_2P_4 - \frac{1}{288}P_3^2 + \frac{1}{128}P_1^2P_4 + \frac{1}{32}P_1P_2P_3 + \frac{1}{128}P_2^3 - \frac{5}{192}P_1^3P_3 - \frac{15}{256}P_1^2P_2^2 + \frac{35}{512}P_1^4P_2 - \frac{63}{3072}P_1^6$$

Proof: one-factor Bergomi model

- To complete the proof, since

$$\mathbb{E}[\mathrm{VIX}_T] = \mathbb{E}[\sqrt{f(T, X_T)}] = \sqrt{\Xi_0} \sum_{n=0}^{6} \mathbb{E}[Q_n(X_T, v_T)]\omega^n + O(\omega^7),$$

  it is enough to compute $\mathbb{E}[Q_n(X_T, v_T)]$ for $n \in \{0, 1, \ldots, 6\}$.
- P_{2n} (resp. P_{2n+1}) being an even (resp. odd) polynomial in x, Q_1, Q_3 and Q_5 are odd polynomials in x. As X_T is a symmetric random variable, this implies that $\mathbb{E}[Q_n(X_T, v_T)] = 0$ for $n \in \{1, 3, 5\}$.
- For the computation of $\mathbb{E}[Q_n(X_T, v_T)]$, $n \in \{2, 4, 6\}$, remember that, from the **orthogonality property of Hermite polynomials**, $\mathbb{E}[P_m P_n(X_T, v_T)] = 0$ whenever $m \neq n$ (in particular, $\mathbb{E}[P_n(X_T, v_T)] = 0$ for $n \neq 0$).
- The other terms can be computed using that $\mathbb{E}[X_T^{2n}] = \frac{(2n)!}{2^n n!} v_T^n$.

Proof: two-factor Bergomi model

$$
\begin{aligned}
\xi_t^u &= \xi_0^u f^u(t, x_t^u) = \xi_0^u g^u(t, X_t^1, X_t^2) \\
dX_t^i &= -k_i X_t^i \, dt + dZ_t^i, \qquad X_0^i = 0, \qquad i \in \{1, 2\} \\
x_t^u &:= \alpha_\theta \left\{ \theta_1 e^{-k_1(u-t)} X_t^1 + \theta_2 e^{-k_2(u-t)} X_t^2 \right\} \\
f^u(t, x) &:= \exp\left(\omega x - \frac{\omega^2}{2} v_t(u) \right) \\
v_t(u) &:= \mathrm{Var}(x_t^u) = \alpha_\theta^2 \Big\{ \theta_1^2 e^{-2k_1(u-t)} v_t^1 + \theta_2^2 e^{-2k_2(u-t)} v_t^2 \\
&\qquad + 2\theta_1\theta_2 e^{-(k_1+k_2)(u-t)} v_t^{1,2} \Big\} \\
v_t^i &:= \frac{1 - e^{-2k_i t}}{2k_i}, \qquad v_t^{1,2} := \rho \frac{1 - e^{-(k_1+k_2)t}}{k_1 + k_2} \\
(X_t^1, X_t^2) &\sim \mathcal{N}\left(0, \begin{pmatrix} v_t^1 & v_t^{1,2} \\ v_t^{1,2} & v_t^2 \end{pmatrix} \right)
\end{aligned}
$$

Proof: two-factor Bergomi model

- Denote $\omega_\theta := \omega\alpha_\theta$,

$$X_t := \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix}, \quad V_t := \mathrm{Cov}(X_t) = \begin{pmatrix} v_t^1 & v_t^{1,2} \\ v_t^{1,2} & v_t^2 \end{pmatrix}, \quad \lambda(\delta) := \omega_\theta \begin{pmatrix} \theta_1 e^{-k_1\delta} \\ \theta_2 e^{-k_2\delta} \end{pmatrix}.$$

- With these notations (prime $=$ transpose)

$$\xi_t^u = \xi_0^u \exp\left(\lambda(u-t)' X_t - \frac{1}{2}\lambda(u-t)' V_t \lambda(u-t)\right).$$

- Then $\mathrm{VIX}_T^2 = \frac{1}{\tau}\int_T^{T+\tau} \xi_T^u \, du = f(T, X_T)$ with

$$f(T,x) := \frac{1}{\tau}\int_T^{T+\tau} \xi_0^u \exp\left(\lambda(u-T)' x - \frac{1}{2}\lambda(u-T)' V_T \lambda(u-T)\right) du, \quad x \in \mathbb{R}^2.$$

Proof: two-factor Bergomi model

$$f(T,x) := \frac{1}{\tau}\int_T^{T+\tau} \xi_0^u \exp\left(\lambda(u-T)'x - \frac{1}{2}\lambda(u-T)'V_T\lambda(u-T)\right) du, \quad x \in \mathbb{R}^2.$$

- For clarity, T being fixed, denote

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} := X_T, \qquad V = \begin{pmatrix} v_1 & v_{1,2} \\ v_{1,2} & v_2 \end{pmatrix} := V_T, \qquad \lambda_u := \lambda(u-T).$$

- Expand the above exponential term in powers of λ_u:

$$\exp\left(\lambda_u' x - \frac{1}{2}\lambda_u' V \lambda_u\right) = \sum_{\nu \in \mathbb{N}^2} H_\nu(x,V)\frac{\lambda_u^\nu}{\nu!}$$

where $\lambda^\nu := \lambda_1^{\nu_1}\lambda_2^{\nu_2}$, $\nu! := \nu_1!\nu_2!$, and the $H_\nu(x,V)$ are the **dual bivariate Hermite polynomials** (see Takemura and Takeuchi 1988).

Proof: two-factor Bergomi model

- The first $H_\nu(x, V)$ are given by

$$
\begin{aligned}
H_{0,0}(x,V) &= 1 \\
H_{1,0}(x,V) &= x_1 \\
H_{2,0}(x,V) &= x_1^2 - v_1 \\
H_{1,1}(x,V) &= x_1 x_2 - v_{1,2} \\
H_{3,0}(x,V) &= x_1^3 - 3 v_1 x_1
\end{aligned}
\qquad
\begin{aligned}
H_{2,1}(x,V) &= x_1^2 x_2 - 2 v_{1,2} x_1 - v_1 x_2 \\
H_{4,0}(x,V) &= x_1^4 - 6 v_1 x_1^2 + 3 v_1^2 \\
H_{3,1}(x,V) &= x_1^3 x_2 - 3 v_{1,2} x_1^2 - 3 v_1 x_1 x_2 + 3 v_1 v_{1,2} \\
H_{2,2}(x,V) &= x_1^2 x_2^2 - v_2 x_1^2 - 4 v_{1,2} x_1 x_2 - v_1 x_2^2 + v_1 v_2 + 2v \\
H_{5,0}(x,V) &= x_1^5 - 10 v_1 x_1^3 + 15 v_1^2 x_1
\end{aligned}
$$

$$
\begin{aligned}
H_{4,1}(x,V) &= x_1^4 x_2 - 4 v_{1,2} x_1^3 - 6 v_1 x_1^2 x_2 + 12 v_1 v_{1,2} x_1 + 3 v_1^2 x_2 \\
H_{3,2}(x,V) &= x_1^3 x_2^2 - v_2 x_1^3 - 6 v_{1,2} x_1^2 x_2 - 3 v_1 x_1 x_2^2 + 3\left(v_1 v_2 + 2 v_{1,2}^2\right) x_1 + 6 v_1 v_{1,2} x_2 \\
H_{6,0}(x,V) &= x_1^6 - 15 v_1 x_1^4 + 45 v_1^2 x_1^2 - 15 v_1^3 \\
H_{5,1}(x,V) &= x_1^5 x_2 - 5 v_{1,2} x_1^4 - 10 v_1 x_1^3 x_2 + 30 v_1 v_{1,2} x_1^2 + 15 v_1^2 x_1 x_2 - 15 v_1^2 v_{1,2} \\
H_{4,2}(x,V) &= x_1^4 x_2^2 - v_2 x_1^4 - 6 v_1 x_1^2 x_2^2 + 3 v_1^2 x_2^2 - 8 v_{1,2} x_1^3 x_2 + 6\left(v_1 v_2 + 2 v_{1,2}^2\right) x_1^2 \\
&\quad + 24 v_1 v_{1,2} x_1 x_2 - 3 v_1^2 v_2 - 12 v_1 v_{1,2}^2 \\
H_{3,3}(x,V) &= x_1^3 x_2^3 - 9 v_{1,2} x_1^2 x_2^2 - 3 v_2 x_1^3 x_2 - 3 v_1 x_1 x_2^3 + 9\left(v_1 v_2 + 2 v_{1,2}^2\right) x_1 x_2 + 9 v_{1,2} v_2 x_1^2 \\
&\quad + 9 v_1 v_{1,2} x_2^2 - 9 v_1 v_{1,2} v_2 - 6 v_{1,2}^3.
\end{aligned}
$$

Proof: two-factor Bergomi model

$$f(T,x) := \frac{1}{\tau}\int_T^{T+\tau} \xi_0^u \exp\left(\lambda_u' x - \frac{1}{2}\lambda_u' V \lambda_u\right) du, \quad x \in \mathbb{R}^2$$

$$\exp\left(\lambda_u' x - \frac{1}{2}\lambda_u' V \lambda_u\right) = \sum_{\nu\in\mathbb{N}^2} H_\nu(x,V)\frac{\lambda_u^\nu}{\nu!}$$

$$\begin{aligned}
f(T,x) &= \sum_{\nu\in\mathbb{N}^2} H_\nu(x,V)\frac{1}{\nu!}\frac{1}{\tau}\int_T^{T+\tau} \xi_0^u \lambda_u^\nu\, du \\
&= \sum_{\nu\in\mathbb{N}^2} H_\nu(x,V)\omega_\theta^{\nu_1+\nu_2}\frac{\theta_1^{\nu_1}\theta_2^{\nu_2}}{\nu_1!\nu_2!}\frac{1}{\tau}\int_T^{T+\tau} \xi_0^u e^{-(\nu_1 k_1+\nu_2 k_2)(u-T)} du \\
&= \sum_{\nu\in\mathbb{N}^2} H_\nu(x,V)\omega_\theta^{\nu_1+\nu_2}\frac{\theta_1^{\nu_1}\theta_2^{\nu_2}}{\nu_1!\nu_2!}\Xi_{\nu_1,\nu_2} \\
&= \sum_{n=0}^{\infty}\frac{\omega_\theta^n}{n!}\sum_{p=0}^{n}\frac{n!}{p!(n-p)!}\theta_1^p\theta_2^{n-p}\Xi_{p,n-p}H_{p,n-p}(x,V).
\end{aligned}$$

Proof: two-factor Bergomi model

$$f(T,x) = \sum_{n=0}^{\infty} \frac{\omega_\theta^n}{n!} \sum_{p=0}^{n} \frac{n!}{p!(n-p)!} \theta_1^p \theta_2^{n-p} \Xi_{p,n-p} H_{p,n-p}(x,V), \quad x \in \mathbb{R}^2$$

- Let us denote $f(T,x) = \Xi_{0,0}(1+\varepsilon)$ with (recall $I_{m,n} := \frac{\Xi_{m,n}}{\Xi_{0,0}}$)

$$\begin{aligned} \varepsilon &:= \sum_{n=1}^{\infty} \frac{\omega_\theta^n}{n!} P_n(x,V) \\ P_n(x,V) &:= \sum_{p=0}^{n} \frac{n!}{p!(n-p)!} \theta_1^p \theta_2^{n-p} I_{p,n-p} H_{p,n-p}(x,V) \end{aligned}$$

- Then

$$\sqrt{f(T,x)} = \sqrt{\Xi_{0,0}}\sqrt{1+\varepsilon} = \sqrt{\Xi_{0,0}} \sum_{n=0}^{6} Q_n(x,V) \omega_\theta^n + O(\omega^7)$$

where $Q_n(x,V)$ are built from $P_n(x,V)$ as seen in the one-factor case.

$$\mathbb{E}[\mathrm{VIX}_T] = \mathbb{E}[\sqrt{f(T,X)}] = \sqrt{\Xi_{0,0}} \sum_{n=0}^{6} \mathbb{E}[Q_n(X,V)] \omega_\theta^n + O(\omega^7)$$

Proof: two-factor Bergomi model

$$\mathbb{E}[\mathrm{VIX}_T] = \mathbb{E}[\sqrt{f(T,X)}] = \sqrt{\Xi_{0,0}} \sum_{n=0}^{6} \mathbb{E}[Q_n(X,V)]\omega_\theta^n + O(\omega^7)$$

- $H_\nu(x,V)$, as a polynomial in x, has same parity as $|\nu| := \nu_1 + \nu_2$ $\Longrightarrow P_n(x,V)$ has same parity as $n \Longrightarrow Q_1(x,V)$, $Q_3(x,V)$ and $Q_5(x,V)$ are odd polynomials in x. Since X is a centered random variable, $\mathbb{E}[Q_n(X,V)] = 0$ for $n \in \{1,3,5\}$.
- To compute $\mathbb{E}[Q_n(X,V)]$, $n \in \{2,4,6\}$, use the **weak orthogonality property of Hermite polynomials**: $\mathbb{E}[H_\mu H_\nu(X,V)] = 0$ whenever $|\mu| \neq |\nu|$. In particular, $\mathbb{E}[P_m P_n(X,V)] = 0$ whenever $m \neq n$, and $\mathbb{E}[P_n(X,V)] = 0$ for $n \neq 0$. For the other terms use

$$\mathbb{E}[X_1^2] = v_1, \quad \mathbb{E}[X_1X_2] = v_{1,2}, \quad \mathbb{E}[X_1^4] = 3v_1^2, \quad \mathbb{E}[X_1^3X_2] = 3v_1v_{1,2}, \quad \mathbb{E}[X_1^2X_2^2] = v_1v_2 + 2v_{1,2}^2,$$

$$\mathbb{E}[X_1^6] = 15v_1^3, \quad \mathbb{E}[X_1^5X_2] = 15v_1^2v_{1,2}, \quad \mathbb{E}[X_1^4X_2^2] = 3v_1v_2 + 12v_1v_{1,2}^2, \quad \mathbb{E}[X_1^3X_2^3] = 9v_1v_{1,2}v_2 + 6v_{1,2}^3.$$

$$\mathbb{E}[X_1^{2m}X_2^{2n}] = \frac{(2n)!}{2^{m+n}} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{(-1)^{n-i-j}(2m+2i)!}{(2i)!(m+i)!j!(n-i-j)!} v_1^{m-n+j} v_2^j v_{1,2}^{2(n-j)}, \quad m \geq n$$

$$\mathbb{E}[X_1^{2m+1}X_2^{2n+1}] = \frac{(2n+1)!}{2^{m+n+1}} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{(-1)^{n-i-j}(2m+2i+2)!}{(2i+1)!(m+i+1)!j!(n-i-j)!} v_1^{m-n+j} v_2^j v_{1,2}^{2(n-j)+1}, \quad m \geq n$$

Joint SPX/VIX smile calibration

The joint SPX/VIX smile calibration puzzle

- **It looks impossible to jointly calibrate the SPX and VIX smiles using continuous-time stochastic vol models with continuous SPX paths.**
- **In those models, large ATM SPX skew $\Longrightarrow$ large vol-of-vol, inconsistent with the relatively low VIX implied vols**, especially for short maturities.
- However, mean-reversion also comes into play. Increasing mean-reversion means that ATM SPX skew flattens and VIX implied vol decreases. **At different speeds?**
- Objective: precisely pinpoint the roles of vol-of-vol and mean-reversion.
- Bergomi-G. (2012): Expansion of SPX smile in small vol-of-vol in generic stochastic vol models.
- This talk: Expansion of VIX futures in small vol-of-vol in Bergomi models.
- Putting together both expansions sheds light on the **structural joint constraints on SPX and VIX imposed by stochastic vol models** in general, using the example of Bergomi models.

The joint SPX/VIX smile calibration puzzle

- In particular G. (2017) has shown that SPX/VIX market data shows **inversion of convex ordering** for short maturities T:

$$\mathrm{VIX}^2_{\mathsf{mkt},T} \leq_c \mathrm{VIX}^2_{\mathsf{loc},T}.$$

- G. (2018) has shown that **in the Bergomi models inversion of convex ordering requires large mean-reversion and large vol-of-vol**.
- Here we directly use approximate formulas of SPX skew and VIX futures in the one-factor Bergomi model to prove that **in the Bergomi models joint calibration requires large k and ω**.
- Make this statement more precise: **How big should $\frac{\omega}{k}$ be? $\frac{\omega^2}{k}$?**

Reminder on the ergodic regime:

- The limiting regime where k and ω tend to $+\infty$ while $\frac{\omega^2}{k}$ is kept **constant** corresponds to an **ergodic** limit where (ωX_t) quickly reaches its stationary distribution $\mathcal{N}(0, \frac{\omega^2}{2k})$. Cf Fouque, Papanicolaou and Sircar (2000).
- Only regime where k, ω are large and the variance of σ_t^2 has a finite limit, which is the natural regime in finance.

The SPX smile in the one-factor Bergomi model

- Bergomi-G. expansion (2012) gives the smile of generic stochastic volatility models at order 2 in vol-of-vol:

$$\widehat{\sigma}(T,K) = \widehat{\sigma}_T^{\mathrm{ATM}} + \mathcal{S}_T \ln\left(\frac{K}{S_0}\right) + \mathcal{C}_T \ln^2\left(\frac{K}{S_0}\right) + O(\omega^3)$$

- In the case of the one-factor Bergomi model with a flat initial term structure of variance swaps ($\xi_0^u \equiv \xi$), coefficients are explicit functions of ω, k, ρ, ξ, T. In particular, the ATM skew

$$\mathcal{S}_T = \frac{\rho\omega}{2}\mathcal{J}(kT) + \frac{\rho^2\omega^2\sqrt{\xi T}}{8}\left(2\mathcal{H}(kT) + 4\frac{\mathcal{J}(kT) - \mathcal{J}(2kT)}{kT} - 3\mathcal{J}(kT)^2\right)$$

where

$$\mathcal{I}(\alpha) = \frac{1-e^{-\alpha}}{\alpha}, \qquad \mathcal{J}(\alpha) = \frac{\alpha - 1 + e^{-\alpha}}{\alpha^2}$$

$$\mathcal{K}(\alpha) = \frac{1-e^{-\alpha}-\alpha e^{-\alpha}}{\alpha^2}, \qquad \mathcal{H}(\alpha) = \frac{\mathcal{J}(\alpha) - \mathcal{K}(\alpha)}{\alpha}$$

SPX skew and implied vol of VIX^2 at first order in ω (Bergomi1F)

$$\begin{aligned}
\mathcal{S}_T &= \frac{\rho\omega}{2}\frac{kT-1+e^{-kT}}{(kT)^2}+O(\omega^2)\\
\sigma_{\mathsf{VIX}_T^2} &= \omega\frac{1-e^{-k\tau}}{k\tau}\sqrt{\frac{1-e^{-2kT}}{2kT}}+O(\omega^3)
\end{aligned}$$

Small mean-reversion: cannot jointly calibrate

- $\mathcal{S}_T \approx \frac{\rho\omega}{4}$. Calibration to very short-term SPX smile: $\mathcal{S}_T \approx -1.5$ $\Longrightarrow \rho\omega \approx -6 \Longrightarrow \omega \geq 6$.
- $\sigma_{\mathsf{VIX}_T^2} \approx \omega \geq 6$: too large compared to market data (≈ 3)!
- Vol-of-vol implied by SPX skew $\approx 2 \times$ vol-of-vol implied by VIX futures!

SPX skew and implied vol of VIX^2 at first order in ω (Bergomi1F)

$$\mathcal{S}_T = \frac{\rho\omega}{2}\frac{kT - 1 + e^{-kT}}{(kT)^2} + O(\omega^2)$$

$$\sigma_{\mathsf{VIX}_T^2} = \omega\frac{1 - e^{-k\tau}}{k\tau}\sqrt{\frac{1 - e^{-2kT}}{2kT}} + O(\omega^3)$$

Large mean-reversion:

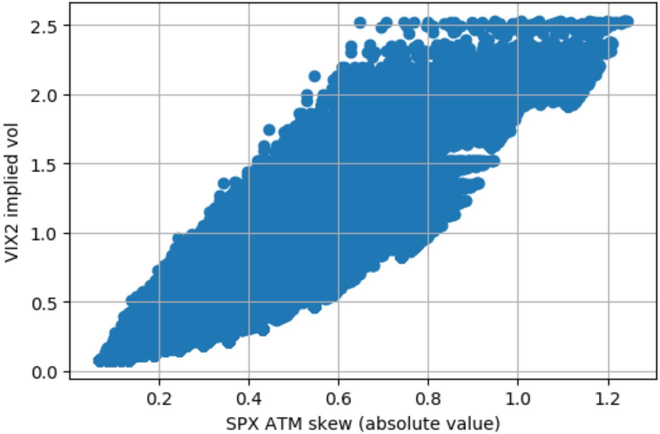
- $\mathcal{S}_T \approx \frac{\rho\omega}{2kT}$, $kT \gg 1$. Calibration to SPX smile, $T = \frac{1}{4}$: $\frac{\rho\omega}{2kT} \approx -0.6 \Longrightarrow 2\frac{\rho\omega}{k} \approx -0.6 \Longrightarrow \frac{\omega}{k} \geq 0.3$: ω and k are large. Numerical example: $k = 20, \rho = -1 \Longrightarrow \omega \geq 6$
- $\sigma_{\mathsf{VIX}_T^2} \approx \frac{\omega}{k^{3/2}\tau\sqrt{2T}} \approx \frac{\sqrt{2T}}{\rho\tau\sqrt{k}}\mathcal{S}_T$ behaves like $\frac{\omega}{k^{3/2}} \ll \frac{\omega}{k}$! Because of mean-reversion, implied vol of VIX_T^2 is much smaller. Numerical example with $\omega = 6$: $\sigma_{\mathsf{VIX}_T^2} \approx 1$.

$\Longrightarrow$ **Both ω and k must be large, with $\omega \approx k$ so $\frac{\omega^2}{k}$ large!**
Large stationary standard deviation of instantaneous vol.

Problems

- $\frac{\omega^2}{k}$ large $\Longrightarrow$ the small vol-of-vol expansions may be inaccurate, and the volatility is difficult to simulate (very large variance).
- Calibration only to VIX future, not to the full VIX smile. Use skewed Bergomi model (Bergomi 2008).
- Term-structure of SPX ATM skew requires at least two mean-reversion scales. The slow mean-reversion component ruins the $\frac{\omega}{k^{3/2}}$ behavior.

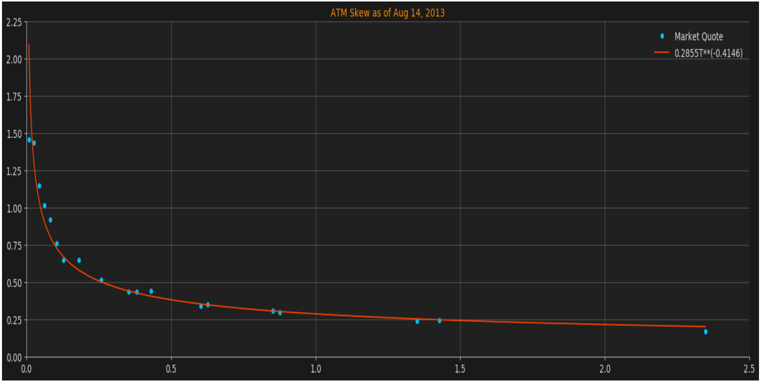
Two-factor Bergomi model: varying all parameters



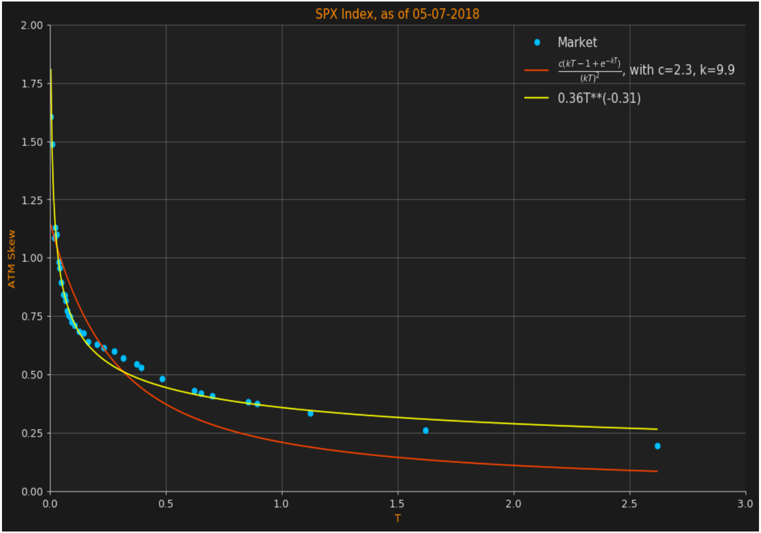
$$\omega \in [3, 8], k_1 \in [20, 100], k_2 \in [8, 20], \theta_2 \in [0, 0.3], \rho_{S1}, \rho_{S2} \in [-0.99, -0.5],\ T = 0.1$$

Term-structure of SPX ATM skew

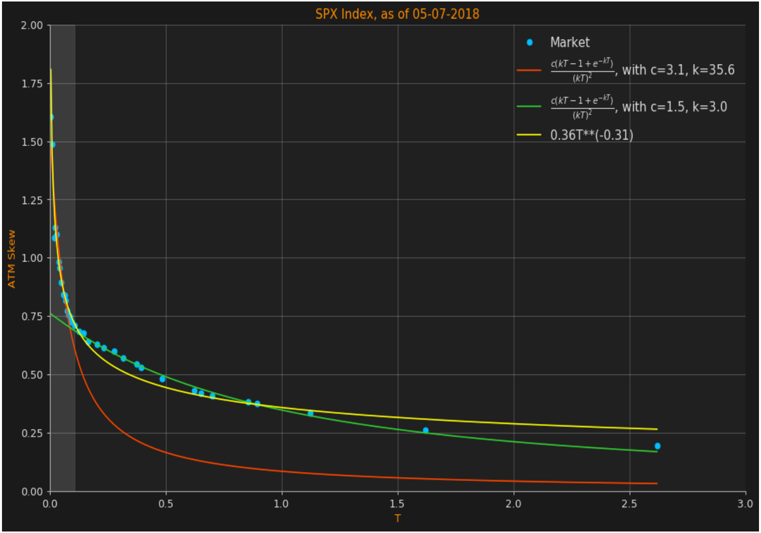
One-factor Bergomi model with large mean-reversion and vol-of-vol: $\mathcal{S}_T \sim \frac{1}{T}$.
To mimic a power-law decay $\mathcal{S}_T \sim \frac{1}{T^\alpha}$: 2-factor Bergomi model and rough volatility model.



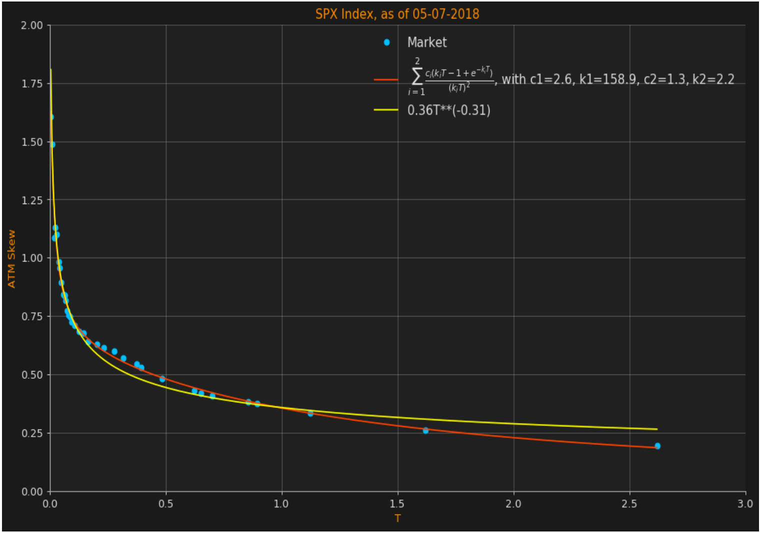
SPX ATM skew, May 7, 2018



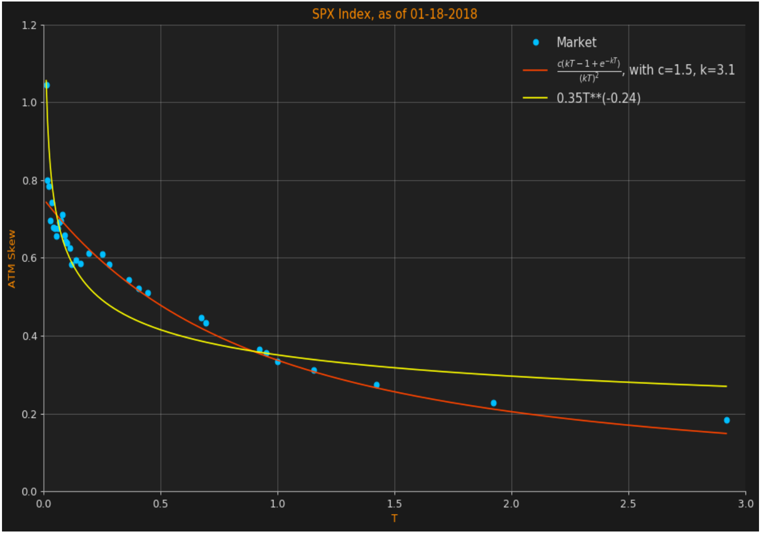
SPX ATM skew, May 7, 2018



SPX ATM skew, May 7, 2018



However... SPX ATM skew, Jan 18, 2018



Rough Bergomi model: Power-law kernel $K(\theta) = \nu\theta^{H-\frac{1}{2}}$

- **No Markov representation** for ξ_t^u.
- **Instantaneous variance $\sigma_t^2 := \xi_t^t$ is not a semimartingale**. One cannot write Itô dynamics $d\xi_t^t = \cdots dt + \cdots dZ_t$ for the instantaneous variance. **No notion of a dynamic volatility of instantaneous spot variance**.
- However we can compare the values of $\mathrm{Var}\left(\ln \frac{\xi_t^u}{\xi_0^u}\right)$ in the power-law and exponential kernel models:

$$\nu^2 \frac{u^{2H} - (u-t)^{2H}}{2H} \quad \longleftrightarrow \quad \omega^2 e^{-2k(u-t)} \frac{1 - e^{-2kt}}{2k} \tag{4.1}$$

$$u = t \to 0 : \qquad \nu^2 \frac{t^{2H}}{2H} \quad \longleftrightarrow \quad \omega^2 \frac{1 - e^{-2kt}}{2k} \approx \omega^2 t \tag{4.2}$$

$$\nu \frac{t^{H-\frac{1}{2}}}{\sqrt{2H}} \quad \longleftrightarrow \quad \omega \tag{4.3}$$

- **$\nu \frac{t^{H-\frac{1}{2}}}{\sqrt{2H}}$ can be interpreted as a short term volatility of instantaneous spot variance.**
- $[\nu] = \text{time}^{-H}$; $\left[\nu\theta^{H-\frac{1}{2}}\right] = \left[\nu \frac{t^{H-\frac{1}{2}}}{\sqrt{2H}}\right] =$ vol.

Rough Bergomi model: Power-law kernel $K(\theta) = \nu\theta^{H-\frac{1}{2}}$

- Short-term ATM skew in SV models $\sim \rho\omega$. **Explains why the ATM skew in such rough volatility models behaves like $T^{H-\frac{1}{2}}$ for short maturities T** (Alós, Fukasawa...), which is one of the reasons why this model has been introduced (Gatheral, Jaisson, Rosenbaum, Friz, Bayer).
- In the limit $H \to 0$, for fixed ν, $\nu^2 \frac{t^{2H}}{2H} \to +\infty$ for any $t > 0$.
- In order for $\mathrm{Var}(\sigma_t^2)$ to tend to a finite limit, we must impose that $\frac{\nu^2}{2H}$ tend to a finite limit $\Longrightarrow$ A natural limiting regime, analogous to the **ergodic regime** described above for the exponential kernel, is **$H, \nu \to 0$, with $\frac{\nu^2}{2H}$ kept constant**.
- However in this ergodic limit the SPX skew is $\sim \sqrt{H}T^{H-\frac{1}{2}}$...

Joint calibration with continuous SPX models? Numerical tests

Joint calibration: Calibrating first to VIX market

Skewing the models on ξ_t^u:

- Following Bergomi (2008), we use a linear combination of two lognormal random variables to model the instantaneous variance σ_t^2 so as to generate positive VIX skew:

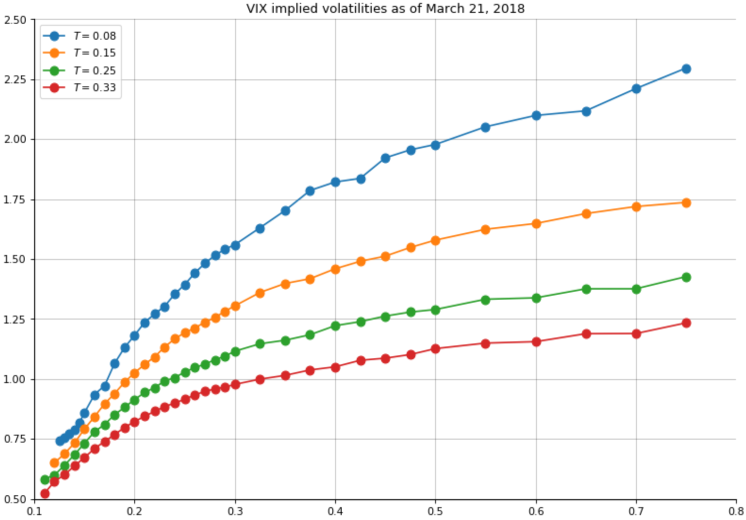
$$\sigma_t^2 = \xi_0^t \left((1-\lambda)\mathcal{E}\left(\omega_0 \int_0^t e^{-k(t-s)} dZ_s\right) + \lambda \mathcal{E}\left(\omega_1 \int_0^t e^{-k(t-s)} dZ_s\right)\right)$$

  or

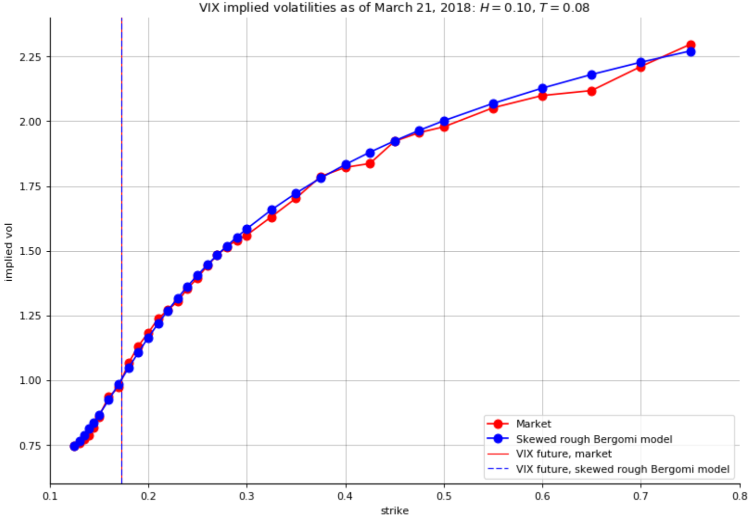
$$\sigma_t^2 = \xi_0^t \left((1-\lambda)\mathcal{E}\left(\nu_0 \int_0^t (t-s)^{H-\frac{1}{2}} dZ_s\right) + \lambda \mathcal{E}\left(\nu_1 \int_0^t (t-s)^{H-1/2} dZ_s\right)\right)$$

  with $\lambda \in [0,1]$.
- $\mathcal{E}(X)$ is simply a shorthand notation for $\exp\left(X - \frac{1}{2}\mathrm{Var}(X)\right)$.
- Also (independently) introduced by De Marco.

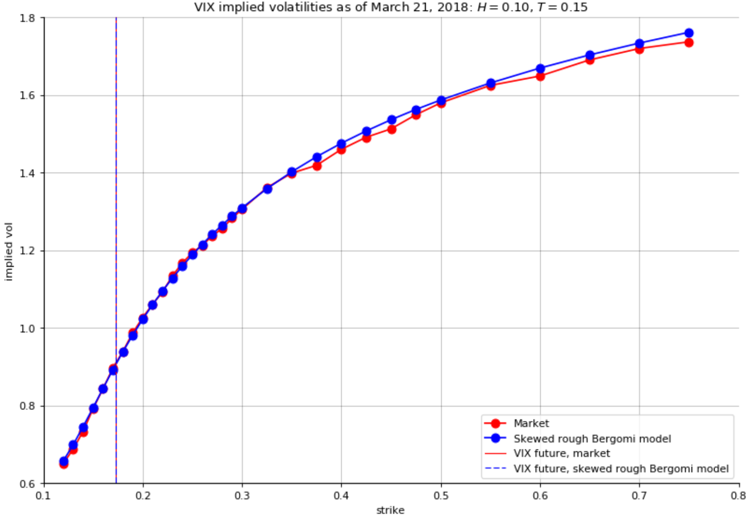
Skewed rough Bergomi: Calibration to VIX future and VIX options (March 21, 2018



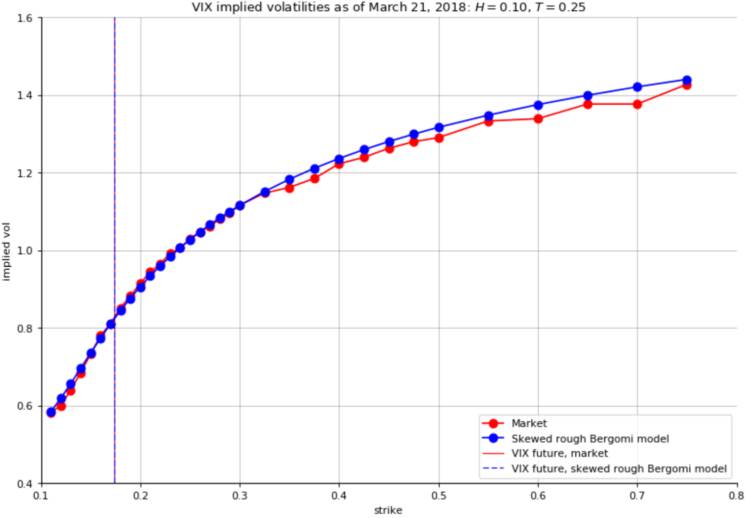
Skewed rough Bergomi: Calibration to VIX future and VIX options (March 21, 2018



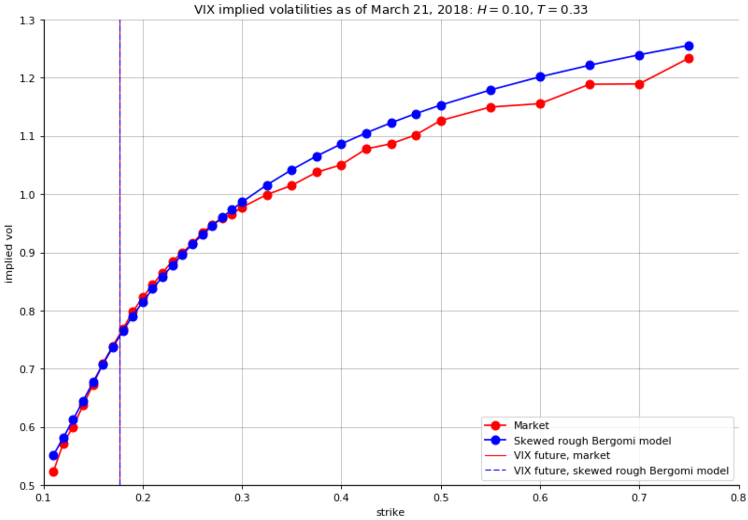
Skewed rough Bergomi: Calibration to VIX future and VIX options



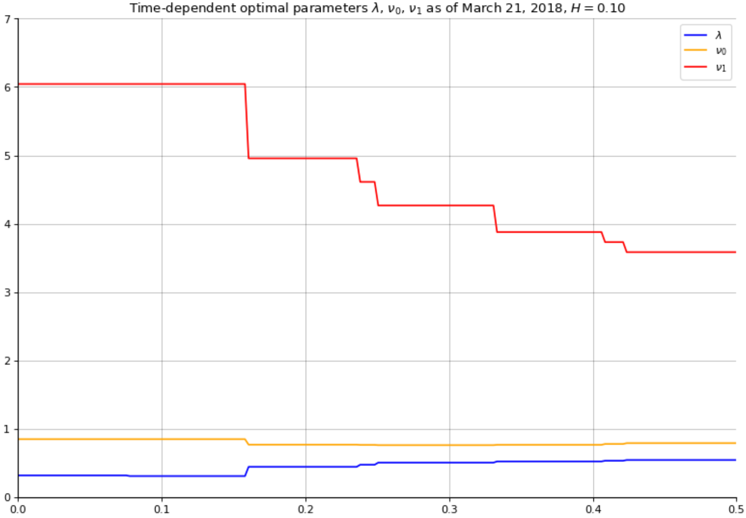
Skewed rough Bergomi: Calibration to VIX future and VIX options



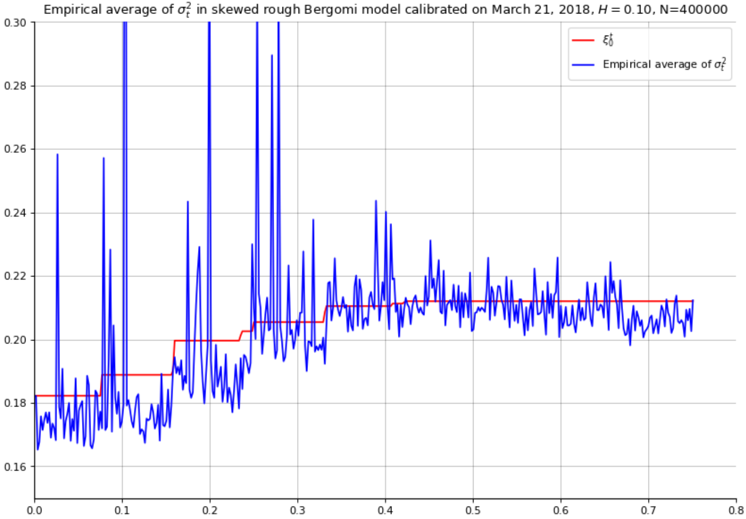
Skewed rough Bergomi: Calibration to VIX future and VIX options



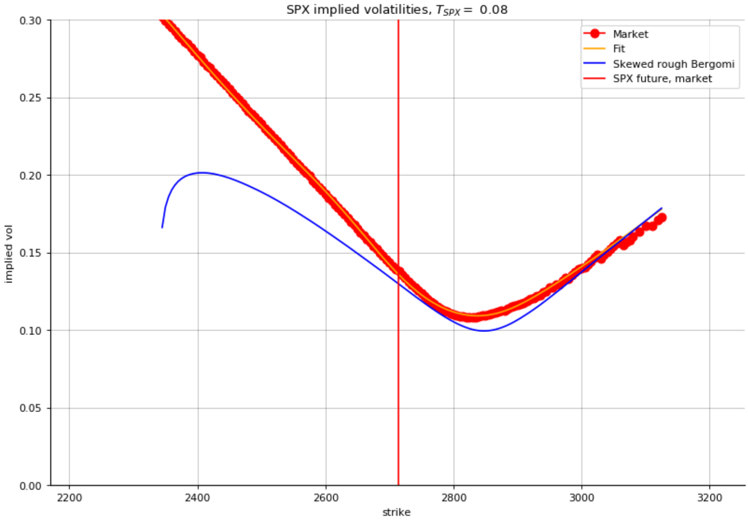
Skewed rough Bergomi: Calibration to VIX future and VIX options (March 21, 2018



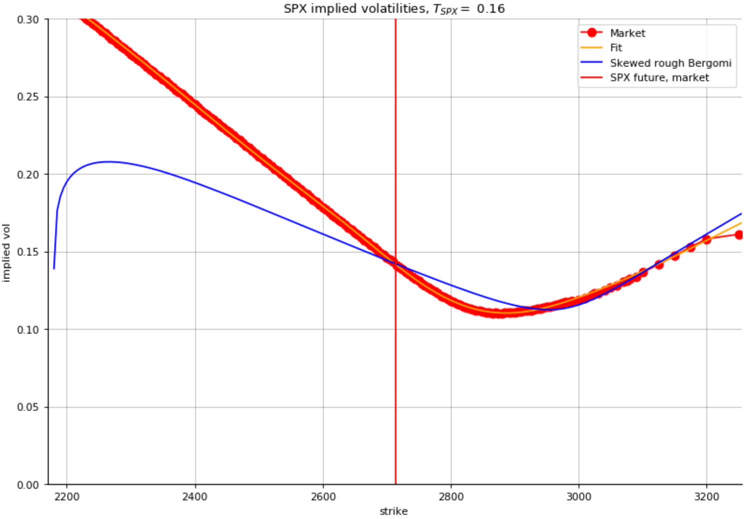
Skewed rough Bergomi: Calibration to VIX future and VIX options (March 21, 2018



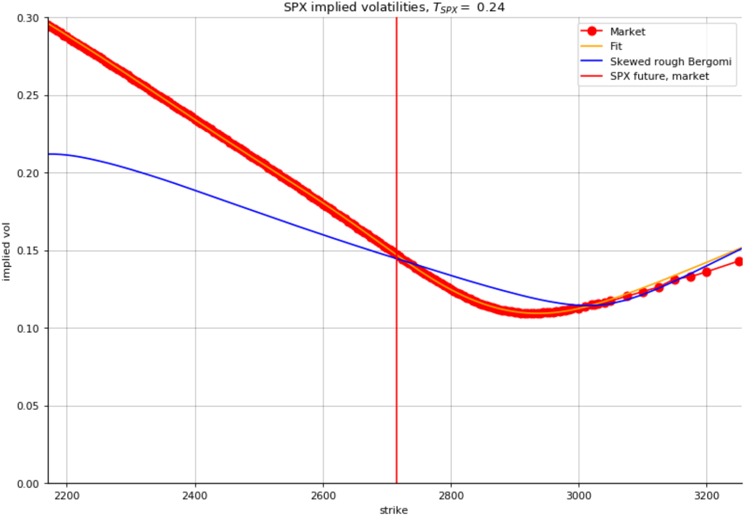
Skewed rough Bergomi calibrated to VIX: SPX smile



Skewed rough Bergomi calibrated to VIX: SPX smile



Skewed rough Bergomi calibrated to VIX: SPX smile



Skewed rough Bergomi calibrated to VIX: SPX smile

Skewed rough Bergomi calibrated to VIX: SPX smile

- **Not enough ATM skew for SPX**, despite pushing negative spot-vol correlation as much as possible.
- I get **similar results** when I use the **skewed 2-factor Bergomi model** instead of the skewed rough Bergomi model.

Joint calibration: Calibrating first to SPX market

Consider only continuous models on SPX that are calibrated to SPX smile:

$$\frac{dS_t}{S_t} = \frac{a_t}{\sqrt{\mathbb{E}[a_t^2|S_t]}} \sigma_{\mathrm{lv}}(t, S_t)\, dW_t$$

and optimize on (a_t) so as to match VIX options — or compute the infimum of VIX implied vols within those models.

Natural candidates for (a_t): skewed rough or 2-factor Bergomi model.
More generally: $a_t = \sigma_i(X_t)$, $t \in [T_i, T_i + \tau]$

- The **leverage function**

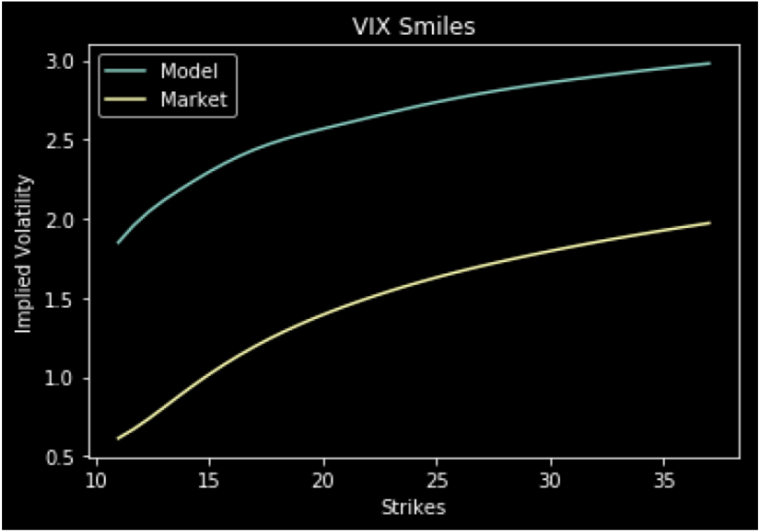
$$l(t, S_t) = \frac{\sigma_{\mathrm{loc}}(t, S_t)}{\sqrt{\mathbb{E}[a_t^2|S_t]}}$$

  does not mean revert; it **fights against inversion of convex ordering**.
- Numerically estimate

$$\mathrm{VIX}_{T_i}^2 = \frac{1}{\tau} \int_{T_i}^{T_i+\tau} \mathbb{E}\left[\frac{\sigma_i(X_t)^2}{\mathbb{E}[\sigma_i(X_t)^2|S_t]} \sigma_{\mathrm{loc}}(t, S_t)^2 \middle| \mathcal{F}_{T_i} \right] dt$$

  (use least squares Monte Carlo or neural networks)

Joint calibration: Calibrating first to SPX market (Aug 1, 2018)



Why jumps can help

- For a continuous model to calibrate jointly to SPX and VIX options, the distribution of $\mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau}\sigma_t^2\,dt\middle|\mathcal{F}_T\right]$ should be as narrow as possible, but without killing the SPX skew. The problem of ergodic/stationary (σ_t) is that they produce flat SPX skew.
- Jump-Lévy processes are precisely examples of processes that can generate deterministic realized variance together with a smile on the underlying.
- This explains why jumps have proved useful in this problem.

Conjecture

- Consider continuous models on SPX that are calibrated to SPX smile:

$$\frac{dS_t}{S_t} = \frac{a_t}{\sqrt{\mathbb{E}[a_t^2|S_t]}}\sigma_{\mathrm{loc}}(t, S_t)\, dW_t.$$

- Define

$$\mathsf{VIX}_T^2 = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}\left[\frac{a_t^2}{\mathbb{E}[a_t^2|S_t]}\sigma_{\mathrm{loc}}^2(t, S_t)\middle|\mathcal{F}_T\right] dt.$$

- **Conjecture: Continuous-time continuous-paths models for the SPX cannot fit VIX smile for small T:**

$$\inf_{(a_t)} \mathbb{E}\left[(\mathsf{VIX}_T - K)_+\right] > C_{\mathsf{VIX}}^{\mathsf{mkt}}(T, K).$$

- Controlled singular Mc-Kean equation, mean-field HJB PDE.

The joint SPX/VIX smile calibration puzzle solved

- **Exact joint calibration of SPX and VIX smiles**.
- **Completely different approach**: instead of parametric continuous-time models we use **nonparametric discrete-time models.**
- Discrete-time allows to decouple SPX skew and VIX implied vol.
- Nonparametric gives flexibility to fit the whole smiles.
- The model is solution to a **dispersion-constrained martingale transport problem**.
- Numerically built using the Sinkhorn algorithm.

Talk tomorrow at 3:15pm.

A few selected references

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