F-EQUIVALENCE FOR PARABOLIC SYSTEMS AND APPLICATION TO THE STABILIZATION OF NONLINEAR PDE

Abstract. We consider the F-equivalence problem for parabolic systems: under which conditions a control system, governed by a parabolic operator A and a control operator B, can be made equivalent to an arbitrarily exponentially stable evolution system through an appropriate control feedback law? While this problem has been resolved for finite-dimensional systems fifty years ago, good conditions for infinite-dimensional systems remain a challenge, especially for systems in spatial dimension larger than one. Our main result establishes optimal conditions for the existence of an F-equivalence pair (T, K) for a given parabolic control system (A, B). We introduce an extended framework for F-equivalence of parabolic operators, addressing key limitations of existing approaches, and we prove that the pair (T, K) is unique if and only if (A, B) is approximately controllable. As a consequence, this provides a method to construct feedback operators for the rapid stabilization of semilinear parabolic systems, possibly multi-dimensional in space. We provide several illustrative examples, including the rapid stabilization of the heat equation, the Kuramoto-Sivashinsky equation, and the Fisher-KPP equation.

MSC2020 classification: 93D15, 35K58

Keywords: system equivalence; feedback stabilization; rapid stabilization; exponential stability; parabolic systems.

Contents

1. Introduction 2
2. Settings and notations 5
2.1. Functional setting 5
2.2. Generalized Sobolev spaces 7
2.3. Frequency decomposition 7
2.4. Control setting 7
3. Main results 8
3.1. F-equivalence results 8
3.2. Rapid stabilization results 10
4. Applications and examples 11
4.1. Laplacian on manifolds 11
4.2. Nonlinear examples 14
5. Main proofs 18
5.1. Existence of parabolic F-equivalence 18
5.2. Stabilization of nonlinear systems 26
6. Approximate controllability and uniqueness 31
6.1. Weak F-equivalence 31
6.2. Approximate controllability 33
7. Acknowledgements 35
Appendix A. Series of functions in a Hilbert space 36
Appendix B. Properties of the operator A 36

B.1. A is normal 36
B.2. Extension for normal operators 37
Appendix C. Finite dimensional F-equivalence 37
Appendix D. Proof of Lemma 2.5 38
Appendix E. Proof of Proposition 3.3 39
References 40

1. Introduction

Consider the following nonlinear control problem

$$\partial_t u(t) = Au(t) + \mathcal{F}(u(t)) + Bw(t), \tag{1.1}$$

where A is an unbounded linear operator, $\mathcal{F}$ is a nonlinear perturbation that can also be unbounded, B is a given control operator, w is a control that can be chosen, and $u(t) \in H$, where H is a given Hilbert space. An interesting question in control theory is to know whether we can rapidly stabilize the system with the control w, that is whether the following holds.

PROBLEM 1. *For any $\lambda > 0$, there exists an operator $K = K_\lambda \in \mathcal{L}(D(A); U)$ such that, by choosing $w(t) = K_\lambda u(t)$, the system* (1.1) *is (locally) exponentially stable with decay rate λ.*

"Locally" here only makes sense when $\mathcal{F} \neq 0$ and means locally around the equilibrium $u^* = 0$ and U refers to a given Hilbert space such that $BK \in \mathcal{L}(D(A); H)$.

This question has been extensively investigated in the last decades in different frameworks and under different assumptions on A, B and $\mathcal{F}$. Even when the system is linear (i.e. $\mathcal{F} = 0$), answering this question in all generality is challenging. The first works date back (at least) to Slemrod [47] in 1972 in the case where $\mathcal{F} = 0$ and B is a bounded operator. Many results in this framework were obtained using tools from optimal control and Linear-Quadratic (LQ) theory by Lions, Barbu, Lasiecka, Triggiani and many others [34, 5, 32, 33, 52, 53]. This question was considered in the semilinear framework, i.e. $\mathcal{F}(H) \subset H$ in [48]. Other approaches successfully obtained results even when B is unbounded; one can cite, for instance, the observability approach of [49, 36, 38], in particular [49] shows that when (A, B) is exactly null controllable the system can be rapidly stabilized[1] and [36] gives a very nice characterization of stabilizability in terms of observability. Other results, inspired by optimal control approaches and using Riccati or Hamilton-Jacobi-Bellman equations, were obtained on semilinear systems either on particular systems of interest (for instance, [7]) or when A is parabolic [5]. One can also cite the Gramian approach (see, for instance, [52]) and in particular the recent result of [42] where the author managed modify the Gramian approach to get a quantitative rapid stabilization and even a finite-time stabilization for a semilinear system (i.e. $\mathcal{F}(H) \subset H$) when A is skew-adjoint (see [41] on the application to the 1D Schrödinger equation). Nevertheless, in these approaches, it often happens that the control feedback laws are not explicit, as they rely either on solving a minimization problem and an algebraic Riccati equation [28] (or even a Hamilton-Jacobi-Bellman equation [6], see also [30] for a learning approach to

[1]See also [57] for exponential stability without requesting rapid stabilization, i.e. arbitrary λ.

alleviate this difficulty) or because they rely on the knowledge of the semigroup $e^{\mathcal{A}^* t}$.

Other methods have been introduced, specifically for the parabolic cases. For instance, the impressive Frequency Lyapunov which obtains a quantitative rapid stabilization and even a finite-time stabilization for the multidimensional heat equation [56] (and later for the 2D Navier-Stokes equation [55]) by relying on Carleman estimates and a specific Lyapunov function. However, it requires the control to be distributed, that is K takes value in H. For this most challenging case, where B is unbounded and the control belongs to a finite-dimensional space (that is K takes value in $\mathbb{R}^k$ for some $k \in \mathbb{N} \setminus \{0\}$) it is worth highlighting, still in the parabolic framework, the work of [3] where the authors manage to deal both with a wide class of linear parabolic systems and, notably, with some nonlinear systems where the nonlinear perturbation $\mathcal{F}$ is not semilinear (i.e. $\mathcal{F}(H)$ does not belong to H) as long as the system is approximately controllable [2].

Another method was introduced to tackle the aforementioned limitations and deal with this most challenging case in a general framework: the F-*equivalence* (for *feedback equivalence*[2]). The principle is simple: instead of trying directly to find a feedback K, this method solves a different mathematical problem:

PROBLEM 2. *Given an operator $\widetilde{A}$, find $(T, K) \in \mathcal{L}(H) \times \mathcal{L}(D(A), \mathbb{R}^k)$ such that T is an isomorphism from H into itself and maps (in H) the system*

$$\partial_t u(t) = \mathcal{A}u(t) + BKu(t) \tag{1.2}$$

to the system

$$\partial_t u(t) = \widetilde{\mathcal{A}}u(t). \tag{1.3}$$

Of course, $\widetilde{\mathcal{A}}$ generates an exponentially stable semigroup on H, the existence of such a pair implies the exponential stability of the original system in H. This approach is sometimes called *generalized backstepping* or *Fredholm backstepping* for, when T is a Volterra transform of the second kind, it coincides with the well investigated *backstepping* method for 1D systems [4, 29, 26] (see also [54] or [10] and references therein). In the last ten years, F-equivalence approaches have been used to achieve rapid stabilization of many systems, first for particular systems [16, 17, 15, 13, 18, 23, 21, 14, 35] and recently in increasingly general settings [22, 24].

By definition, the F-equivalence problem is *a priori* asking for more than the rapid stabilization, which is only its consequence. However, it actually turns out that for many systems the sufficient conditions of existence of an F-equivalence are relatively permissive and for skew-adjoint systems they were shown to be even better than the usual known sufficient conditions for rapid stabilization [24, Section 3.1]. This can be explained since the F-equivalence allows to look at the problem directly as a stabilization problem rather than deriving a feedback from the resolution of the (optimal) control problem. This avoids usual admissibility conditions on B (see, for instance, [11, Section 2.3] or [50]) which usually ensures that the system (1.1) is well-posed for a whole class of control, which is not needed for the stabilization problem (one only needs the system to be well-posed along $w = Ku$). By considering the equivalence with the simpler system (1.3), it also avoids regularity additional conditions on K as long as (1.1) is well-posed. This, together with the explicitness

[2]This name was, in fact, first introduced by Brunovsky in [9] for linear finite-dimensional system.

of the feedback constructed makes the F-equivalence interesting both as a problem and for its application to the rapid stabilization.

However, when it comes to parabolic systems, the existing F-equivalence conditions are likely too conservative compared to usual condition of rapid stabilization: they are stronger than asking for the exact null controllability of (A, B) of [49], let alone the approximate controllability as in [3].

Another limit is that all the existing results of F-equivalence assume uniformly bounded multiplicities of the eigenvalues of A. For skew-adjoint systems this condition is necessary as soon as there is a finite number of controls (i.e. K takes value in $\mathbb{R}^k$). For parabolic system, however, this is likely too conservative as well. While, strictly speaking, this does not restrict this approach to 1D systems, it is still a strong limitation in practice when looking at systems that are multidimensional in space.

In this paper, we tackle these two limitations. We show the following (see Theorem 3.2 for a more detailed version):

THEOREM 1.1. *Let A be a parabolic (unbounded) operator on a Hilbert space H with a Riesz basis of eigenvectors and $B \in (D(A'))^k$. For all $\lambda \in \mathbb{R}_{>0}$, there exists an explicitly computable $m(\lambda)$ such that either*

- *$k < m(\lambda)$ and there is no exponentially stable operator $\widetilde{A}$ such that there exists a solution to the F-equivalence problem 2.*
- *$k \geq m(\lambda)$, in this case if B satisfies the λ-approximate controllability condition (H_B), there exists an explicit $\widetilde{A}$, T and $K \in \mathcal{L}(H; \mathbb{R}^k)$ such that $\widetilde{A}$ is exponentially stable with decay rate λ and (T, K) are solutions of the F-equivalence problem 2.*

This implies in particular that the original system (1.1) with $w(t) = Ku$ and $\mathcal{F} = 0$ is well-posed (see Proposition 3.3).

A consequence of this theorem is the exponential stability of the nonlinear system (1.1) (see also Theorem 3.4):

THEOREM 1.2. *Let A be a parabolic (unbounded) operator on a Hilbert space H with a Riesz basis of eigenvectors and let $B \in (D(A)')^k$ satisfies the λ-approximate controllability condition (H_B). If $\mathcal{F}$ satisfies the following assumption: there exists $\gamma \in [0, \frac{1}{2}]$ such that $\mathcal{F} : H \to D_{-\gamma}(A)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\forall u, v \in H, \quad \|u\|_H,\ \|v\|_H \leq \delta \Rightarrow \|\mathcal{F}u - \mathcal{F}v\|_{D_{-\gamma}(A)} \leq \varepsilon \|u - v\|_H, \tag{1.4}$$

then there exists an explicit $K \in \mathcal{L}(H, \mathbb{R}^k)$ such that the system (1.1) *is exponentially stable with $w(t) = Ku$.*

As intended, the λ-approximate controllability assumption (H_B) is much less restrictive than the F-equivalence conditions in a general setting given by [24]. This either significantly improves or recovers the recent results of [17, 23, 21, 35, 24]. As an illustration, in the case of the 1d heat equation and Burgers' equation on a torus studied in [21, 24] with $B = (B_1, B_2)$ where $B_1 :\ x \mapsto \sum_{n\geq 1} b_n^1 \sin(nx)$ is odd and B_2 $x \mapsto \sum_{n\geq 0} b_n^2 \cos(nx)$ is even, the F-equivalence condition of [23, 24] amounts to

$$b_0^2 \neq 0 \text{ and } \exists \gamma \in [0, 1/2),\ \forall j \in \{1, 2\},\ \forall n \geq 1,\ c \leq |b_n^j| \leq C n^{\gamma}, \tag{1.5}$$

and our λ-approximate controllability assumption (H_B) amounts to the (much) less restrictive condition

$$\forall j \in \{1,2\},\ \forall n \leq \sqrt{\lambda},\ b_n^j \neq 0. \tag{1.6}$$

In particular, if one wants to stabilize at any rate λ, the previous conditions amount to the approximate controllability of (A, B) by the generalized Fattorini criterion (see [3] and Lemma 6.8). The example of the Kuramoto-Shivashinski system of [17] is discussed in Section 4.2.1.

In Theorem 1.2 there is also no requirement on the multiplicity of the eigenvalues of A, which makes it suitable, for instance, to multidimensional systems in space, in contrast to essentially all the previous F-equivalence approaches [16, 17, 23, 35, 21, 22, 24].

While our main goal of this work is to improve the conditions for the F-equivalence problem for parabolic systems and extend them to multidimensional systems, one can note that the rapid stabilization result Theorem 3.4 can still be compared to previous results for parabolic systems that use different approaches. In particular, compared to [44], the system is not necessarily linear and, compared to [3, 1], the conditions on the nonlinearity $\mathcal{F}$ are different (in particular thanks to the F-equivalence our conditions on the nonlinearity are expressed only with respect to A and do not depend *a priori* on the feedback and can be weaker). Note that compared to [3], and similarly to [2], B can belong to $(D(A^*)')^k$ and does not have to belong to the smaller space $D((A^*)^{-s})^k$ for some $s < 1$.

In Section 4 we illustrate this result on several examples of applications. Among others, we study the rapid stabilization of a heat equation with potential on a Riemannian manifold, the (nonlinear) Kuramoto-Sivashinsky and Fisher-KPP equations.

Overall, the paper is organized as follows: in Section 2 we introduce our setting, notations and useful propositions. In Section 3 we state our main results, i.e. Theorems 3.2 and 3.4 that are shown in Section 5, and in Section 4 we give examples of applications on concrete systems.

2. Settings and notations

2.1. **Functional setting.** Let $(H, \langle \cdot\,,\cdot \rangle_H)$ be a Hilbert space and A be an unbounded operator on H satisfying the following conditions:

- (A1) There exists a family of eigenvectors $(e_n)_{n\geq 1}$ of A that is a Riesz basis of H. Then for every $x \in H$, Ax has a meaning (not necessarily in H) and we define the domain of A as
$$D(A) = \{x \in H \mid Ax \in H\}.$$
- (A2) The sequence $(\mathrm{Re}(\lambda_n))_{n\geq 1}$, where λ_n are the eigenvalues associated to $(e_n)_{n\in\mathbb{N}^*}$, is non-increasing and we have
$$\mathrm{Re}(\lambda_n) \underset{n\to+\infty}{\rightarrow} -\infty.$$
- (A3) There exists $C > 0$ such that
$$\forall n \geq 1,\ |\,\mathrm{Re}(\lambda_n)| \geq C|\,\mathrm{Im}(\lambda_n)|.$$

DEFINITION 2.1. *Let A be an unbounded operator on H. We will say that A is diagonal parabolic if it satisfies (A1), (A2) and (A3).*

In particular, any parabolic self-adjoint operator is a diagonal parabolic operator.

REMARK 2.2. *Hypothesis (A2) implies that the multiplicity of each eigenvalue of A is finite, but the supremum of all such multiplicities can be infinite. Hypothesis (A2) is here to ensure that A is the infinitesimal generator of an analytic semigroup.*

REMARK 2.3. *As we will see in Subsection 2.2 with the Sobolev scale, the growth rate of* $\mathrm{Re}(\lambda_n)$ *is closely linked to the nature of the space $D(A)$.*

For A diagonal parabolic operator, hypothesis (A2) ensures that $\{\mathrm{Re}(\lambda_n) \mid n \geq 1\}$ has a maximum m_A. We define $c_A := \max(0, m_A)$, this constant will be useful for stating our main result. Besides, from (A1) there exists an inner product of H such that $(e_n)_{n\geq 1}$ is orthonormal and the associated norm is equivalent to the norm associated with $\langle \cdot\,, \cdot\rangle_H$ (see [24, Section 2]). Therefore in the following we will assume, without loss of generality, that e_n is an orthonormal basis of H. Also, for every $x \in H$ we define x_n to be $\langle x\,, e_n\rangle_H$.

Since A is closed, $D(A)$ is an Hilbert space endowed with the inner product

$$\forall y, z \in D(A),\ \langle y\,, z\rangle_{D(A)} := \langle y\,, z\rangle_H + \langle Ay\,, Az\rangle_H. \tag{2.1}$$

Notice that $(e_n/\sqrt{1 + |\lambda_n|^2})_{n\geq 1}$ forms an orthonormal basis of $D(A)$. We see that $(D(A), H, D(A)')$ is a Gelfand triple.

Finally, note that a parabolic diagonal operator A is normal (see Proposition B.1), in particular we have $D(A) = D(A^*)$. This allows us to see it as an element of $\mathcal{L}(H, D(A)')$. This is formalized in Appendix B.2.

Later, it will be useful to consider AM where $M \in \mathcal{L}(D(A)')$, and we wish that $AM \in \mathcal{L}(H, D(A)')$. However, this is not generally the case, so we need to work with another operator algebra. This is the reason for the following proposition.

PROPOSITION 2.4. *We define the following algebra*

$$\mathcal{L}_H(D(A)') = \{M \in \mathcal{L}(D(A)') \mid M_{|H} \in \mathcal{L}(H)\}, \tag{2.2}$$

which, endowed with the norm,

$$||M||_{\mathcal{L}_H(D(A)')} := \sup_{\substack{||x||^2_{D(A)'} + ||y||^2_H = 1 \\ (x,y)\in D(A)'\times H}} \sqrt{||Mx||^2_{D(A)'} + ||My||^2_H}, \tag{2.3}$$

is a Banach algebra. Furthermore, we have the embedding of $\mathcal{L}_H(D(A)')$ in $\mathcal{L}(D(A)' \times H)$

$$\forall M \in \mathcal{L}_H(D(A)'),\ \forall (x, y) \in D(A)' \times H,\ \varphi(M)(x, y) := (Mx, My). \tag{2.4}$$

Proof. It is clear that $\mathcal{L}_H(D(A)')$ is an algebra and that φ is injective. Notice that the norm given above was designed such that

$$\forall M \in \mathcal{L}_H(D(A)'),\ ||\varphi(M)||_{\mathcal{L}(D(A)'\times H)} = ||M||_{\mathcal{L}_H(D(A)')}.$$

Hence, $\mathcal{L}_H(D(A)')$ is a Banach algebra. □

If we denote by $\mathcal{GL}_H(D(A)')$ the group of invertible elements in $\mathcal{L}_H(D(A)')$, then the above embedding gives us the following characterization:

$$M \in \mathcal{GL}_H(D(A)') \iff (M, M_{|H}) \in \mathcal{GL}(D(A)') \times \mathcal{GL}(H). \tag{2.5}$$

Notice that now, for every $M \in \mathcal{L}_H(D(A)')$, we have $AM \in \mathcal{L}(H, D(A)')$.

2.2. **Generalized Sobolev spaces.** To define the generalized Sobolev spaces, let us fix $\delta \geq 0$ such that $-A+\delta$ is invertible.[3] We denote here and in the following $-A+\delta$ for $-A+\delta I$. Then we can define

$$\forall s \in \mathbb{R}, \quad D_s(A) := D((-A+\delta)^s). \tag{2.6}$$

Endowed with the usual graph norm, these become Hilbert spaces. Let $s \in \mathbb{R}$, one can show that the following norm is equivalent

$$\forall x = \sum_{n\geq 1} x_n e_n \in D_s(A), \quad \|x\|^2_{D_s(A)} := \sum_{n\geq 1} (1+|\lambda_n|^2)^s |x_n|^2. \tag{2.7}$$

We can identify $D_{-s}(A)$ with $D_s(A)'$, and the triple $(D_{-s}(A), H, D_s(A))$ forms a Gelfand triple. Note that $D_1(A) = D(A)$ and $D_0(A) = H$.

We refer to this scale of spaces as generalized Sobolev spaces, as they share the same properties as classical Sobolev spaces. Moreover, when A is a power of the Laplace operator on a closed manifold, these spaces coincide with the usual Sobolev spaces. For more details, see Subsection 4.1 and in particular (4.3).

2.3. **Frequency decomposition.** Let $\lambda > 0$. Here, we focus on defining a decomposition of our spaces into low and high frequencies. We define the low and high frequency spaces as

$$\begin{aligned} L_\lambda &= \operatorname{span}\{e_n \mid \operatorname{Re}(\lambda_n) \geq -\lambda\}, \\ H_\lambda &= \overline{\operatorname{span}\{e_n \mid \operatorname{Re}(\lambda_n) < -\lambda\}}^{H}, \\ D(A)'_\lambda &= \overline{\operatorname{span}\{e_n \mid \operatorname{Re}(\lambda_n) < -\lambda\}}^{D(A)'}. \end{aligned} \tag{2.8}$$

We have the orthogonal decomposition $H = L_\lambda \oplus H_\lambda$ and $D(A)' = L_\lambda \oplus D(A)'_\lambda$. We set $N(\lambda) := \dim L_\lambda < +\infty$ and define $m(\lambda)$ to be the greatest multiplicity of any eigenvalue in L_λ.

2.4. **Control setting.** In this paper, we seek to stabilize A using only a finite number of scalar controls, which means that our control system looks like

$$\partial_t u = Au + Bw(t),$$

with $w(t) \in \mathbb{C}^m$ and B a given control operator. Let E be a normed vector space. We will use the canonical isomorphism from E^m to $\mathcal{L}(\mathbb{C}^m, E)$ to identify an element $B \in \mathcal{L}(\mathbb{C}^m, E)$ with $(B_1, \dots, B_m) \in E^m$ in the following way

$$B : z \in \mathbb{C}^m \mapsto \sum_{j=1}^{m} z_j B_j \in E. \tag{2.9}$$

Let $B \in (D(A)')^m$ for some fixed $m \geq 1$, when the supremum of the multiplicities of A is infinite, the pair (A, B) is not approximately controllable, making it impossible to achieve stabilization at any desired rate. Instead, we set a target stabilization rate $\lambda > 0$ and seek to determine whether we can stabilize the system at this rate. Note that, for any given $\lambda > 0$, we have the following lemma

[3]Note that is it sectorial since A is a diagonal parabolic operator.

LEMMA 2.5. *There exist* $N_1(\lambda), \dots, N_{m(\lambda)}(\lambda) \in \mathbb{N}^*$ *and a partition* $(e_n^1)_{n\geq 1}, \dots, (e_n^{m(\lambda)})_{n\geq 1}$ *of* $(e_n)_{n\geq 1}$ *such that, if we set* $L_\lambda^j = span((e_n^j)_{1\leq n\leq N_j(\lambda)})$ *and* $\mathcal{H}^j = \overline{span((e_n^j)_{n\geq 1})}^H$ *(thus* $H = \bigoplus_{j=1}^{m(\lambda)} \mathcal{H}^j$ *) , then the multiplicities of eigenvalues are simple in* L_λ^j *and*

$$L_\lambda = \bigoplus_{k=1}^{m(\lambda)} L_\lambda^k, \quad H = \bigoplus_{j=1}^{m(\lambda)} \mathcal{H}^j.$$

Moreover, for each $j \in \{1, \dots, m(\lambda)\}$*,* A *induces a diagonal parabolic operator on* $\mathcal{H}^j$ *such that*

$$A = A_1 + \cdots + A_{m(\lambda)}, \quad D(A) = \bigoplus_{j=1}^{m(\lambda)} D(A_j), \quad D(A)' = \bigoplus_{j=1}^{m(\lambda)} D(A_j)'.$$

This is shown in Appendix D.

In the following, we define P_L^j as the orthogonal projection onto L_λ^j in $\mathcal{H}^j$. We then set $P_H^j = Id_{\mathcal{H}^j} - P_L^j$. In order to achieve a stabilization at decay rate λ, we make the following assumption on our control operator:

(H_B) $m \geq m(\lambda)$, and for all $j \in \{1, \dots, m(\lambda)\}$, we have $B_j \in D(A_j)'$ and

$$\langle B_j \,, e_n^j\rangle_{D(A)'} \neq 0, \ \ \forall\, n \in \{1, \dots, N_j(\lambda)\},$$

DEFINITION 2.6. *Let* $B = (B_1, \dots, B_m) \in (D(A)')^m$*. We say that* B *is* F_λ*-admissible if it satisfies* (H_B)*.*

Let us briefly comment on this assumption (H_B). We first allow our control operators to be unbounded, which means they belong to a larger space than H. For well-posedness reasons, we know that $D(A)'$ is optimal, and here it is allowed, which makes it slightly less restrictive than the condition of [3] where $B \in D_s(A)^m$ with $s < 1$ and similar to the condition of [1] (which consider in addition a non-autonomous setting) . Then the condition on the scalar product of the B_j in (H_B) is here to ensure that the low-frequency system is controllable.

Finally, we define the concept of target operator, which we will need to define the concept of F-equivalence.

DEFINITION 2.7. *Let* A *be a diagonal parabolic operator and* $\lambda > 0$*. We say that an unbounded normal operator* D *on* H *is a* λ*-target if it is the infinitesimal generator of a differentiable semigroup on* H *with a growth rate of at most* $-\lambda$*. This means that*

$$\exists C > 0, \forall x \in H, \forall t \geq 0, \ ||e^{tD}x||_H \leq Ce^{-\lambda t}||x||_H.$$

REMARK 2.8. *In Definition 2.7, the operator* D *could have a domain different from* A*, and this happens, for instance, in* [14]*. In the following, however, we will only consider* λ*-targets having the same domain as* A*.*

3. Main results

3.1. ***F*-equivalence results.** Let $\lambda > 0$, A be a diagonal parabolic operator in H, D be a λ-target, and $B \in (D(A)')^m$. Notice that D only depend on A and λ. We can now define the concept of F-equivalence.

DEFINITION 3.1 (F-equivalence)**.** *Let $(T, K) \in \mathcal{GL}_H(D(A)') \times \mathcal{L}(H, \mathbb{C}^m)$. We say that (T, K) is an F-equivalence of (A, B, D), or that it is an F-equivalence between (A, B) and D, if*

$$\begin{cases} T(A + BK) = DT \text{ in } \mathcal{L}(H, D(A)'), \\ TB = B \text{ in } D(A)'. \end{cases} \tag{3.1}$$

Furthermore, if $K \in \mathcal{L}(L_\lambda, \mathbb{C}^m)$, we say that (T, K) is a parabolic F-equivalence.

Before presenting our main result, we want to emphasize few points. As one can imagine, finding an F-equivalence is a challenging problem. The condition $TB = B$ in 3.1 is included for two reasons. First, to make the problem linear in (T, K), and secondly, in the hope of achieving uniqueness, i.e., that there exists one and only one F-equivalence of (A, B, D), which greatly aids in finding a solution. As Theorem C.1 shows (see also [12]), if (A, B) is finite-dimensional and with $D = A - \lambda$, then there exists one and only one F-equivalence of (A, B, D). Unfortunately, in our case, we will show in Section 6 that in general, there is no uniqueness to the F-equivalence problem. However, at the same time, we introduce a new formalism that we call *weak F-equivalence*, which allows us to regain uniqueness. More precisely, we show that the uniqueness is linked with the approximate controllability of (A, B). For more details, see Section 6. Our first main theorem is the following

THEOREM 3.2 (Parabolic F-equivalence)**.** *Let A be a diagonal parabolic operator, and let $\lambda \in \mathbb{R}_{>0}$. Suppose $B \in (D(A)')^{m(\lambda)}$ is an F_λ-admissible control operator (see Definition 2.6). For $\mu \geq \lambda + c_A$, we define*

$$D = \begin{pmatrix} A_L - \mu & 0 \\ 0 & A_H \end{pmatrix}. \tag{3.2}$$

Then, D is a λ-target and for almost every $\mu \geq \lambda + c_A$, there exists a parabolic F-equivalence (T, K) between (A, B) and D.

The proof is provided in Subsection 5.1. Let us comment on the above theorem. First, the definition of D is very natural if the goal is to obtain a λ-target, that is a operator with growth rate at most $-\lambda$. The hope behind is to find a feedback operator K acting only on the low-frequency space, as one can expect intuitively and as is classically used for parabolic systems see, for instance, [3, 48, 56]. Secondly, besides D being intuitive, it is novel to perform F-equivalence in a generic framework with a target operator different from[4] $A - \lambda$, as done in [21, 22, 24]. This is, of course, made possible by the parabolic nature of the system.

In fact, here, it would have been impossible to take $D = A - \lambda$, because if, for example, $B \in H$, then BK would be a compact operator on H. If there exists $T \in \mathcal{GL}(H)$ such that

$$T(A + BK) = (A - \lambda)T,$$

we should have $\sigma(A + BK) = \sigma(A - \lambda)$. However, we know that if BK is compact, A and $A + BK$ should asymptotically have the same spectrum (see, for instance, [19, Chapter IV, Sec. 1]), which would be absurd.

[4]Note that another target operator was also used in [14] in the particular case of the Saint-Venant system.

3.2. **Rapid stabilization results.** Here, we apply F-equivalence to the stabilization of parabolic systems. We start with the linear case, then we consider semilinear equations. As before, let $\lambda > 0$, A be a diagonal parabolic operator in H, D be a λ-target, and $B \in (D(A)')^m$. As expected, finding a solution to the F-equivalence problem also ensures rapid stabilization of the linear system:

PROPOSITION 3.3. *Let* $(T, K) \in \mathcal{GL}_H(D(A)') \times \mathcal{L}(H, \mathbb{C}^m)$ *be a* F*-equivalence of* (A, B, D)*. Then* $A + BK$ *is a unbounded operator on* H *with dense domain* $T^{-1}(D(A))$ *which generate a differentiable semigroup with a growth of at most* $-\lambda$*. In particular, the Cauchy problem*

$$\begin{cases} \partial_t u = (A + BK)u, \quad \forall t > 0, \\ u(0) = u_0 \in H, \end{cases} \tag{3.3}$$

is well-posed in $C^0([0, +\infty); H) \cap C^\infty(0, +\infty; H)$ *with* $u(t) \in D(A + BK)$, $\forall t > 0$*, and we have the following exponential stability estimate:*

$$\exists C > 0, \forall u_0 \in H, \forall t \geq 0, \ ||u(t)||_H \leq Ce^{-\lambda t}||u_0||_H.^5 \tag{3.4}$$

Proof. See Appendix E. □

The above proposition demonstrates the utility of the F-equivalence approach for the stabilization of linear systems. To show that (A, B) is exponentially stable at rate λ, one only needs to demonstrate the existence of an F-equivalence between (A, B) and some target operator D as described above. Additionally, notice that this F-equivalence approach ensures that the problem is well-posed.

Now, using F-equivalence we aim to stabilize the following type of nonlinear control system

$$\partial_t u = Au + Bw(t) + \mathcal{F}(u). \tag{3.5}$$

Where $\mathcal{F}$ is a nonlinear map which satisfies some conditions that we will define right after. Let us first give an informal presentation of the idea. Suppose there exists an F-equivalence (T, K) of (A, B, D) where D is a known λ-target. First we set $w = Ku$, and we show that the following initial value problem is well-posed

$$\begin{cases} \partial_t u = (A + BK)u + \mathcal{F}(u), \\ u(0) = u_0 \in D(A + BK). \end{cases} \tag{3.6}$$

Then, we set $v := Tu$ to transform the above equation into

$$\partial_t v = Dv + T\mathcal{F}(T^{-1}v). \tag{3.7}$$

Finally, we obtain stability estimates on $v(\cdot)$, which we then transfer back to $u(\cdot)$ using T^{-1}. We hope that if v is small enough, the $T\mathcal{F}T^{-1}$ term will behave correctly enough not to disturb the stability of the system in v too much. When the system is semilinear, i.e. $\mathcal{F} : H \to H$, this is a relatively classical approach (see, for instance, [48]) as long as $\mathcal{F}$ is small enough around 0, which is typically the case for a nonlinear perturbation. A more difficult case occurs when $\mathcal{F}$ itself is unbounded. We introduce the following assumption on $\mathcal{F}$.

[5] Here $u(\cdot)$ is the solution with initial condition u_0.

Assumption 1. *It is a map from H to $D_{-1/2}(A)$, it is Lipschitz on bounded sets and for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\forall u, v \in H, \quad ||u||_H,\ ||v||_H \leq \delta \Rightarrow ||\mathcal{F}(u) - \mathcal{F}(v)||_{D_{-1/2}(A)} \leq \varepsilon ||u - v||_H. \tag{3.8}$$

This is typically satisfied for quadratic functions but also holds for much more generic classes of operators. The next theorem formalizes the above discussion.

THEOREM 3.4. *Let A be a diagonal parabolic operator and $\mathcal{F}$ be a map satisfying Assumption 1. Let $\lambda \in \mathbb{R}_{>0}$. Suppose $B \in (D(A)')^m$ is an F_λ-admissible control operator. Let (T, K) be a parabolic F-equivalence of (A, B, D) given by Theorem 3.2 for some $\mu \geq \lambda + c_A$. Then for all $u_0 \in H$ there exists a unique maximal solution $u \in C^0([0, \tau_{u_0}); H)$ (with $\tau_{u_0} \in (0, +\infty]$) to the following system*

$$\begin{cases} \partial_t u = (A + BK)u + \mathcal{F}(u), \\ u(0) = u_0. \end{cases} \tag{3.9}$$

Moreover it is locally exponentially stable in H. More specifically, there exists $C_\lambda > 0$ and $\delta > 0$ such that for all $u_0 \in H$ with $||u_0||_H \leq \delta$, there exists a unique solution $u(\cdot) \in C^0([0, +\infty); H)$ to (3.9) and

$$\forall t \geq 0,\ ||u(t)||_H \leq C_\lambda e^{-\frac{\lambda}{2}t} ||u_0||_H. \tag{3.10}$$

REMARK 3.5. *If one only wanted the local exponential stability in Theorem 3.4 without the existence of a unique maximal solution for any initial condition, then Assumption 1 could be weakened and $\mathcal{F}$ only has to be locally Lipschitz, instead of Lipschitz on bounded sets.*

The proof of this result is done in Section 5.2. Note that, as shown in the proof of Theorem 3.2, K is obtained by solving a finite-dimensional linear system of equations. Thus, the above theorem provides a very simple and explicit way to stabilize a whole class of nonlinear parabolic PDE.

4. Applications and examples

4.1. **Laplacian on manifolds.** Let $(\mathcal{M}, g)$ be a compact oriented and connected d-dimensional riemannian manifold, and set $H = L^2(\mathcal{M})$, $A = \Delta_g$ and $(e_n)_{n \geq 1}$ an orthonormal basis of eigein vector such that

$$0 = -\lambda_1 \leq -\lambda_2 \leq \cdots \leq -\lambda_k \to +\infty. \tag{4.1}$$

The Weyl law tells us that $N(\lambda) \underset{\lambda \to +\infty}{\sim} \frac{\mathrm{Vol}(\mathcal{M})\omega_d}{(2\pi)^d} \lambda^{\frac{d}{2}}$, where we recall that $N(\lambda) = \dim(L_\lambda)$ (see (2.8)). Because the eigenvalues are non-decreasing, and from the definition of L_λ, we have $N(\lambda_n) = n$. Hence, we have

$$|\lambda_n| \underset{n \to +\infty}{\sim} \frac{4\pi^2}{(\omega_d \mathrm{Vol}(\mathcal{M}))^{\frac{2}{d}}} n^{\frac{2}{d}}. \tag{4.2}$$

Using the classical characterization of Sobolev spaces on manifolds as in [40], we have

$$\forall s \in \mathbb{R}, \quad D_s(A) = H^{2s}(\mathcal{M}). \tag{4.3}$$

Notice that Δ_g is a diagonal parabolic operator on $L^2(\mathbb{T})$, so we have the following immediate corollary of our main result.

COROLLARY 4.1. *Let $(\mathcal{M}, g)$ be a compact oriented and connected d-dimensional Riemannian manifold. We set $H = L^2(\mathcal{M})$, $A = \Delta_g$, and fix $\lambda \in \mathbb{R}_{>0}$. Let $B \in (H^{-2}(\mathcal{M}))^{m(\lambda)}$ be an F_λ-admissible control operator. Then there exists $K \in \mathcal{L}(L^2(\mathcal{M}), \mathbb{R}^{m(\lambda)})$ such that the Cauchy problem*

$$\begin{cases} \partial_t u = \Delta_g u + BKu, & \forall t > 0, \\ u(0) = u_0 \in L^2(\mathcal{M}), \end{cases} \tag{4.4}$$

is well-posed,[6] *and we have the following stability estimate:*

$$\exists C > 0, \forall u_0 \in L^2(\mathcal{M}), \forall t \geq 0, \; ||u(t)||_{L^2} \leq Ce^{-\lambda t}||u_0||_{L^2}. \tag{4.5}$$

Before presenting concrete examples, notice that we have all the information we need about the asymptotic behavior of $N(\lambda)$ and λ_n. Interestingly, we see that the growth rate of the eigenvalues is entirely governed by the topology of the manifold, while the prefactor depends on its geometry, specifically its volume.

Despite all this, we still have no information on $m(\lambda)$. In fact, it is well-known that Δ_g has "generically" simple eigenvalues, which implies $m(\lambda) = 1$ for all $\lambda > 0$. For a more precise definition, see [39, 51]. For clarity, we will also demonstrate a similar result in a simpler framework in the following paragraph.

Let $d \geq 2$ and $l = (l_1, \dots, l_d) \in \mathbb{R}^d_{>0}$. We define the d-torus with sides $l_1, \dots, l_d$ as

$$\mathbb{T}^d_l := \mathbb{R}/2\pi l_1\mathbb{Z} \times \cdots \times \mathbb{R}/2\pi l_d\mathbb{Z}.$$

It is a flat Riemannian manifold, and in the usual coordinates $(\theta_1, \dots, \theta_d)$, we have as expected $\Delta_{\mathbb{T}^d_l} = \sum_{k=1}^d \partial^2_{\theta_k}$. Let $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, we set in local coordinates

$$e_n(\theta) = \frac{1}{\sqrt{(2\pi)^d l_1 \dots l_d}} e^{i \sum_{k=1}^d \frac{n_k}{l_k}\theta_k}. \tag{4.6}$$

We can easily verify that this defines smooth functions on $\mathbb{T}^d_l$ and that they form an orthonormal basis of $L^2(\mathbb{T}^d_l)$. Additionally, we have

$$\Delta_{\mathbb{T}^d_l} e_n = \left(\sum_{k=1}^d \frac{n_k^2}{l_k^2}\right) e_n.$$

Hence, we explicitly have the eigenvalues of $\Delta_{\mathbb{T}^d_l}$.

PROPOSITION 4.2. *Let $D \subset \mathbb{R}^d_{>0}$ be a bounded, non-negligible measurable set, endowed with the normalized Lebesgue measure denoted by $\mathbb{P}$. We define a Bernoulli random variable $S : D \to \mathbb{R}$ to be 1 for all $l \in D$ such that $\Delta_{\mathbb{T}^d_l}$ has only simple eigenvalues. Then we have*

$$\mathbb{P}(S = 1) = 1. \tag{4.7}$$

Proof. We set

$$N = \{S = 0\} = \{l \in D \,|\, \exists n \neq m \in \mathbb{Z}^d, \sum_{k=1}^d \frac{n_k^2}{l_k^2} = \sum_{k=1}^d \frac{m_k^2}{l_k^2}\} = \bigcup_{\substack{n \neq m \\ (n,m) \in \mathbb{Z}^d}} \{l \in D \,|\, \sum_{k=1}^d \frac{n_k^2}{l_k^2} = \sum_{k=1}^d \frac{m_k^2}{l_k^2}\},$$

which is measurable as a countable union of measurable sets. We will show that $N \subset C_\mathbb{Q}$, with $C_\mathbb{Q} := \{l \in D \,|\, (\frac{1}{l_1^2}, \dots, \frac{1}{l_d^2})$ is $\mathbb{Q}$-linearly dependent$\}$, and that $\mathbb{P}(C_\mathbb{Q}) = 0$.

---

[6] In the sense of Proposition 3.3.

Let $l \in N$. Then there exist $n \neq m \in \mathbb{Z}^d$ such that

$$\sum_{k=1}^{d} \frac{n_k^2 - m_k^2}{l_k^2} = 0.$$

Hence, this implies that $l \in C_{\mathbb{Q}}$. Now for all $q \in \mathbb{Q}^d \setminus \{0\}$, we set $L_q(x) = \sum_{k=1}^{d} q_k x_k$ for $x \in \mathbb{R}^d$. Notice that we have

$$\{l \in D \,|\, (l_1, \dots, l_d) \text{ is } \mathbb{Q}\text{-linearly dependent}\} = D \cap \bigcup_{q \in \mathbb{Q}^d \setminus \{0\}} \ker(L_q).$$

Hence, the above set is measure zero as a countable union of measure zero sets, and because $x \mapsto \left(\frac{1}{x_1^2}, \dots, \frac{1}{x_d^2}\right)$ is a diffeomorphism on $\mathbb{R}^d_{>0}$, we deduce that $C_{\mathbb{Q}}$ is measure zero too.

□

Another way to state the above result is to say that if one takes D as in the proposition, it can be seen as a parameter space for randomly selecting a d-torus. Then, almost surely, we will have $m(\lambda) = 1$ for all $\lambda > 0$. This simple framework allows us to give precise meaning to the statement "Δ_g has generically simple eigenvalues".

We conclude this subsection with some examples. Let $\mathcal{M}$ be a compact oriented and connected Riemannian manifold of dimension $d \leq 3$. If we make no further assumptions on it, we have the following result.

PROPOSITION 4.3. *Let $p \in \mathcal{M}$, and denote by δ_p the Dirac distribution at p. Let $\lambda > 0$, and define $K \in \mathcal{L}(L^2(\mathcal{M}), \mathbb{R})$ as*[7]

$$\forall f \in L^2(\mathcal{M}), \quad Kf = -\lambda \int_{\mathcal{M}} f \, d\mu_g. \tag{4.8}$$

Then, if λ is small enough (more precisely, $\lambda < -\lambda_2$), the Cauchy problem

$$\begin{cases} \partial_t u = \Delta_g u + \delta_p K u, & \forall t > 0, \\ u(0) = u_0 \in L^2(\mathcal{M}), \end{cases} \tag{4.9}$$

is well-posed, and there exists $C > 0$ such that

$$\forall u_0 \in L^2(\mathcal{M}), \quad ||u(t)||_{L^2} \leq C e^{-\lambda t} ||u_0||_{L^2}. \tag{4.10}$$

Proof. Since $\mathcal{M}$ is connected, $\lambda_1 = 0$ is a simple eigenvalue, and hence $e_1 = \frac{1}{\sqrt{\mathrm{Vol}(\mathcal{M})}}$ and $\lambda_2 < 0$. Let $\lambda \in (0, -\lambda_2)$, then $m(\lambda) = 1$. By the Sobolev embedding theorem, $\delta_p \in H^{-\frac{d}{2}-\varepsilon}(\mathcal{M}) \subset H^{-2}(\mathcal{M})$. Since $(e_n)_{n \geq 1}$ forms a Hilbert basis of $L^2(\mathcal{M})$, it is straightforward to show that, in $H^{-\frac{d}{2}-\varepsilon}(\mathcal{M})$, we have

$$\delta_p = \sum_{n \geq 1} e_n(p) e_n. \tag{4.11}$$

Now we can apply Theorem 3.2. Since $L_\lambda = \mathrm{span}(e_1)$, we have $(\Delta_g)_L = 0$ and $(\delta_p)_L = \frac{1}{\sqrt{\mathrm{Vol}(\mathcal{M})}}$. Solving the one-dimensional F-equivalence is then trivial, and we obtain

$$K = -\lambda \sqrt{\mathrm{Vol}(\mathcal{M})} \langle \cdot, e_1 \rangle_{L^2}. \tag{4.12}$$

□

[7]Here, $d\mu_g$ is the measure induced by the Riemannian volume form.

To stabilize the heat equation at any desired rate using only a Dirac control, we assume that $(\mathcal{M}, g)$ is such that Δ_g has only simple eigenvalues. As we discussed earlier, this situation typically arises when a manifold is chosen randomly. We now present the following stronger result.

PROPOSITION 4.4. *Suppose that dim $\mathcal{M} \leq 3$ and that Δ_g has only simple eigenvalue, then for almost every $p \in \mathcal{M}$, the pair (Δ_g, δ_p) is approximately controllable. This implies that for every $\nu > 0$, there exists $K \in \mathcal{L}(L^2(\mathcal{M}), \mathbb{R})$ such that the Cauchy problem*

$$\begin{cases} \partial_t u = \Delta_g u + \delta_p K u, \quad \forall t > 0, \\ u(0) = u_0 \in L^2(\mathcal{M}), \end{cases} \tag{4.13}$$

is well-posed, and there exists a constant $C > 0$ such that

$$\forall u_0 \in L^2(\mathcal{M}), \quad ||u(t)||_{L^2} \leq C e^{-\nu t} ||u_0||_{L^2}. \tag{4.14}$$

Proof. The nodal set of e_n (i.e. $e_n^{-1}(\{0\})$) has measure zero for every $n \geq 1$, so for almost every $p \in \mathcal{M}$, we have

$$\forall n \geq 1, \quad e_n(p) \neq 0. \tag{4.15}$$

Recall that $\delta_p \in H^{-\frac{d}{2}-\varepsilon}(\mathcal{M})$ and that $\delta_p = \sum_{n \geq 1} e_n(p) e_n$ with $\langle \delta_p, e_n \rangle = e_n(p) \neq 0$. Using Lemma 6.8, we deduce that for almost every $p \in \mathcal{M}$, the pair (Δ_g, δ_p) is approximately controllable. Now let $\lambda > \nu$ as in Theorem 3.2. Since $m(\lambda) = 1$, the approximate controllability implies that δ_p is F_λ-admissible. Therefore, Theorem 3.2 together with Proposition 3.3 allows us to conclude.

□

REMARK 4.5. *As always with any F-equivalence, the feedback K can be easily constructed using the same method as in the example following Corollary 4.7.*

4.2. **Nonlinear examples.** In this section we provides some concrete applications of Theorem 3.4 to some classical PDE.

4.2.1. *Kuramoto–Sivashinsky equation.* Here, we focus on the Kuramoto–Sivashinsky equation on the one-dimensional torus, which is given by

$$\partial_t u + \Delta^2 u + \Delta u + \frac{1}{2} \partial_x (u^2) = 0. \tag{4.16}$$

This equation was introduced by Yoshiki Kuramoto and Gregory Sivashinsky to study flame front propagation, for more details see [31, 45, 46]. To apply Theorem 3.4, we need to establish the appropriate setting. We work in $H = L^2(\mathbb{T})$, and define $A = -(\Delta^2 + \Delta)$, hence we have

$$\forall s \in \mathbb{R}, \quad D_s(A) = H^{4s}(\mathcal{M}). \tag{4.17}$$

The eigenbasis of A is $(e_n)_{n \in \mathbb{Z}}$, defined in (4.6). In order to apply our result we can reindex by $\mathbb{N}_{>0}$ in the following way

$$\forall n \geq 1, \ \tilde{e}_n := \begin{cases} e_k \text{ if } n = 2k+1, \ k \geq 0, \\ e_{-k} \text{ if } n = 2k, \ k \geq 1. \end{cases} \tag{4.18}$$

With this notation we similarly define $\tilde{\lambda}_n$, we deduce from the previous section that

$$\tilde{\lambda}_n = -\left\lfloor \frac{n}{2} \right\rfloor^4 + \left\lfloor \frac{n}{2} \right\rfloor^2 \underset{n \to +\infty}{\sim} -\frac{n^4}{16}. \tag{4.19}$$

Hence A is a diagonal parabolic operator on H. For simplicity, we will continue to use the family $(e_n)_{n\in\mathbb{Z}}$ for the Sobolev norms.

We want to define $\mathcal{F}(u) = -\frac{1}{2}\partial_x(u^2)$ as a map from $L^2(\mathbb{T})$ to $H^{-2}(\mathbb{T})$, we will use the following lemma.

LEMMA 4.6. *Let $u, v \in L^2(\mathbb{T})$. Then there exists a constant $C > 0$ such that*

$$||\partial_x(uv)||_{H^{-2}} \leq C||u||_{L^2}||v||_{L^2}. \tag{4.20}$$

Proof. Let $u, v \in C^\infty(\mathbb{T})$, we have

$$uv = \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} u_k v_{n-k} \right) e_n. \tag{4.21}$$

We define $\langle n \rangle = \sqrt{1 + |n|^2}$ for all $n \in \mathbb{Z}$. Then by definition of Sobolev norms we have

$$||\partial_x(uv)||^2_{H^{-2}} = \sum_{n\in\mathbb{Z}} n^2 \left| \sum_{k\in\mathbb{Z}} u_k v_{n-k} \right|^2 \langle n \rangle^{-4}. \tag{4.22}$$

Now, applying Cauchy-Schwarz inequality gives

$$||\partial_x(uv)||^2_{H^{-2}} \leq ||u||^2_{L^2}||v||^2_{L^2} \sum_{n\in\mathbb{Z}} \langle n \rangle^{-2}. \tag{4.23}$$

Which concludes the proof. □

Now, as $\mathcal{F}$ is quadratic, the previous lemma ensures us that $\mathcal{F}$ satisfies Assumption 1. Note that for all $\lambda > 0$, we have $m(\lambda) = 3$. Hence, applying Theorem 3.4 leads to the following immediate corollary.

COROLLARY 4.7. *Let $\lambda \in \mathbb{R}_{>0}$. Suppose that $(f_1, f_2, f_3) \in (H^{-2}(\mathbb{T}))^3$ is an F_λ-admissible control operator. Then there exist $K_1, K_2, K_3 \in \mathcal{L}(L^2(\mathbb{T}), \mathbb{C})$ such that for every $u_0 \in L^2(\mathbb{T})$, there is a $\tau > 0$ for which there is a unique maximal solution $u(\cdot) \in C^0([0, \tau); L^2(\mathbb{T}))$ to*

$$\begin{cases} \partial_t u + \Delta^2 u + \Delta u + \frac{1}{2}\partial_x(u^2) + f_1 K_1 u + f_2 K_2 u + f_3 K_3 u = 0, \\ u(0) = u_0. \end{cases} \tag{4.24}$$

Furthermore, there exist $C_\lambda > 0$ and $\delta > 0$ such that for all $u_0 \in L^2(\mathbb{T})$ with $||u_0||_{L^2} \leq \delta$, the solution exists for all time $t \geq 0$ and satisfies

$$\forall t \geq 0, \quad ||u(t)||_{L^2} \leq C_\lambda e^{-\frac{\lambda}{2}t}||u_0||_{L^2}. \tag{4.25}$$

We now provide a concrete application of the above corollary to demonstrate that the feedback derived from the parabolic F-equivalence is easily constructible.

Suppose we want to stabilize the system at a rate $\nu := 10$. Hence we select $\lambda = 20$, and then $N(\lambda) = 5$. For this example, we define

$$\forall x \in \mathbb{T}, \quad f_1(x) = \frac{1}{\sqrt{2\pi}}, \quad f_2(x) = \frac{1}{\sqrt{2\pi}}(e^{-ix} + e^{-2ix}), \quad f_3(x) = \frac{1}{\sqrt{2\pi}}(e^{ix} + e^{2ix}). \tag{4.26}$$

Thus, we have $f_1 = \tilde{e}_1,\ f_2 = \tilde{e}_2 + \tilde{e}_4,\ f_3 = \tilde{e}_3 + \tilde{e}_5 \in L_\lambda$, and $B = (f_1, f_2, f_3)$ is clearly λ-admissible.

To find our feedbacks, we only need to solve a finite-dimensional F-equivalence problem. Identifying $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_4, \tilde{e}_3, \tilde{e}_5)$ with the canonical basis of $\mathbb{C}^5$, we have

$$A_L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}. \tag{4.27}$$

Let $\mu \geq \lambda$. If we denote by $(\tilde{T}, \tilde{K})$ the solution of

$$\begin{cases} \tilde{T}(A_L + B\tilde{K}) = (A_L - \mu)\tilde{T}, \\ TB = B, \end{cases} \tag{4.28}$$

then since $TB = B$ is equivalent to $Tf_1 = f_1$, $Tf_2 = f_2$, and $Tf_3 = f_3$, we can decompose the problem into three subproblems. It is straightforward to solve these either manually or numerically, and we find that

$$\begin{aligned} \tilde{T} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{12} + 1 & -\frac{\mu}{12} & 0 & 0 \\ 0 & \frac{\mu}{12} & 1 - \frac{\mu}{12} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu}{12} + 1 & -\frac{\mu}{12} \\ 0 & 0 & 0 & \frac{\mu}{12} & 1 - \frac{\mu}{12} \end{pmatrix}, \\ \tilde{K} &= \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu(\mu+12)}{12} & \frac{\mu(\mu-12)}{12} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\mu(\mu+12)}{12} & \frac{\mu(\mu-12)}{12} \end{pmatrix}. \end{aligned} \tag{4.29}$$

By Theorem 3.2, we know that for almost every $\mu \geq \lambda$, we can use $\tilde{K}$ to define our feedbacks in Corollary 4.7. Then for all $f \in L^2(\mathbb{T})$, the above corollary applies with

$$\begin{aligned} K_1 f &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} -\mu f(x)\, dx, \\ K_2 f &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) \left(-\frac{\mu(\mu+12)}{12} e^{ix} + \frac{\mu(\mu-12)}{12} e^{2ix} \right) dx, \\ K_3 f &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) \left(-\frac{\mu(\mu+12)}{12} e^{-ix} + \frac{\mu(\mu-12)}{12} e^{-2ix} \right) dx. \end{aligned} \tag{4.30}$$

As we will now briefly show, our result can also be applied to the Kuramoto-Shivansky system studied in [17], as it was one of the first examples of Fredholm backstepping, that is

$$\begin{cases} \partial_t u + \Delta^2 u + \nu \Delta u + \frac{1}{2}\partial_x(u^2) = 0 \quad \text{in } (0, +\infty) \times (0,1), \\ u(t,0) = u(t,1) = 0, \quad \forall t > 0, \\ \Delta u(t,0) = w(t), \ \Delta u(t,1) = 0, \quad \forall t > 0, \\ u(0,\cdot) = u_0 \in L^2(0,1), \end{cases} \tag{4.31}$$

with $\nu > 0$. Thus, we work in $H = L^2(0,1)$, with $A = -\Delta^2 - \nu\Delta$ and

$$D(A) = \{u \in H^4(0,1) \mid u(0) = u(1) = \Delta u(0) = \Delta u(1) = 0\}.$$

Setting $e_n(x) = \sqrt{2}\sin(\pi n x)$ for all $x \in (0,1)$ and for $n \geq 1$, we observe that $(e_n)_{n\geq 1}$ forms an orthonormal basis of H consisting of eigenvectors of A, with eigenvalues

$$\forall n \geq 1, \quad \lambda_n = -\pi^4 n^4 + \nu \pi^2 n^2. \tag{4.32}$$

Thus, A is a self-adjoint diagonal parabolic operator on H. Multiplying (4.31) by a smooth function in $D(A)$ and integrating by parts, we obtain

$$\forall u \in D(A), \quad B^* u = -\partial_x u(0). \tag{4.33}$$

which defines, by duality, $B \in \mathcal{L}(\mathbb{R}, D(A)')$. Moreover, we have

$$\forall n \geq 1, \quad \langle B\,, e_n\rangle_{D(A)',D(A)} = -\pi n. \tag{4.34}$$

Following [17], we assume that

$$\nu \notin \{n^2\pi^2 + k^2\pi^2 \mid n, k \geq 1,\, n \neq k\}, \tag{4.35}$$

which ensures that A has only simple eigenvalues (thus, (A, B) is approximately controllable by Lemma 6.8). Consequently, (4.34) implies that B is F_λ-admissible for all $\lambda > 0$. The nonlinearity $\mathcal{F}$ can be handled similarly to the previous example. Therefore, applying Theorem 3.4, we obtain the following corollary.

COROLLARY 4.8. *Let $\lambda \in \mathbb{R}_{>0}$. There exists $K \in \mathcal{L}(L^2(0,1), \mathbb{R})$ such that for every $u_0 \in L^2(0,1)$, there exists $\tau > 0$ for which there is a unique maximal solution $u \in C^0([0,\tau); L^2(0,1))$ to*

$$\begin{cases} \partial_t u + \Delta^2 u + \nu\Delta u + \frac{1}{2}\partial_x(u^2) = 0 \quad \text{in } (0,\tau)\times(0,1), \\ u(t,0) = u(t,1) = 0, \quad \forall t \in (0,\tau), \\ \Delta u(t,0) = Ku(t,\cdot),\ \Delta u(t,1) = 0, \quad \forall t \in (0,\tau), \\ u(0,\cdot) = u_0 \in L^2(0,1). \end{cases} \tag{4.36}$$

Furthermore, there exist constants $C_\lambda > 0$, and $\delta > 0$ such that for all $u_0 \in L^2(0,1)$ with $||u_0||_{L^2} \leq \delta$, the solution exists for all $t \geq 0$ and satisfies

$$\forall t \geq 0, \quad ||u(t,\cdot)||_{L^2} \leq C_\lambda e^{-\frac{\lambda}{2}t}||u_0||_{L^2}. \tag{4.37}$$

REMARK 4.9. *As before, the feedback K is constructed by solving a finite-dimensional linear system (see Remark C2). Compared to* [17]*, this approach is significantly simpler. One could similarly use our approach for the Kuramoto-Shivashinski system found in* [37].

4.2.2. *Fischer-KPP equation.* The Fischer-KPP equation, introduced independently in 1937 by Ronald Fischer and by Andrey Kolmogorov, Ivan Petrovski, and Nikolai Piskunov, was developed to model biological processes, see [20, 27]. In particular it describes phenomena such as the spread of advantageous genes in a population. Here, we focus on the Fischer-KPP equation on compact oriented and connected d-dimensional riemannian manifold $\mathcal{M}$, which is given by

$$\partial_t u = \Delta u + u(1-u). \tag{4.38}$$

We follow the same approach as for the previous equation. Let us fix $s > d/2$, and work in $H = H^s(\mathcal{M})$ with $A = \Delta + I$. Thus, $D_r(A) = H^{s+2r}(\mathcal{M})$. Since we are using $H = H^s(\mathcal{M})$, we cannot directly work with $(e_n)_{n\geq 1}$ given in 4.1. Instead, we will use the following basis [8] of $H^s(\mathcal{M})$

$$\forall n \geq 1, \quad f_n = e_n n^{-\frac{s}{d}}. \tag{4.39}$$

[8] As discussed before, Weyl's law ensures that this is a Riesz basis of $H^s(\mathcal{M})$.

From Weyl's law there exists a constant $c_d > 0$ such that

$$\lambda_n \underset{n\to+\infty}{\sim} -c_d n^{\frac{2}{d}}. \tag{4.40}$$

Therefore, A is diagonal parabolic on H. Recall that by Sobolev embedding theorem $H^s(\mathcal{M})$ is a Banach algebra since $s > d/2$, which means that there exists $C > 0$ such that

$$\forall u, v \in H^s(\mathcal{M}), \quad ||uv||_{H^s} \leq C||u||_{H^s}||v||_{H^s}. \tag{4.41}$$

Hence the map $\mathcal{F} : H^s(\mathcal{M}) \to H^s(\mathcal{M})$ defined as

$$\forall u \in H^s(\mathcal{M}),\ \mathcal{F}(u) = -u^2, \tag{4.42}$$

is well-defined and satisfies Assumption 1. Hence using (4.3), we can apply Theorem 3.4 to get the following corollary.

COROLLARY 4.10. *Let $\lambda \in \mathbb{R}_{>0}$. Suppose that $B \in (H^{s-2}(\mathcal{M}))^{m(\lambda)}$ is an F_λ-admissible control operator. Then there exist $K \in \mathcal{L}(H^s(\mathcal{M}), \mathbb{R}^{m(\lambda)})$ such that for every $u_0 \in H^s(\mathcal{M})$, there is a $\tau > 0$ for which there is a unique maximal solution $u(\cdot) \in C^0([0,\tau]; H^s(\mathcal{M}))$ to*

$$\begin{cases} \partial_t u = \Delta u + u(1-u) + BKu, \\ u(0) = u_0. \end{cases} \tag{4.43}$$

Furthermore, there exists $C_\lambda > 0$ and $\delta > 0$ such that for all $u_0 \in H^s(\mathcal{M})$ with $||u_0||_{H^s} \leq \delta$, the solution exists for all time $t \geq 0$ and satisfies

$$\forall t \geq 0, \quad ||u(t)||_{H^s} \leq C_\lambda e^{-\frac{\lambda}{2}t}||u_0||_{H^s}. \tag{4.44}$$

Note that if $\dim \mathcal{M} = 1$, since $f_1 = e_1$, we can choose s close enough to $\frac{1}{2}$ such that the argument in the proof of Proposition 4.3 also applies here. Hence for $\lambda \in (0, -\lambda_2)$, $\delta_p \in H^{s-2}(\mathcal{M})$ is F_λ-admissible and then defining

$$\forall f \in H^s(\mathcal{M}), \quad Kf = -\lambda \int_{\mathcal{M}} f\, d\mu_g, \tag{4.45}$$

we can apply the above corollary with $B = \delta_p$ and this feedback operator K. In other words, one can stabilize the Fischer-KPP equation using a Dirac control and the above feedback.

5. Main proofs

5.1. **Existence of parabolic F-equivalence.** In this subsection we prove Theorem 3.2.

5.1.1. *Proof strategy.* Let A, λ, D, and B be as in Theorem 3.2. In particular, our frequency decomposition of spaces are always with respect to λ. Below, we briefly outline the proof steps for Theorem 3.2:

1. We begin by establishing necessary conditions on the form of (T, K) for it to be a parabolic F-equivalence of (A, B, D), as detailed in Proposition 5.2.
2. In Subsection 5.1.3, we first apply Lemma 2.5 using the partition induced by the F_λ-admissibility of B. Hence, for each $j \in \{1, \dots, m(\lambda)\}$, we have A_j, a diagonal parabolic operator on $\mathcal{H}^j$, with $B_j \in D(A_j)'$. Exploiting the fact that A_j has only simple eigenvalues (see, for instance, [17, (2.10)]) in L^j_λ, we show in Proposition 5.5 that for almost every $\mu \geq \lambda + c_A$ and for all $j \in \{1, \dots, m(\lambda)\}$, there exists a parabolic F-equivalence (T_j, K_j) between (A_j, B_j) and D_j, where

$$D_j = \begin{pmatrix} A_{j_L} - \mu & 0 \\ 0 & A_{j_H} \end{pmatrix}. \tag{5.1}$$

The core of the proof lies in this step, with the main technical challenge being to establish that T_j is an isomorphism. To achieve this, we make essential use of the polynomial properties of finite-dimensional F-equivalence feedback, see Theorem C.1. Note that D_j explicitly depends on μ, but for notational convenience, we do not indicate this dependence explicitly.

(3) Finally, in Subsection 5.1.4, we prove Theorem 3.2. To this end we set

$$T = T_1 + \cdots + T_{m(\lambda)}, \quad K = (K_1, \ldots, K_{m(\lambda)}). \tag{5.2}$$

Then we demonstrate that (T, K) indeed forms a parabolic F-equivalence between (A, B) and D.

5.1.2. *Necessary conditions on* (T, K). The goal of this subsection is to show some conditions that (T, K) should satisfy to be a parabolic F-equivalence between (A, B) and D (see Proposition 5.2). This will greatly help us understand the form of (T, K) in Subsection 5.1.3, and we will also reuse it in Section 6.

First, let us introduce the following notation: we denote by P_L and P_H the orthogonal projections on L_λ and H_λ in H, and by a slight abuse of notation, we use the same symbols for the orthogonal projections on L_λ and $D(A)'_\lambda$ in $D(A)'$.

Then for every $x \in D(A)'$, we have $x = x_L + x_H$ with $x_L = P_L x$ and $x_H = P_H x$. Now, for every normed vector space E, the previous decompositions give us the following decomposition on the space of bounded operators

$$\mathcal{L}(H, E) = \mathcal{L}(L_\lambda, E) \oplus \mathcal{L}(H_\lambda, E).$$

This is also true when replacing H with $D(A)'$. Notice that $A(L_\lambda) \subset L_\lambda$ and that $A(H_\lambda \cap D(A)) \subset H_\lambda$, so similarly we can define $A_L = AP_L$ and $A_H = AP_H$, and we have $A = A_L + A_H$.

We now decompose $\mathcal{L}(H, D(A)')$ in terms of frequency. Let $M \in \mathcal{L}(H, D(A)')$. We can write

$$M = P_L M P_L + P_L M P_H + P_H M P_L + P_H M P_H, \tag{5.3}$$

which allows us to define the direct sum corresponding to the above decomposition

$$\mathcal{L}(H, D(A)') = LL_\lambda(H, D(A)') \oplus HL_\lambda(H, D(A)') \oplus LH_\lambda(H, D(A)') \oplus HH_\lambda(H, D(A)'), \tag{5.4}$$

where, for instance,

$$HL_\lambda(H, D(A)') = \{M \in \mathcal{L}(H, D(A)') \mid M = P_L M P_H\}, \tag{5.5}$$

and $LL_\lambda(H, D(A)')$, $LH_\lambda(H, D(A)')$ and $HH_\lambda(H, D(A)')$ are defined accordingly. We could apply the same approach to $\mathcal{L}(H)$ or $\mathcal{L}(D(A)')$ and the subspaces defined earlier. In these cases, this decomposition allows us to use a matrix formalism. For example, if $M \in \mathcal{L}(H)$, we write

$$M = \begin{pmatrix} M_{LL} & M_{HL} \\ M_{LH} & M_{HH} \end{pmatrix}, \tag{5.6}$$

with

$$M_{LL} = P_L M P_L, \quad M_{HL} = P_L M P_H, \quad M_{LH} = P_H M P_L, \quad M_{HH} = P_H M P_H,$$

and we verify that the usual matrix multiplication rules apply. For instance, for $x \in H$, we write $x = \begin{pmatrix} x_L \\ x_H \end{pmatrix}$, and we have

$$Mx = M_{LL}x_L + M_{HL}x_H + M_{LH}x_L + M_{HL}x_H = \begin{pmatrix} M_{LL} & M_{HL} \\ M_{LH} & M_{HH} \end{pmatrix} \begin{pmatrix} x_L \\ x_H \end{pmatrix}.$$

Notice that we can define $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$ as in Proposition 2.4. The algebra defined in the proposition below arises naturally in parabolic F-equivalence, as we will see in Sections 5 and 6.

PROPOSITION 5.1. *We define the commutant algebra of* A_H *as*

$$C(A_H) = \{M \in \mathcal{L}_{H_\lambda}(D(A)'_\lambda) \mid A_H M = M A_H \text{ in } \mathcal{L}(H_\lambda, D(A)'_\lambda)\}. \tag{5.7}$$

It is a sub-Banach algebra of $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$.

Proof. It's clear that $C(A_H)$ is a subalgebra of $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$. We just need to show that it is closed. Let $(M_n)_{n\geq 1}$ be a sequence in $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$ such that $M_n \to M$ in $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$. Recall that the norm of $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$ is equivalent to the following

$$\forall M \in \mathcal{L}_{H_\lambda}(D(A)'_\lambda), \ |||M||| := \|M\|_{\mathcal{L}(D(A)'_\lambda)} + \|M_{|H_\lambda}\|_{\mathcal{L}(H_\lambda)}. \tag{5.8}$$

We have

$$\|MA_H - A_H M\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} \leq \|MA_H - M_n A_H\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} + \|M_n A_H - A_H M\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)},$$

for the first them notice that

$$\|MA_H - M_n A_H\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} \leq \|A_H\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} \|M - M_n\|_{\mathcal{L}(D(A)'_\lambda)}.$$

Now using that $M_n A_H = A_H M_n$, we get

$$\|M_n A_H - A_H M\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} \leq \|A_H\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} \|M - M_n\|_{\mathcal{L}(H_\lambda)}.$$

Then passing to the limit, we finally get $\|MA_H - A_H M\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} = 0$, and hence $M \in C(A_H)$. □

We can now show the following necessary conditions on (T, K):

PROPOSITION 5.2. *Let* $\mu \in \mathbb{R}_{>0} \setminus \{\lambda_l - \lambda_h\}_{h \geq l \geq 1}$*, we set*

$$D_\mu = \begin{pmatrix} A_L - \mu & 0 \\ 0 & A_H \end{pmatrix}.$$

Let (T, K) *be a parabolic* F*-equivalence between* (A, B) *and* D *(hence* $K \in \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$*). Then, if we denote by* $(\tilde{T}, \tilde{K})$ *the unique finite-dimensional* F*-equivalence between* (A_L, B_L) *and* $A_L - \mu$ *given by Theorem C.1,*[9] *we have*

$$T = \begin{pmatrix} \tilde{T} & 0 \\ \tau & C \end{pmatrix}, \quad K = \tilde{K}, \tag{5.9}$$

with $C \in C(A_H)$ *and* $\tau \in LH_\lambda(D(A)')$ *and is defined as*

$$\forall n \in \{1, \ldots, N(\lambda)\}, \ \tau(e_n) = \sum_{k \geq 1} \frac{\langle B_H K e_n \,, \sqrt{1 + |\lambda_k|^2} e_k \rangle_{D(A)'}}{\lambda_k - \lambda_n} \sqrt{1 + |\lambda_k|^2} e_k. \tag{5.10}$$

[9]We can apply this theorem using the isomorphism between $\mathbb{C}^{N(\lambda)}$ and L_λ, which sends the canonical basis to $(e_n)_{1 \leq n \leq N(\lambda)}$.

Furthermore, we have $\tau(L_\lambda) \subset H_\lambda$.

Proof. Let $n > N(\lambda)$. Then $Ke_n = 0$, and by F-equivalence, we have

$$T(A+BK)e_n = \lambda_n T e_n = DTe_n. \tag{5.11}$$

Hence, Te_n is an eigenvector of D associated to the eigenvalue λ_n. Note that $(e_j)_{j\geq 1}$ form a basis of eigenvectors of D, from its definition. Now, because $\mu \notin \{\lambda_l - \lambda_h\}_{h\geq l\geq 1}$, we have $Te_n \in \operatorname{span}(e_k)_{k>N(\lambda)}$. Otherwise, it should exists $l \leq N(\lambda)$ such that $\langle Te_n\,, e_l\rangle_H \neq 0$, but by (5.11) we would have

$$\lambda_n \langle Te_n\,, e_l\rangle_H = (\lambda_l - \mu)\langle Te_n\,, e_l\rangle_H,$$

which would be a contradiction. Thus, we have

$$\forall v \in H_\lambda,\ TA_H v = A_H T v. \tag{5.12}$$

Therefore, $T_{HL} = 0$ and $T_{HH} \in C(A_H)$.

Now let $n \leq N(\lambda)$. Applying this to the F-equivalence equation and using $TB = B$ gives us

$$\lambda_n Te_n + B_L K e_n + B_H K e_n = DTe_n. \tag{5.13}$$

Projecting this on L_λ and $D(A)'_\lambda$ gives us

$$\begin{cases} \lambda_n (Te_n)_L + B_L K e_n = (A_L - \mu)(Te_n)_L, \\ \lambda_n (Te_n)_H + B_H K e_n = A_H (Te_n)_H. \end{cases} \tag{5.14}$$

Now, as $T_{HL} = 0$, we have $T_{LL} B_L = B_L$ and hence the first equation above is equivalent to

$$T_{LL}(A_L + B_L K) = (A_L - \mu) T_{LL}. \tag{5.15}$$

Then, identifying L_λ with $\mathbb{C}^{N(\lambda)}$ using the isomorphism that sends the canonical basis to $(e_n)_{1\leq n\leq N(\lambda)}$, we have, from Theorem C.1, $T_{LL} = \tilde{T}$ and $K = \tilde{K}$. Now let's use the second equality in (5.14). We set $h_n := (Te_n)_H$. Note that, given the choice of n, $h_n = T_{LH} e_n$. For $k > N(\lambda)$, we have

$$\lambda_n \langle h_n\,, e_k\rangle_{D(A)'} + \langle B_H K e_n\,, e_k\rangle_{D(A)'} = \langle A_H h_n\,, e_k\rangle_{D(A)'}. \tag{5.16}$$

Using $A^* e_k = \overline{\lambda_k} e_k$, we get

$$\langle h_n\,, e_k\rangle_{D(A)'} = \frac{\langle B_H K e_n\,, e_k\rangle_{D(A)'}}{\lambda_k - \lambda_n}. \tag{5.17}$$

Notice that by definition of $N(\lambda)$, we have $\operatorname{Re}(\lambda_{N(\lambda)+1} - \lambda_{N(\lambda)}) \neq 0$, hence the above expression is well-defined (recall that $k > N(\lambda) \geq n$). Recall that $(\sqrt{1+|\lambda_m|^2} e_m)_{m\geq 1}$ is an orthonormal basis of $D(A)'$, and so

$$\tau(e_n) := h_n = \sum_{k\geq 1} \frac{\langle B_H K e_n\,, \sqrt{1+|\lambda_k|^2} e_k\rangle_{D(A)'}}{\lambda_k - \lambda_n} \sqrt{1+|\lambda_k|^2} e_k.$$

Now, to prove that $\tau(L_\lambda) \subset H$, it suffices to show that $\tau(e_n) \in H$ for each $n \in \{1, \ldots, N(\lambda)\}$.

Let $k \geq 1$. We set $I_k := \langle B_H K e_n\,, \sqrt{1+|\lambda_k|^2} e_k\rangle_{D(A)'}$. This forms an ℓ^2 sequence since $B_H \in D(A)'$, and we have

$$\frac{|I_k|}{|\lambda_k - \lambda_n|} \sqrt{1+|\lambda_k|^2} \underset{k\to+\infty}{\sim} |I_k|. \tag{5.18}$$

Thus $\tau(e_n) \in H$.

□

REMARK 5.3. *Here we emphasize that* $T_{HL} = 0$ *reflects the internal structure of the equation*

$$\partial_t u = Au + BKu, \tag{5.19}$$

with $K \in \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$. *Notice that because* $Kx = Kx_L$, *this equation can be decoupled as follows:*

$$\partial_t u = Au + BKu \iff \begin{cases} \partial_t u_L = A_L u_L + B_L K u_L, \\ \partial_t u_H = A_H u_H + B_H K u_L. \end{cases} \tag{5.20}$$

The first evolution equation on u_L *in the system is independent of* u_H *and can be solved on its own. This fact is then reflected by* $T_{HL} = 0$ *in the* F*-equivalence. Similarly, one can observe that* $T_{LH} \neq 0$ *if* $B_H \neq 0$, *as the low-frequency part* u_L *influences the evolution of* u_H.

REMARK 5.4. *Notice that* K *is entirely determined as soon as* A, B *and* D *are given.* T *is not, however, and the only free component of* T *is* C. *As we will discuss in Section 6, this is what prevents achieving uniqueness.*

5.1.3. *Simple multiplicity case.* In this section, we aim to establish the following proposition.

PROPOSITION 5.5. *For almost every* $\mu \geq \lambda + c_A$ *and for all* $j \in \{1, \dots, m(\lambda)\}$, *there exists a parabolic* F*-equivalence* (T_j, K_j) *between* (A_j, B_j) *and the* λ*-target* D_j.

To clarify, here we work with the Gelfand triple $(D(A_j), \mathcal{H}^j, D(A_j)')$, and we redefine accordingly all the necessary operator spaces. Then $T_j \in \mathcal{GL}_{\mathcal{H}^j}(D(A_j)')$, $K_j \in \mathcal{L}(L^j_\lambda, \mathbb{C})$, and equation 3.1 becomes

$$\begin{cases} T_j(A_j + B_j K_j) = D_j T_j \text{ in } \mathcal{L}(\mathcal{H}^j, D(A_j)') \\ T_j B_j = B_j \text{ in } D(A_j)'. \end{cases} \tag{5.21}$$

Notice that Proposition 5.2 also applies here for (A_j, B_j, D_j), and we denote by $\tilde{T}_j$, τ_j, and K_j the operators in (5.9). For the proof of Proposition 5.5, we will need the following lemma.

LEMMA 5.6. *Let* $\mu \in \mathbb{R}_{>0} \setminus \{\lambda_l - \lambda_h\}_{h \geq l \geq 1}$. *There exists a parabolic* F*-equivalence of* (A_j, B_j, D_j) *if and only if there exists* $C_j \in C(A_{jH}) \cap \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$ *such that*

$$\tau_j B_{jL} + C_j B_{jH} = B_{jH}. \tag{5.22}$$

Proof. Let (T_j, K_j) be a parabolic F-equivalence between (A_j, B_j, D_j). By definition, we have $T_j B_j = B_j$. Using Proposition 5.2 for (A_j, B_j, D_j), we obtain:

$$T_j B_j = B_j \iff \begin{pmatrix} \tilde{T}_j B_{jL} = B_{jL} \\ \tau_j B_{jL} + C_j B_{jH} = B_{jH} \end{pmatrix}. \tag{5.23}$$

Proposition 5.2 ensures that $C_j \in C(A_{jH})$, and we have (recall that $D(A_j)'_\lambda$ correspond to the projection on the high frequencies):

$$\forall v \in D(A_j)'_\lambda, \quad T_j v = C_j v. \tag{5.24}$$

Since $T_j \in \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$, the equality above implies $C_j \in \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$.

Conversely, suppose that there exists $C_j \in C({A_j}_H) \cap \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$ such that $\tau_j {B_j}_L + C_j {B_j}_H = {B_j}_H$ holds. Then, we can define T_j and K_j as in Proposition 5.2. By Theorem C.1, we have that $\tilde{T}_j {B_j}_L = {B_j}_L$, so by Equation (5.23), we have $T_j B_j = B_j$.

Next, let's verify that $T_j \in \mathcal{GL}_{\mathcal{H}^j}(D(A_j)')$. Define

$$R_j = \begin{pmatrix} \tilde{T}_j^{-1} & 0 \\ -C_j^{-1}\tau_j \tilde{T}_j^{-1} & C_j^{-1} \end{pmatrix}. \tag{5.25}$$

Noticing that $-C_j^{-1}\tau_j\tilde{T}_j^{-1} \in \mathcal{L}(L^j_\lambda, H_\lambda)$, and using the fact that $C_j \in \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$, we have $R_j \in \mathcal{L}_{\mathcal{H}^j}(D(A_j)')$. Then $T_j \in \mathcal{GL}_{\mathcal{H}^j}(D(A_j)')$ immediately follows from the relation

$$T_j R_j = R_j T_j = \mathrm{Id}_{D(A_j)'}. \tag{5.26}$$

We now need to demonstrate that the first equation in (5.21) holds. Let $n > N_j(\lambda)$, then we have

$$\begin{aligned} T_j(A_j + B_j K_j)e^j_n &= T_j A_j e^j_n \\ &= C_j A_j e^j_n \\ &= A_j C_j e^j_n \\ &= D_j T_j e^j_n. \end{aligned} \tag{5.27}$$

Next, consider the case where $n \leq N_j(\lambda)$. Using $T_j B_j = B_j$, we have

$$\begin{aligned} T_j(A_j + B_j K_j)e^j_n &= \lambda^j_n T_j e^j_n + {B_j}_L K_j e^j_n + {B_j}_H K_j e^j_n \\ &= \underbrace{(\tilde{T}_j {A_j}_L + {B_j}_L K_j)e^j_n}_{\in L^j_\lambda} + \underbrace{(\lambda^j_n \tau_j + {B_j}_H K_j)e^j_n}_{\in D(A_j)'_\lambda}. \end{aligned} \tag{5.28}$$

By Theorem C.1, we have

$$(\tilde{T}_j {A_j}_L + {B_j}_L K_j)e^j_n = ({A_j}_L - \lambda^j_n)\tilde{T}_j e^j_n. \tag{5.29}$$

Then, by the definition of τ_j, and setting $f^j_k := \sqrt{1 + |\lambda^j_k|^2} e^j_k$, we have

$$(\lambda^j_n \tau_j + {B_j}_H K_j)e^j_n = \sum_{k \geq 1} \lambda^j_k \frac{\langle {B_j}_H K_j e^j_n, f^j_k \rangle_{D(A_j)'}}{\lambda^j_k - \lambda^j_n} f^j_k = {A_j}_H \tau_j e^j_n. \tag{5.30}$$

Thus

$$T_j(A_j + B_j K_j)e^j_n = ({A_j}_L - \lambda^j_n)\tilde{T}_j e^j_n + {A_j}_H \tau_j e^j_n = D_j T_j e^j_n. \tag{5.31}$$

Finally, by continuity and linearity, we have

$$\forall x \in \mathcal{H}^j, \quad T_j(A_j + B_j K_j)x = D_j T_j x. \tag{5.32}$$

□

The preceding lemma is very useful for constructing parabolic F-equivalences, as it shows that we only need to construct C_j. Notice that

$$\forall M \in \mathcal{L}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda), \quad M \in C({A_j}_H) \iff \forall n > N_j(\lambda),\ C_j e^j_n \in \ker({A_j}_H - \lambda^j_n). \tag{5.33}$$

Hence, it is natural to try C_j as a diagonal operator, which means

$$\forall n > N_j(\lambda),\ C_j e^j_n = c^j_n e^j_n. \tag{5.34}$$

Then the condition $C_j \in \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$ simply becomes

$$\exists c_1^j, c_2^j > 0, \text{ such that } c_1^j \le |c_n^j| \le c_2^j. \tag{5.35}$$

Note, however, that it is not clear yet that there would exists a parabolic F-equivalence of (A_j, B_j, D_j) with such a diagonal C_j since it also have to satisfy (5.22). In fact, we are going to show that not only it is possible to have a parabolic F-equivalence of (A_j, B_j, D_j) with C_j, but in addition it is nearly always possible even without additional assumption of B_H. More precisely, we define Λ_j to be the set of all $\mu > 0$ for which there exists a parabolic F-equivalence (T_j, K_j) of (A_j, B_j, D_j)[10] with C_j being diagonal. The following proposition describes important topological properties of Λ_j.

PROPOSITION 5.7. *Λ_j is open and dense in $\mathbb{R}_{>0} \setminus \{\lambda_l - \lambda_h\}_{h \ge l \ge 1}$, and $\mathbb{R}_{>0} \setminus \Lambda_j$ is negligible.*

Proof. First, we begin by demonstrating the openness of Λ_j. To avoid confusion, we will explicitly indicate every dependency on μ in our notation throughout this proof. Let $\mu \in \Lambda_j$, then there exists a parabolic F-equivalence (T_j^μ, K_j^μ) of (A_j, B_j, D_j^μ). Let $k > N_j(\lambda)$, and we set

$$f_k^j = \sqrt{1 + |\lambda_k^j|}e_k^j, \quad b_k^j := \langle B_j \,, f_k^j \rangle_{D(A_j)'}, \quad K_n^j := K_j e_n^j. \tag{5.36}$$

Then, projecting (5.22) onto e_k, we obtain

$$(1 - c_k^{\mu,j})b_k^j = \sum_{n=1}^{N_j(\lambda)} b_n^j \langle \tau_j^\mu(e_n) \,, f_k^j \rangle_{D(A_j)'} = b_k^j \sum_{n=1}^{N_j(\lambda)} \frac{b_n^j K_n^{\mu,j}}{\lambda_k^j - \lambda_n^j}. \tag{5.37}$$

Hence, if $b_k^j \neq 0$, this imposes

$$c_k^{\mu,j} = 1 - \sum_{n=1}^{N_j(\lambda)} \frac{b_n^j K_n^{\mu,j}}{\lambda_k^j - \lambda_n^j}. \tag{5.38}$$

Now, let $\delta \in \mathbb{R}$ such that $\mu + \delta \in \mathbb{R}_{>0} \setminus \{\lambda_l - \lambda_h\}_{h \ge l \ge 1}$. We will show that if $|\delta|$ is small enough, then $\mu + \delta \in \Lambda_j$, thereby proving that Λ_j is an open set of $\mathbb{R}_{>0} \setminus \{\lambda_l - \lambda_h\}_{h \ge l \ge 1}$.

Let $k > N_j(\lambda)$, and define $C_j^{\mu+\delta}$ as follows:

- If $b_k^j = 0$, then set $c_k^{\mu+\delta,j} = 1$.
- Otherwise, set $c_k^{\mu+\delta,j} = 1 - \sum_{n=1}^{N_j(\lambda)} \frac{b_n^j K_n^{\mu+\delta,j}}{\lambda_k^j - \lambda_n^j}$.

Now, by Lemma 5.6, we only need to check that (5.35) holds for $(c_k^{\mu+\delta,j})_{k>N(\lambda)}$. Notice that the above expressions imply, under Hypothesis (A2), that (note that the number of terms in the sum is finite and does not depend on k)

$$\lim_{k \to +\infty} c_k^{\mu+\delta,j} = 1. \tag{5.39}$$

Hence, the sequence is bounded from above. Since $\mu \in \Lambda_j$, then (5.35) holds for $(c_k^{\mu,j})_{k \ge N(\lambda)}$. Now observe that, from Theorem C.1, the $K_n^{\mu+\delta,j}$ are continuous in δ. Also notice that the

[10] Recall that D_j depends on μ.

number of terms in the sum defining $c_k^{\mu+\delta,j}$ is finite and independent of k, and that we have (as $(\mathrm{Re}(\lambda_m^j))_{m\geq 1}$ is non-increasing)

$$\forall k > N_j(\lambda),\ \forall n \leq N_j(\lambda),\ |\lambda_k^j - \lambda_n^j| \geq \mathrm{Re}(\lambda^j_{N_j(\lambda)+1} - \lambda^j_{N_j(\lambda)}) > 0. \tag{5.40}$$

Thus, we deduce that the $c_k^{\mu+\delta,j}$ are continuous in δ, uniformly in k. Therefore, using (5.35) with $(c_k^{\mu,j})_{k\geq N(\lambda)}$ if $|\delta|$ is small enough, we obtain that there exists $c_j > 0$ such that

$$|c_k^{\mu+\delta,j}| > c_j,\ \ \forall k > N(\lambda). \tag{5.41}$$

which shows that $\mu + \delta \in \Lambda_j$.

Now we prove that $\mathbb{R}_{>0}\setminus\Lambda_j$ is discrete in $\mathbb{R}_{>0}$, hence countable, and this will demonstrate the last two assertions of Proposition 5.7. We proceed by contradiction, suppose there exists $\mu_\infty \in \mathbb{R}_{>0} \setminus \Lambda_j$ such that there exists a injective sequence $(\mu_m)_{m\geq 1}$ with $\mu_m \in \mathbb{R}_{>0} \setminus \Lambda_j$ and $\mu_m \to \mu_\infty$ as $m \to \infty$. By the previous discussion and Lemma 5.6, for each $m \geq 1$, there exists $k_m > N(\lambda)$ such that

$$c_{k_m}^{\mu_m,j} = 1 - \sum_{n=1}^{N_j(\lambda)} \frac{b_n^j K_n^{\mu_m,j}}{\lambda_{k_m}^j - \lambda_n^j} = 0. \tag{5.42}$$

Recall that, from (5.39), there exists k_0 large enough such that $c_k^{\mu_\infty,j} \geq 1/2$ for all $k > k_0$, and by the previous discussion, $c_k^{\mu_m,j}$ converge to $c_k^{\mu_\infty,j}$ uniformly in k, when $m \to +\infty$. Therefore, there exists m_1 such that

$$\forall m \geq m_1, \forall k > k_0, c_k^{\mu_m,j} \neq 0. \tag{5.43}$$

Therefore for $m \geq m_1$ there can only be a finite number of k where (5.42) holds. Hence we can find $k_0 > N(\lambda)$ and extract a subsequence ψ such that $k_{\psi_m} = k_0$, namely

$$\forall m \geq 1,\ c_{k_0}^{\mu_{\psi(m)},j} = 0. \tag{5.44}$$

However, by Theorem C.1, we know that $K_n^{\mu,j}$, is a polynomial in μ without constant term. Hence $c_{k_0}^{\mu,j}$ must be a non zero polynomial in μ, but the isolated zero theorem and (5.44) imply

$$\forall \mu \in \mathbb{R},\ c_{k_0}^{\mu,j} = 0, \tag{5.45}$$

which is a contradiction.

□

Finally, we can prove Proposition 5.5.

Proof of Proposition 5.5. We set $\Lambda = \bigcap_{j\in\{1,\dots,m(\lambda)\}} \Lambda_j$. Proposition 5.7 ensures us that Λ is full measure and dense in $\mathbb{R}_{>0}$, so let $\mu > \lambda + c_A$ within Λ, then for all $j \in \{1,\dots,m(\lambda)\}$ there exists a parabolic F-equivalence of (A_j, B_j, D_j). Now as $\mu \geq \lambda + c_A$, we have

$$\forall \nu \in \sigma(D_j),\ \nu \leq -\lambda.$$

This implies that D_j is a λ-target. This concludes the proof of Proposition 5.5. □

5.1.4. *Last step.* We now finalize the proof of our main result. By Proposition 5.5, for almost every $\mu \geq \lambda + c_A$ and for all $j \in \{1, \dots, m(\lambda)\}$, there exists a parabolic F-equivalence (T_j, K_j) between (A_j, B_j) and D_j. Hence, by Lemma D.1, if we set

$$T = T_1 + \cdots + T_{m(\lambda)}, \quad K = (K_1, \dots, K_{m(\lambda)}), \tag{5.46}$$

we have $T \in \mathcal{GL}_H(D(A)')$ and $K \in \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$.

Again, by Lemma D.1, notice that

$$D = D_1 + \cdots + D_{m(\lambda)}, \tag{5.47}$$

and because $\mu \geq \lambda + c_A$, we know that D is a λ-target. Finally, let $x \in H$. We have $x = x_1 + \cdots + x_{m(\lambda)}$ with $x_j \in \mathcal{H}^j$. By the definition of (T_j, K_j), we have $TB = B$ in $D(A)'$, and

$$T(A + BK)x = (TA + BK)x = \sum_{j=1}^{m(\lambda)} (T_j A_j + B_j K_j) x_j = \sum_{j=1}^{m(\lambda)} D_j T_j x_j = DTx. \tag{5.48}$$

This concludes the proof of Theorem 3.2.

5.2. **Stabilization of nonlinear systems.** In this subsection we prove Theorem 3.4. To do this we fix $\lambda > 0$ and apply our main F-equivalence result, then we set $G = T \circ \mathcal{F} \circ T^{-1}$ and we work with the transformed system

$$\begin{cases} \partial_t v = Dv + G(v), \\ v(0) = v_0 \in H. \end{cases} \tag{5.49}$$

For simplicity, in the whole proof we assume that $0 \in \rho(A)$, thus we use the following equivalent norm

$$\forall s \in \mathbb{R},\ \forall u \in D_s(A),\ ||u||^2_{D_s(A)} := \sum_{n \geq 1} |\lambda_n|^{2s} |u_n|^2. \tag{5.50}$$

Notice that since $T \in \mathcal{GL}_H(D(A)')$, we can show that for every $s \in [0, 1]$ we have $T \in \mathcal{GL}(D_{-s}(A))$, therefore G satisfies Assumption 1. We denote by $(d_n)_{n \geq 1}$ the eigenvalues of D. Let $s \in \mathbb{R}$, it is clear that $D_s(A) = D_s(D)$, and that there exits $c_\lambda > 0$ such that

$$\forall v \in D(A),\ {c_\lambda}^{-1} ||v||_{D_s(A)} \leq ||v||_{D_s(D)} \leq c_\lambda ||v||_{D_s(A)}. \tag{5.51}$$

We start by showing that (5.49) is locally well-posed. More precisely we want to show the next proposition.

PROPOSITION 5.8. *Let $v_0 \in H$, there exists a unique maximal solution $v \in C^0([0, \tau_v), H) \cap L^2((0, \tau_v); D_{1/2}(A))$ to (5.49), with $\tau_v \in (0, +\infty]$.*

The fact that $v \in L^2((0, \tau); D_{1/2}(A))$ is classical for parabolic systems and will be crucial to show the stability of solutions with small initial data. In order to prove this proposition we will need two lemmas.

The following result ensures uniqueness and will allow us to define maximal solutions.

LEMMA 5.9. *Let $\tau > 0$ and $v, w \in C^0([0, \tau); H)$ be solutions of (5.49), if $v(0) = w(0)$ then $v = w$.*

Proof. Let $t > 0$, as D is diagonal parabolic $e^{tD}(D_{-1/2}(A)) \subset H$. We first show that there exists $K > 0$ such that

$$||e^{tD}||_{\mathcal{L}(D_{-1/2}(A),H)} \leq \frac{K}{\sqrt{t}}. \tag{5.52}$$

By hypothesis (A3) there exists $c > 0$ such that $c|d_n| \leq |\operatorname{Re} d_n|$ holds for all $n \geq 1$. Let $x \in D_{-1/2}(D)$, we have

$$||e^{tD}x||_H^2 \leq \sum_{n\geq 1} e^{-2tc|d_n|}|x_n|^2 \leq \sum_{n\geq 1} |d_n|e^{-2tc|d_n|}|d_n|^{-1}|x_n|^2. \tag{5.53}$$

As the maximum of $y \mapsto ye^{-2tcy}$ on $\mathbb{R}_+$ is $\frac{e^{-1}}{2tc}$ and using (5.51) we get that there exists $K > 0$ such that

$$||e^{tD}x||_H^2 \leq \frac{K^2}{t}||x||^2_{D_{-1/2}(A)}. \tag{5.54}$$

This gives the desired estimate. Now let $\tau > 0$ and $v, w \in C^0([0, \tau); H)$ as in the statement. Let $\tau_0 < \tau$, by continuity $(v - w)([0, \tau_0])$ lies in some bounded set, we denote by L the Lipchitz constant of G on this bounded set. Now as $v(0) = w(0)$ we have for all $t \in [0, \tau_0]$

$$||v(t) - w(t)||_H = ||\int_0^t e^{(t-s)D}(G(v(s)) - G(w(s)))\, ds||_H \tag{5.55}$$

$$\leq \int_0^t K(t-s)^{-\frac{1}{2}}||G(v(s)) - G(w(s))||_{D_{-1/2}(A)}\, ds \tag{5.56}$$

$$\leq \int_0^t KL(t-s)^{-\frac{1}{2}}||v(s) - w(s)||_H\, ds. \tag{5.57}$$

Now, a variant of Gronwall lemma for parabolic equations (see [25, Chapter 1]) ensure us that $||v(t) - w(t)||_H = 0$, which concludes. ☐

Now for the existence part, we use a fixed point approach. To this end we will need results on the following inhomogeneous Cauchy problem

$$\begin{cases} \partial_t v = Dv + g(t), \ \forall t \in (0, \tau), \\ v(0) = v_0 \in H, \end{cases} \tag{5.58}$$

where $g \in L^2((0, \tau); D_{-1/2}(A))$.

LEMMA 5.10. *Let $\tau > 0$ and $g \in L^2((0, \tau); D_{-1/2}(A))$, for every $v_0 \in H$, there exists a unique solution $v \in C^0([0, \tau]; H) \cap L^2((0, \tau); D_{1/2}(A))$ to (5.58). Moreover, there exists $C \geq 1$ independent of v_0, v, g and τ such that*

$$||v||_{C^0([0,\tau];H)} + ||v||_{L^2((0,\tau);D_{1/2}(A))} \leq C(||v_0||_H + ||g||_{L^2((0,\tau);D_{-1/2}(A))}). \tag{5.59}$$

REMARK 5.11. *In the special case where A is a self-adjoint operator, this result can be found in* [8, Chapter 10] *and is attributed to Lions.*

Proof. We set $X_\tau = C^0([0, \tau]; H) \cap L^2((0, \tau); D_{1/2}(A))$. We want to define v as

$$v(t) = e^{tD}v_0 + \int_0^t e^{(t-s)D}g(s)\, ds. \tag{5.60}$$

Therefore, we need to check that $e^{(\cdot)D}v_0 \in L^2((0,\tau), D_{1/2}(A))$ (semigroup properties ensures that $e^{(\cdot)D}v_0 \in C^0([0,\tau];H)$) and that the integral part is in X_τ. At the same time, we show the different estimates.

For all $n \geq 1$ we have $\mathrm{Re}(d_n) \leq -\lambda$, hence $||e^{tD}v_0||_H \leq ||v_0||_H$ for all $t \in [0,\tau]$. We set $v_n := \langle v_0 \, , e_n \rangle$, then we have

$$\int_0^\tau ||e^{tD}v_0||^2_{D_{1/2}(A)}\, dt = \sum_{n\geq 1} \int_0^\tau e^{-2t|\operatorname{Re} d_n|}|v_n|^2|d_n|\, dt. \tag{5.61}$$

Now by hypothesis (A3) there exists $c > 0$ such that $c|d_n| \leq |\operatorname{Re} d_n|$ holds for all $n \geq 1$. Thus $e^{-2t|\operatorname{Re} d_n|} \leq e^{-2tc|d_n|}$ and after integration we get

$$\int_0^\tau ||e^{tD}v_0||^2_{D_{1/2}(A)}\, dt \leq \frac{1}{2c}\sum_{n\geq 1}(1 - e^{-2\tau c|d_n|})|v_n|^2\, dt \leq \frac{1}{2c}||v_0||^2_H. \tag{5.62}$$

Now we show that $I : t \mapsto \int_0^t e^{(t-s)D}g(s)\, ds$ is in $C^0([0,\tau];H)$. Let $t \in [0,\tau]$, we write $g(t) = \sum_{n\geq 1} g_n(t)e_n$, and the hypothesis on g ensures that

$$||g||^2_{L^2((0,\tau);D_{-1/2}(A))} = \sum_{n\geq 1}\int_0^\tau |d_n|^{-1}|g_n(t)|^2\, dt < \infty. \tag{5.63}$$

On the other hand, we have

$$I(t) = \sum_{n\geq 1}\int_0^t e^{(t-s)d_n}g_n(s)\, ds\, e_n. \tag{5.64}$$

By Cauchy-Schwarz inequality we get

$$\left|\int_0^t e^{(t-s)d_n}g_n(s)\, ds\right|^2 \leq \frac{1}{2c|d_n|}(1 - e^{-2c|d_n|t})\int_0^\tau |g_n(s)|^2\, ds. \tag{5.65}$$

Hence we get the continuity using Lemma A.1, and we have by the previous inequality

$$||I(t)||^2_H \leq \frac{1}{2c}||g||^2_{L^2((0,\tau);D_{-1/2}(A))}. \tag{5.66}$$

Now using (5.64) we have

$$||I||^2_{L^2((0,\tau);D_{1/2}(A))} = \sum_{n\geq 1}\int_0^\tau |d_n|\left|\int_0^t e^{(t-s)d_n}g_n(s)\, ds\right|^2 dt \tag{5.67}$$

$$\leq \sum_{n\geq 1}|d_n|\, ||\mathbf{1}_{[0,\tau]}e^{(\cdot)d_n} * \mathbf{1}_{[0,\tau]}g_n(\cdot)||^2_{L^2(\mathbb{R})}. \tag{5.68}$$

Hence applying Young's convolution inequality gives

$$||I||^2_{L^2((0,\tau);D_{1/2}(A))} \leq \sum_{n\geq 1}\frac{1}{c^2|d_n|}(1 - e^{-c\tau|d_n|})^2\int_0^\tau |g_n(s)|^2\, ds \tag{5.69}$$

$$\leq \frac{1}{c^2}||g||^2_{L^2((0,\tau);D_{-1/2}(A))}, \tag{5.70}$$

Which concludes the proof. □

Now, we can use a fixed point argument to prove Proposition 5.8.

Proof of Proposition 5.8. Let $v_0 \in H$. It suffices to demonstrate the existence of a solution to (5.49). Once this is established, the maximal solution with the initial value v_0 can be defined using Lemma 5.9.

Let $\tau > 0$ to be chosen later on. As before, we define $X_\tau = C^0([0,\tau]; H) \cap L^2((0,\tau); D_{1/2}(A))$, and it defines a Banach space with the following norm

$$\forall v \in X_\tau, \ ||v||_{X_\tau} := ||v||_{C^0([0,\tau];H)} + ||v||_{L^2((0,\tau);D_{1/2}(A))}. \tag{5.71}$$

Wet set $\delta = 2C||v_0||_H$ where C is the constant of Lemma 5.10 and we define the following closed subset

$$B_\tau(\delta) = \{v \in X_\tau \mid ||v||_{X_\tau} \leq \delta\}. \tag{5.72}$$

We denote by L the Lipschitz constant of G on the closed ball of radius δ of H. Let us define the mapping M which, to each $v \in B_\tau(\delta)$, associates w, the solution to

$$\begin{cases} \partial_t w = Dw + G(v(t)), \ \forall t \in (0,\tau), \\ w(0) = v_0. \end{cases} \tag{5.73}$$

Let $v \in B_\tau(\delta)$, we have (recall that $\delta = 2C\|v_0\|_H$ and $G(0) = 0$)

$$||G(v)||^2_{L^2((0,\tau);D_{-1/2}(A))} \leq L^2 \int_0^\tau ||v(s)||^2_H \, ds \leq L^2\tau ||v||^2_{C^0([0,\tau];H)} \leq 4C^2L^2\tau ||v_0||^2_H. \tag{5.74}$$

Hence by Lemma 5.10 we get

$$||Mv||_{X_\tau} \leq C(||v_0||_H + ||G(v)||_{L^2((0,\tau);D_{-1/2}(A))}) \leq (C + 2\sqrt{\tau}LC)||v_0||_H. \tag{5.75}$$

Then we set $\tau = \frac{1}{4L^2C^2}$ and as $C \geq 1$, M become a mapping from $B_\tau(\delta)$ to itself. Moreover, let $v_1, v_2 \in B_\tau(\delta)$, we have again by Lemma 5.10 and (5.75)

$$||Mv_1 - Mv_2||_{X_\tau} \leq CL||v_1 - v_2||_{L^2((0,\tau);H)} \tag{5.76}$$

$$\leq \sqrt{\tau} CL ||v_1 - v_2||_{X_\tau} \tag{5.77}$$

$$= \frac{1}{2} ||v_1 - v_2||_{X_\tau}. \tag{5.78}$$

This shows that M is a contraction on $B_\tau(\delta)$, allowing us to apply the Picard fixed point theorem to conclude. □

Now, let $v_0 \in H$, as v is a mild solution to (5.49) its weak derivative is $Dv + G(v)$, and as $v \in X_\tau$ and G satisfies Assumption 1 (recall that G satisfies Assumption 1 because $\mathcal{F}$) we have $\partial_t v \in L^2((0,\tau); D_{-1/2}(A))$ and hence $v \in H^1((0,\tau); D_{-1/2}(A))$. This implies by classical approximations argument that $||v(\cdot)||^2_H \in W^{1,1}(0,\tau)$ and

$$\frac{d}{dt} ||v(t)||^2_H = 2\,\mathrm{Re}(\langle \partial_t v \,, v \rangle_{D_{-1/2}(A), D_{1/2}(A)}) \text{ a.e on } (0, \tau_v). \tag{5.79}$$

We now proceed to demonstrate the stability part, for which we will need the following lemma.

LEMMA 5.12. *There exists $\delta > 0$ such that for any $v_0 \in H$ and any $T > 0$, if*

$$||v||_{C^0([0,T];H)} \leq \delta, \tag{5.80}$$

then

$$\forall t \in [0,T], \ ||v(t)||_H \leq e^{-\frac{\lambda}{2}t} ||v_0||_H. \tag{5.81}$$

Proof. Let $\varepsilon > 0$ to be chosen later, from Assumption (1), there exists $\delta > 0$ such that

$$\forall u, v \in H, \quad ||u||_H,\ ||v||_H \leq \delta \Rightarrow ||G(u) - G(v)||_{D_{-1/2}(A)} \leq \varepsilon ||u - v||_H. \tag{5.82}$$

Now as $D_{1/2}(A)$ is continuously embedded in H, there exists $\alpha > 0$ such that

$$\forall u \in D_{1/2}(A),\ \|u\|_H \leq \alpha \|u\|_{D_{1/2}(A)}. \tag{5.83}$$

Let $v_0 \in H$ and $T > 0$ and assume that (5.80) holds. We have for almost every $t \in (0, T)$

$$\frac{1}{2}\frac{d}{dt}||v||_H^2 = \mathrm{Re}(\langle \partial_t v\,, v\rangle_{D_{-1/2}(A), D_{1/2}(A)}) \tag{5.84}$$

$$= \mathrm{Re}(\langle Dv + G(v)\,, v\rangle_{D_{-1/2}(A), D_{1/2}(A)}) \tag{5.85}$$

$$\leq \mathrm{Re}(\langle Dv\,, v\rangle_{D_{-1/2}(A), D_{1/2}(A)}) + \varepsilon\alpha ||v||^2_{D_{1/2}(A)}. \tag{5.86}$$

By definition, we have for almost every $t \in (0, T)$

$$\mathrm{Re}(\langle Dv\,, v\rangle_{D_{-1/2}(A), D_{1/2}(A)}) = -\sum_{n\geq 1} |\operatorname{Re} d_n||v_n(t)|^2. \tag{5.87}$$

Now, as $|\operatorname{Re} d_n| \geq \lambda$ for every $n \geq 1$, we have

$$\mathrm{Re}(\langle Dv\,, v\rangle_{D_{-1/2}(A), D_{1/2}(A)}) \leq -\lambda ||v||_H^2. \tag{5.88}$$

On the other hand, by hypothesis (A3) there exists $c > 0$ such that $c|d_n| \leq |\operatorname{Re} d_n|$ holds for all $n \geq 1$. Thus we have

$$\mathrm{Re}(\langle Dv\,, v\rangle_{D_{-1/2}(A), D_{1/2}(A)}) \leq -c||v||^2_{D_{1/2}(A)}. \tag{5.89}$$

Combining the two previous inequality we get that for almost every $t \in (0, T)$

$$\frac{1}{2}\frac{d}{dt}||v||_H^2 \leq -\frac{c}{2}||v||^2_{D_{1/2}(A)} - \frac{\lambda}{2}||v||_H^2 + \alpha\varepsilon||v||^2_{D_{1/2}(A)}. \tag{5.90}$$

Hence if we choose $\varepsilon = \frac{c}{4\alpha}$, we can apply a classical variant of Gronwall lemma, and as $v \in C^0([0, T]; H)$, we have

$$\forall t \in [0, T],\ ||v(t)||_H^2 \leq e^{-\lambda t}||v_0||_H^2, \tag{5.91}$$

which concludes the proof. $\square$

Now using the previous lemma, we show that for every $v_0 \in H$ such that $||v_0||_H \leq \frac{\delta}{2}$, then $\tau_v = +\infty$ and we have the desired stability estimate. First suppose by contradiction that $\tau_v < +\infty$. By maximality of solutions, there exists $t' > 0$ such that $||v(t')||_H > \delta$, hence we define

$$\tau_c = \inf\{t \in (0, \tau_v) \mid ||v(t)||_H > \delta\}. \tag{5.92}$$

Notice that $\tau_c > 0$ by the continuity of v. Then by definition we have

$$||v||_{C^0([0,\tau_c];H)} \leq \delta. \tag{5.93}$$

Hence, applying the previous lemma leads to

$$\forall t \in [0, \tau_c],\ ||v(t)||_H \leq e^{-\frac{\lambda}{2}t}||v_0||_H. \tag{5.94}$$

Then we have $||v(\tau_c)||_H \leq \frac{\delta}{2}$, which is a contradiction, hence $\tau_v = +\infty$. Now with a similar reasoning, we get that the estimate (5.94) holds for all $t \in [0, +\infty)$.

We have established the desired result for the system (5.49). To extend this result to the system (3.9), it suffices to observe that, for any $\tau > 0$, $u \in C^0([0, \tau); H)$ is a solution to (3.9) if and only if $v := Tu \in C^0([0, \tau); H)$ is a solution to (5.49). From this, we conclude that (3.9) is locally well-posed. Defining $C_\lambda = \|T^{-1}\|_{\mathcal{L}(H)}\|T\|_{\mathcal{L}(H)}$, we observe that if $\|u_0\|_H \leq \frac{\delta}{2\|T\|_{\mathcal{L}(H)}}$, then u is a global solution, and the following holds

$$\forall t \in [0, +\infty), \quad \|u(t)\|_H \leq C_\lambda e^{-\frac{\lambda}{2}t}\|u_0\|_H. \tag{5.95}$$

6. Approximate controllability and uniqueness

The problem of finding an F-equivalence between a control system (A, B) and a target D is challenging, especially when the problem is ill-posed, meaning there may be multiple pairs (T, K) that satisfy the conditions. In this section, we investigate this issue of uniqueness.

First, in subsection 6.1, we introduce an algebraic characterization for the lack of uniqueness on T. This allows us to introduce the weak F-equivalence formalism and to recover a well-posed problem. Then in subsection 6.2, we show that our algebraic characterization can be linked to the approximate controllability of (A, B). This allows us to prove Theorem 6.6, which implies that the parabolic F-equivalence problem is well-posed if and only if (A, B) is approximately controllable.

6.1. **Weak F-equivalence.** Let us fix A, λ, B, μ, and D as in Theorem 3.2.[11] Let $(T, K), (T', K') \in \mathcal{GL}_H(D(A)') \times \mathcal{L}(H, \mathbb{C}^{m(\lambda)})$ be parabolic F-equivalences of (A, B, D). Proposition 5.9 ensures that $K = K'$, and

$$T_{LL} = T'_{LL},\ T_{LH} = T'_{LH},\ T_{HL} = T'_{HL},\ T_{HH}, T'_{HH} \in C(A_H). \tag{6.1}$$

We adopt the same notation as in the proposition, hence we set $C := T_{HH}$ and $C' := T'_{HH}$. Now, notice that by the definition of F-equivalence, we have

$$(T - T')B = B - B = 0. \tag{6.2}$$

And by Lemma 5.6, this gives us

$$(C - C')B_H = 0. \tag{6.3}$$

This leads us to define the following closed subspace of $C(A_H)$:

$$N_{B_H} = \{M \in C(A_H) \,|\, MB_H = 0\}. \tag{6.4}$$

Then, we endow $\mathcal{L}_H(D(A)')/N_{B_H}$ with the quotient norm, and because N_{B_H} is closed, the quotient is a Banach space. We note $\pi : \mathcal{L}_H(D(A)') \to \mathcal{L}_H(D(A)')/N_{B_H}$ the quotient map. Now, we have found all the possible solutions, as illustrated by the following proposition.

PROPOSITION 6.1. *Let $\mathcal{S} \subset \mathcal{GL}_H(D(A)') \times \mathcal{L}(H, \mathbb{C}^{m(\lambda)})$ be the set of all parabolic F-equivalences of (A, B, D). We denote by (T_*, K_*) the solution given by Theorem 3.2. Then we have*

$$\mathcal{S} = (\pi(T_*) \cap \mathcal{GL}_H(D(A)')) \times \{K_*\}. \tag{6.5}$$

Proof. The above discussion shows that $\mathcal{S} \subset (\pi(T_*) \cap \mathcal{GL}_H(D(A)')) \times \{K_*\}$ (as $\pi(T_*) = T_* + N_{B_H}$). Now let $T \in \mathcal{GL}_H(D(A)')$ such that there exists $N \in N_{B_H}$ with $T = T_* + N$. Hence, we have $TB = B$, and if we set $F(T) = T(A + BK) - DT$, using the same notation as in Proposition 5.9, we have

$$F(T) = \begin{pmatrix} \tilde{T}(A_L + B_L K) - (A_L - \mu)\tilde{T} & 0 \\ \tau(A_L + B_L K) + (C + N)B_H K - A_H\tau & (C + N)A_H - A_H(C + N) \end{pmatrix}. \tag{6.6}$$

By definition, $NB_H = 0$, hence we have $F(T) = F(T_*) = 0$, and this concludes the proof. □

[11]Hence, with the notations of Section 5, $\mu \in \Lambda$.

We can deduce from this a simple formalism that allows us to restate the parabolic F-equivalence problem so that it becomes well-posed. To this end let fix $K \in \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$, if we set $\mathcal{T} := \pi(T)$, we first need to make sense of the following equation

$$\mathcal{T}(A + BK) = D\mathcal{T}. \tag{6.7}$$

We define $F_K : \mathcal{L}_H(D(A)') \to \mathcal{L}(H, D(A)')$ as in the previous proof, which is a bounded linear operator

$$\forall T \in \mathcal{L}_H(D(A)'),\ F_K(T) = T(A + BK) - DT. \tag{6.8}$$

Now for every $T \in \mathcal{L}_H(D(A)')$, we have

$$F_K(T) = \begin{pmatrix} T_{LL}(A_L + B_L K) - (A_L - \mu)T_{LL} + T_{HL}B_H K & T_{HL}A_H - (A_L - \mu)T_{HL} \\ T_{LH}(A_L + B_L K) + T_{HH}B_H K - A_H T_{LH} & T_{HH}A_H - A_H T_{HH} \end{pmatrix}. \tag{6.9}$$

Now let $N \in N_{B_H}$. We have

$$F_K(N) = \begin{pmatrix} 0 & 0 \\ NB_H K & NA_H - A_H N \end{pmatrix} = 0. \tag{6.10}$$

Hence, F_K continuously factors through π, which means that there exists a unique bounded operator $\mathcal{F}_K$ from $\mathcal{L}_H(D(A)')/N_{B_H}$ to $\mathcal{L}(H, D(A)')$ such that

$$\forall T \in \mathcal{L}_H(D(A)'),\ \mathcal{F}_K(\pi(T)) = F(T). \tag{6.11}$$

Now this operator allows us to make sense of (6.7), we simply say that $\mathcal{T}(A+BK) = D\mathcal{T}$ if $\mathcal{F}_K\mathcal{T} = 0$. Finally to define weak F-equivalence, we need to make sense of $\mathcal{T}B = B$, for this we define the following affine subspace

$$\mathcal{F}_B = \{T \in \mathcal{L}_H(D(A)') \,|\, TB = B \,\text{in}\, D(A)'\}. \tag{6.12}$$

Then we define $\mathcal{T}B = B$ as $\mathcal{T} \in \pi(\mathcal{F}_B)$.

DEFINITION 6.2 (Weak F-equivalence)**.** *Let $(\mathcal{T}, K) \in \mathcal{L}_H(D(A)')/N_{B_H} \times \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$. We say that $(\mathcal{T}, K)$ is a weak F-equivalence of (A, B, D), or that it is a weak F-equivalence between (A, B) and D, if*

$$\mathcal{T} \in \pi(\mathcal{F}_B) \cap \ker \mathcal{F}_K. \tag{6.13}$$

The above condition can also be written with the previous notations as

$$\begin{cases} \mathcal{T}(A + BK) = D\mathcal{T}, \\ \mathcal{T}B = B. \end{cases} \tag{6.14}$$

Equation (6.14) clearly explain why the previous definition is called a weak F-equivalence. Finally the next proposition show that finding a weak F-equivalence is a well-posed problem and the solution is linked to parabolic F-equivalence.

PROPOSITION 6.3 (Uniqueness of Weak F-Equivalence)**.** *Let (T_*, K_*) be a parabolic F-equivalence of (A, B, D). Then $(\pi(T_*), K_*)$ is the unique weak F-equivalence of (A, B, D).*

Proof. First, note that $(\pi(T_*), K_*)$ is indeed a weak F-equivalence. Now let $K \in \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$, $T \in \mathcal{L}_H(D(A)')$ such that $TB = B$ and

$$\pi(T)(A + BK) = D\pi(T). \tag{6.15}$$

This implies that $\mathcal{F}_K(\pi(T)) = F_K(T) = 0$. The same reasoning as in the proof of Proposition 5.9 shows that $K = K_*$ and $T - T_* \in C(A_H)$. Since $TB = B$ and $T_*B = B$, we have $(T - T_*)B = (T - T_*)_{HH}B = 0$, hence $T - T_* \in N_{B_H}$, which gives $\pi(T) = \pi(T_*)$. ☐

6.2. **Approximate controllability.** One might ask when finding a parabolic F-equivalence becomes a well-posed problem on its own. As we have shown in Proposition 6.1, the issue of uniqueness is entirely due to the size of N_{B_H}, and having a unique solution is equivalent to $N_{B_H} = \{0\}$. Therefore, in this subsection, we fix A, λ, μ, and D as before, but we let B free.

DEFINITION 6.4. *Let $B \in (D(A)')^m$ and $\tau > 0$, we say that (A, B) is approximately controllable (in time τ) if for all $u_0, u_1 \in H$ and any $\varepsilon > 0$, there exists $w \in L^2(0, \tau; \mathbb{C}^m)$ such that the solution of the following system*

$$\begin{cases} \partial_t u(t) = Au(t) + Bw(t), \ \forall t \in (0, \tau), \\ u(0) = u_0, \end{cases} \tag{6.16}$$

satisfies $\|u(\tau) - u_1\| \leq \varepsilon$.

REMARK 6.5. *By Lemma 6.8, as for finite-dimensional systems, the approximate controllability of (A, B) is in fact, independent of τ. Hence, we will simply refer to the approximate controllability of (A, B) without specifying a final time.*

Here our goal is to relate the size of N_{B_H} to the approximate controllability of (A, B) by proving the following theorem.

THEOREM 6.6. *Let $B \in (D(A)')^m$ be F_λ-admissible. Then $N_{B_H} = \{0\}$ if and only if (A, B) is approximately controllable.*

The above theorem immediately answers our question and provides a new characterization of approximate controllability for parabolic systems.

COROLLARY 6.7. *Let $B \in (D(A)')^m$ be F_λ-admissible. Then (A, B) is approximately controllable if and only if there exists a unique parabolic F-equivalence of (A, B, D).*

To prove Theorem 6.6, we will use the generalized Fattorini criterion introduced by Badra and Takahashi in [3]. Let $B = (B_1, \ldots, B_m) \in (D(A)')^m$, since A is normal, we have $\ker(A^* - \overline{\lambda_n}) = \ker(A - \lambda_n)$, hence $D(A)' = \bigoplus_{n \geq 1} \ker(A^* - \overline{\lambda_n})$. We set $l_n := \dim \ker(A - \lambda_n)$, which allows us to find a partition of $(e_n)_{n \geq 1}$ as follows.[12] For each $n \geq 1$, we denote by $(\varepsilon_k^n)_{1 \leq k \leq l_n}$ a basis of $\ker(A - \lambda_n)$ formed by the elements of $(e_k)_{k \geq 1}$, hence (ε_k^n) is a reordering of (e_n). Now, for $j \in \{1, \ldots, m\}$, we can write in $D(A)'$

$$B_j = \sum_{n \geq 1} \sum_{k=1}^{l_n} b_k^{j,n} \varepsilon_k^n. \tag{6.17}$$

With this, we can now state the criterion for approximate controllability.

LEMMA 6.8 (Fattorini-Badra-Takahashi Criterion)**.** *Let $B = (B_1, \ldots, B_m) \in (D(A)')^m$. The pair (A, B) is approximately controllable (for any time $\tau > 0$) if and only if for every $n \geq 1$, $\operatorname{rank}(\mathcal{B}_n) = l_n$, where $\mathcal{B}_n$ is given by*

$$\mathcal{B}_n = \begin{pmatrix} b_1^{1,n} & b_2^{1,n} & \ldots & b_{l_n}^{1,n} \\ \vdots & \vdots & & \vdots \\ b_1^{m,n} & b_2^{m,n} & \ldots & b_{l_n}^{m,n} \end{pmatrix}. \tag{6.18}$$

Proof. This is a direct application of Theorem 1.3 (using Remark 2.1 which allows us to take $\gamma = 1$) in [3]. □

[12] Note that we will work with $(e_n)_{n \geq 1}$, which is orthogonal but not orthonormal in $D(A)'$.

With this criterion we are now able to prove our theorem.

Proof of Theorem 6.6. Let $C \in C(A_H)$. First we give a characterization of $CB_H = 0$ using a collection of infinite scalar linear systems.

Notice that $CB_H = 0$ is equivalent to $C{B_j}_H = 0$ for all $j \in \{1, \dots, m\}$. Now, for $n > N(\lambda)$, if we denote by P_n the orthogonal projection onto $\ker(A_n - \lambda_n)$, we have $CP_nB = P_nCB$ because $CA_H = A_HC$. Hence, we have the following characterization of $CB_H = 0$

$$\forall n > N(\lambda), \forall j \in \{1, \dots, m\},\ CP_n{B_j}_H = 0. \tag{6.19}$$

Thus, for each $n > N(\lambda)$, we have a finite-dimensional linear system. For $n \geq 0$ and for every $k \in \{1, \dots, l_n\}$, we set $c_k^n := C\varepsilon_k^n \in \ker(A - \lambda_n)$. Then, using (6.19), we have

$$CB_H = 0 \iff \forall n > N(\lambda), \quad \begin{cases} b_1^{1,n} c_1^n + \cdots + b_{l_n}^{1,n} c_{l_n}^n = 0, \\ b_1^{2,n} c_1^n + \cdots + b_{l_n}^{2,n} c_{l_n}^n = 0, \\ \vdots \\ b_1^{m,n} c_1^n + \cdots + b_{l_n}^{m,n} c_{l_n}^n = 0. \end{cases} \tag{6.20}$$

Now, to obtain scalar linear equations, we fix $n > N(\lambda)$. Then for each $k \in \{1, \dots, l_n\}$, we set $x_k := (\langle c_1^n, \varepsilon_k^n \rangle,\ \dots\ , \langle c_{l_n}^n, \varepsilon_k^n \rangle)^T \in \mathbb{C}^{l_n}$, and thus we have

$$\begin{cases} b_1^{1,n} c_1^n + \cdots + b_{l_n}^{1,n} c_{l_n}^n = 0, \\ b_1^{2,n} c_1^n + \cdots + b_{l_n}^{2,n} c_{l_n}^n = 0, \\ \vdots \\ b_1^{m,n} c_1^n + \cdots + b_{l_n}^{m,n} c_{l_n}^n = 0. \end{cases} \iff \forall k \in \{1, \dots, l_n\},\ \mathcal{B}_n x_k = 0. \tag{6.21}$$

Now, we show that $N_{B_H} = \{0\}$ is equivalent to the approximate controllability of (A, B). First, suppose that (A, B) is approximately controllable. Then, Lemma 6.8 ensures that for all $n > N(\lambda)$, we have $\ker \mathcal{B}_n = \{0\}$, and by the above discussion, this implies that $N_{B_H} = \{0\}$. Conversely, suppose that $N_{B_H} = \{0\}$. First, we establish the controllability of (A_L, B_L). Using the decomposition from Lemma 2.5, it suffices to show that $({A_j}_L, {B_j}_L)$ is controllable for all $j \in \{1, \dots, m(\lambda)\}$. This follows from (H_B), thanks to the Kalman criterion. Thus, the pair (A_L, B_L) is controllable. Moreover, the Fattorini criterion applies to finite-dimensional systems as well, where approximate controllability coincides with exact controllability, hence we have

$$\forall n \in \{1, \dots, N(\lambda)\},\ \operatorname{rank}(\mathcal{B}_n) = l_n. \tag{6.22}$$

Now, suppose by contradiction that there exists $n > N(\lambda)$ such that $\operatorname{rg}(\mathcal{B}_n) < l_n$. Hence, there exists $z \in \mathbb{C}^{l_n} \setminus \{0\}$ such that $\mathcal{B}_n z = 0$. With this, we can construct $C \in C(A_H)$ such that it is zero everywhere except on $\ker(A_n - \lambda_n)$, where we have

$$\forall k \in \{1, \dots, l_n\},\ C\varepsilon_k^n = z_k \varepsilon_1^n. \tag{6.23}$$

Then, using (6.21) and the above discussion, we would have $CB_H = 0$, which is a contradiction. Hence, by Lemma 6.8, (A, B) is approximately controllable.

□

REMARK 6.9. *Working with an operator* A *such that* $m(\lambda) \underset{\lambda \to +\infty}{\to} +\infty$*, implies that* (A, B) *is not approximately controllable for any* $B \in (D(A)')^m$*. Indeed, if* (A, B) *is approximately controllable, then Lemma 6.8 implies* $\sup_{n\geq 1} l_n < +\infty$*. Hence, in this case, we know that the problem of finding a parabolic* F*-equivalence for* (A, B, D) *is ill-posed.*

7. Acknowledgements

Blank acknowledgements for the blind review process.

Appendix A. Series of functions in a Hilbert space

Let $(f_n)_{n\geq 1}$ be a sequence of $C^0(I,\mathbb{C})$ where I is a real, non-trivial interval. We say that $(f_n)_{n\geq 1}$ is locally uniformly square summable if for all $t_0 \in I$ there exists a neighborhood $J \subset I$ of t_0 and a non-negative sequence $(c_n)_{n\geq 1} \in \ell^2$ such that

$$\forall n \geq 1, \forall t \in J,\ |f_n(t)| \leq c_n.$$

LEMMA A.1. *Let $(H, \langle \cdot\,, \cdot\rangle)$ be a Hilbert space with an orthonormal basis $(e_n)_{n\geq 1}$. Let $(f_n)_{n\geq 1}$ be locally uniformly square summable sequence of continuous functions. Then the function $F : I \to H$ defined as*

$$\forall t \in I,\ F(t) = \sum_{n\geq 1} f_n(t)e_n,$$

is continuous.

Proof. For all $t \in I$, there exists a neighborhood $J \subset I$ of t such that $(f_n : J \to \mathbb{C})_{n\geq 1}$ is uniformly continuous, hence F is continuous on J and thus continuous at t.

□

Appendix B. Properties of the operator A

B.1. A **is normal.** We have the following proposition.

PROPOSITION B.1. *Let A be a diagonal parabolic operator on H. Then A is normal, and we have*

$$\forall x \in D(A),\ A^*x = \sum_{n\geq 1} \overline{\lambda_n} x_n e_n.$$

Furthermore, it is the infinitesimal generator of an analytic semigroup.

Proof. For all $n \geq 1$, we have $e_n \in D(A)$ by (A1), so $\overline{D(A)} = H$. Notice also that $e_n \in D(A^*)$ for all $n \geq 1$, and we have

$$\forall k, n \geq 1,\ \langle A^*e_n\,, e_k\rangle_H = \langle e_n\,, Ae_k\rangle_H = \overline{\lambda_k}\delta_{kn}.$$

Hence, $A^*e_n = \overline{\lambda_n}e_n$, which means that $D(A) = D(A^*)$ and for all $x \in D(A)$, we have

$$A^*x = \sum_{n\geq 1} \overline{\lambda_n} x_n e_n.$$

Thus, $||A^*x||_H = ||Ax||_H$. To conclude that A is normal, we now just need to show that A is closed, but then it is sufficient to prove that A generates a strongly continuous semigroup, see [43, Sec. 1.2].

Let $t > 0$. We define $S(t) : H \to H$ as $S(t)x = \sum_{n\geq 1} x_n e^{\lambda_n t} e_n$ for every $x \in H$. Again by (A1), we have that $S(t)$ is bounded. It is clear that for all $s, t > 0$ we have $S(t + s) = S(t)S(s)$ and that $S(t)x \to x$ as $t \to 0$ by Lemma A.1 using Hypothesis (A2). We then see that

$$\frac{S(t)x - x}{t} = \sum_{n\geq 1} x_n \frac{e^{\lambda_n t} - 1}{t} e_n.$$

Setting $f_n(t) = x_n \frac{e^{\lambda_n t}-1}{t}$ for $t > 0$ and $f_n(0) = x_n\lambda_n$, the inequality of the mean value theorem gives us $|f_n(t)| \leq |x_n\lambda_n|$ for n large enough via (A2). Then, noticing that

$$D(A) = \{x \in H \mid \sum_{n\geq 1} |x_n\lambda_n|^2 < +\infty\},$$

and using Lemma A.1, this implies that the limit as $t \to 0$ exists if and only if $x \in D(A)$. Consequently, we have

$$\forall x \in D(A),\ \frac{S(t)x - x}{t} \to Ax.$$

Finally, we show that A generates an analytic semigroup using the resolvent characterization, see [43, Sec. 2.5] First, we set $\omega = \mathrm{Re}(\lambda_1)$. By Hypothesis (A2), the half-plane $\mathrm{Re}(\lambda) > \omega$ is contained in the resolvent set of A.

Then, let $\lambda \in \mathbb{C}$ be such that $\mathrm{Re}(\lambda) > \omega$. Using Hypothesis (A3), we can see geometrically that there exists a constant $c > 0$ such that

$$\forall n \geq 1,\ c|\lambda - \lambda_n| \geq |\lambda - \omega|. \tag{B.1}$$

Hence, we have

$$\forall x \in H,\ \|(A-\lambda)^{-1}x\|_H^2 = \sum_{n\geq 1} |\lambda - \lambda_n|^{-2}|x_n|^2 \leq \frac{c^2}{|\lambda - \omega|^2}\|x\|_H^2, \tag{B.2}$$

which gives us the desired estimate on the resolvent:

$$\|(A-\lambda)^{-1}\|_{\mathcal{L}(H)} \leq \frac{c}{|\lambda - \omega|}. \tag{B.3}$$

□

B.2. **Extension for normal operators.**

PROPOSITION B.2. *Let N be an unbounded normal operator on H. Then there exists a unique $\tilde{N} \in \mathcal{L}(H, D(N)')$ such that $\tilde{N}_{|D(N)} = N$.*

Proof. This is a straightforward consequence from the normality of the operator N: as N is normal, it is densely defined, so we again have a Gelfand triple $(D(N), H, D(N)')$ as before. The uniqueness follows from the density. Now let $x \in H$, we define $\tilde{N}x$ as

$$\forall y \in D(N),\ (\tilde{N}x)(y) := \langle N^* y\,, x\rangle_H.$$

Thus, for all $y \in D(N)$, since N is normal we have

$$|\langle N^* y\,, x\rangle_H| \leq ||N^*y||_H||x||_H = ||Ny||_H||x||_H \leq ||y||_{D(N)}||x||_H.$$

This shows that $\tilde{N} \in \mathcal{L}(H, D(N)')$. If $x \in D(N)$, then

$$(\tilde{N}x)(y) = \langle y\,, Nx\rangle_H = \langle y\,, Nx\rangle_{D(N),D(N)'}.$$

Which implies that $\tilde{N}x = Nx$ in the sense of the inclusion given by the Gelfand triple. □

Appendix C. Finite dimensional F-equivalence

We consider a finite dimensional control system

$$\dot{x} = Ax + Bu, \tag{C.1}$$

with $x \in \mathbb{C}^n$, $u \in \mathbb{C}$, $A \in M_n(\mathbb{C})$ and $B \in \mathbb{C}^n$. Here is the main result of this section.

THEOREM C.1. *Let $\lambda \in \mathbb{R}$ and (A, B) be controllable, then there exists one and only one $(T, K) \in \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^{1\times n}$ such that*

$$\begin{cases} T(A + BK) = (A - \lambda)T, \\ TB = B. \end{cases} \tag{C.2}$$

Furthermore, $K_1, \ldots, K_n$ are polynomials in λ, and if A is diagonalizable, they have no constant term.

Proof. First, Theorem 4.1 in [12] gives the existence and uniqueness of (T,K) satisfying (C.2), and its proof ensures that $T_{ij} := \langle Te_j\,, e_i\rangle$ and K_i are polynomials in λ (Note that, in [12], A is assumed to be nilpotent, but the proof can be easily generalized using similar arguments).

Now, assume that A is diagonalizable. Without loss of generality, we can assume that A is diagonal and $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ (note that the controllability is conserved by a change of variable and that A does not depend on λ). Let $j \in \{1, \dots, n\}$, by (C.2), we have

$$\lambda_j Te_j + K_j B = ATe_j - \lambda Te_j. \tag{C.3}$$

Next, for any $i \in \{1, \dots, n\}$, projecting the previous equation onto e_i gives (as $\langle ATe_j\,, e_i\rangle = \langle Te_j\,, \bar{A}e_i\rangle$)

$$\lambda_j T_{ij} + K_j B_i = (\lambda_i - \lambda)T_{ij}. \tag{C.4}$$

Setting $i = j$ in this equation yields

$$\forall i \in \{1, \dots, n\},\ K_i = -\lambda \frac{T_{ii}}{B_i}. \tag{C.5}$$

Thus, we conclude that K_i has no constant term as a polynomial in λ. Note that $B_i \neq 0$ is guaranteed by the fact that (A, B) is controllable. □

REMARK C.2. *In the diagonal case, using the same notation as in the proof,* (C.2) *is equivalent to*

$$\begin{cases} (\lambda - \lambda_i + \lambda_j)T_{ij} + K_j B_i = 0, & \forall i, j \in \{1, \dots, n\}, \\ \sum_{j=1}^n T_{ij} B_j = B_i, & \forall i \in \{1, \dots, n\}. \end{cases} \tag{C.6}$$

This system consists of n^2+n *linear scalar equations, which allow us to numerically compute* (T,K) *in practice.*

Appendix D. Proof of Lemma 2.5

Let us first show that such a partition exists. Using Appendix G of [22], we can construct $(e_n^1)_{1\le n\le N_1(\lambda)}, \dots, (e_n^{m(\lambda)})_{1\le n\le N_{m(\lambda)}(\lambda)}$, which forms a partition of $(e_n)_{1\le n\le N(\lambda)}$, such that the restriction of A to $\operatorname{span}((e_n^j)_{1\le n\le N_j(\lambda)})$ has only simple eigenvalues for all $j \in \{1, \dots, m(\lambda)\}$. Now, we extend these families as follows

$$\forall j \in \{1, \dots, m(\lambda)\}, \forall n > N_j(\lambda), \quad e_n^j = e_{m(\lambda)(n-N_j(\lambda))-(j-1)+N(\lambda)}. \tag{D.1}$$

Thus, $(e_n^1)_{n\ge1}, \dots, (e_n^{m(\lambda)})_{n\ge1}$ form a partition of $(e_n)_{n\ge1}$. Note that this partition is far from being unique, and, in particular, any re-arrangement of $(e_n^j)_{j\in\{1,\dots,m(\lambda)\}, n>N_j(\lambda)}$ is suitable. Then, Lemma 2.5 is a consequence of the following Lemma:

LEMMA D.1. *Let* $d \ge 1$ *and let* $(e_n^1)_{n\ge1}, \dots, (e_n^d)_{n\ge1}$ *be a partition of* $(e_n)_{n\ge1}$*. Define* $\mathcal{H}^j = \overline{span((e_n^j)_{n\ge1})}^H$ *thus* $H = \bigoplus_{j=1}^d \mathcal{H}^j$*. Then for each* $j \in \{1, \dots, d\}$*,* A *induces a diagonal parabolic operator on* $\mathcal{H}^j$ *such that*

$$A = A_1 + \cdots + A_d, \quad D(A) = \bigoplus_{j=1}^d D(A_j), \quad D(A)' = \bigoplus_{j=1}^d D(A_j)'.$$

Proof. Let $j \in \{1, \dots, d\}$. For all $n \geq 1$, define λ_n^j as the eigenvalues associated with e_n^j. Then we set $D(A_j) = \{x \in \mathcal{H}^j \mid \sum_{n\geq 1} |x_n^j|^2 |\lambda_n^j|^2 < +\infty\}$, which allows us to define A_j as

$$\forall x \in D(A_j),\ A_j x = \sum_{n\geq 1} x_n^j \lambda_n^j e_n^j.$$

Then A_j satisfies (A1). Notice that since $(e_n^j)_{n\geq 1}$ is a subsequence of $(e_n)_{n\geq 1}$, this implies that A also satisfies (A2). Therefore, we have $A = A_1 + \cdots + A_d$ and $D(A) = \bigoplus_{j=1}^d D(A_j)$. Taking the dual of the orthogonal sum finally gives the last equality. □

Appendix E. Proof of Proposition 3.3

The proof of Proposition 3.3 is relatively straightforward and we give it here for completeness: since $A \in \mathcal{L}(H, D(A)')$, $B \in (D(A)')^{m(\lambda)}$ (hence it can be seen as belonging to $\mathcal{L}(\mathbb{C}^{m(\lambda)}, D(A)')$, see (2.9)) and $K \in \mathcal{L}(H, \mathbb{C}^{m(\lambda)})$, we have $A + BK \in \mathcal{L}(H, D(A)')$.

Now we view $A + BK$ as an unbounded operator on H and define $D(A + BK) = \{x \in H \mid (A + BK)x \in H\}$. Let's show that $D(A + BK) = T^{-1}(D(A))$.

Let $x \in D(A + BK)$. Then $(A + BK)x \in H$, and so equality (3.1) implies $DTx \in H$, hence $x \in T^{-1}(D(A))$. Conversely, let $x \in T^{-1}(D(A))$. Then again by (3.1), we get $T(A + BK)x \in H$, but now because $T \in \mathcal{GL}(D(A)')$ and $T_{|H} \in \mathcal{GL}(H)$, we have $(A + BK)x \in H$. We know that $D(A)$ is dense, then since T is an isomorphism on H, this shows the density of $D(A + BK)$.

Now we set $S(0) = \mathrm{Id}_H$ and for $t > 0$, we define $S(t) : H \to H$ as

$$\forall x \in H,\ S(t)x = T^{-1} e^{tD} T x. \tag{E.1}$$

Hence, $S := (S(t))_{t\in\mathbb{R}_{\geq 0}}$ is a differentiable semigroup with growth rate at most $-\lambda$ as $(e^{tD})_{t\in\mathbb{R}_{\geq 0}}$ is a differentiable semigroup with the same growth rate. Now we have to check that $A + BK$ is the infinitesimal generator of S. Let $x \in H$ such that the limit in H as $t \to 0$ of $\frac{S(t)x - x}{t}$ exists. We have

$$\frac{S(t)x - x}{t} = T^{-1} \frac{e^{tD} T x - T x}{t}, \tag{E.2}$$

then the existence of the limit is equivalent to $Tx \in D(A)$ and hence $x \in D(A + BK)$ by what we have shown before. The last part of the proposition is an immediate consequence of semigroup theory for evolution equations, see for instance [43, Sec. 4.1]

References

[1] Mehdi Badra, Debanjana Mitra, Mythily Ramaswamy, and Jean-Pierre Raymond. Local feedback stabilization of time-periodic evolution equations by finite dimensional controls. *ESAIM: Control, Optimisation and Calculus of Variations*, 26:101, 2020.

[2] Mehdi Badra and Takéo Takahashi. Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers: Application to the navier–stokes system. *SIAM journal on control and optimization*, 49(2):420–463, 2011.

[3] Mehdi Badra and Takeo Takahashi. On the fattorini criterion for approximate controllability and stabilizability of parabolic systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 20:924–956, 2014.

[4] Andras Balogh and Miroslav Krstic. Infinite dimensional backstepping-style feedback transformations for a heat equation with an arbitrary level of instability. *European journal of control*, 8(2):165–175, 2002.

[5] Viorel Barbu. *Controllability and stabilization of parabolic equations*. Springer, 2018.

[6] Richard Bellman. On the theory of dynamic programming. *Proceedings of the national Academy of Sciences*, 38(8):716–719, 1952.

[7] Tobias Breiten and Karl Kunisch. Riccati-based feedback control of the monodomain equations with the fitzhugh–nagumo model. *SIAM Journal on Control and Optimization*, 52(6):4057–4081, 2014.

[8] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer, 2010.

[9] Pavol Brunovskỳ. A classification of linear controllable systems. *Kybernetika*, 6(3):173–188, 1970.

[10] Jean Cauvin-Vila, Virginie Ehrlacher, and Amaury Hayat. Boundary stabilization of one-dimensional cross-diffusion systems in a moving domain: linearized system. *Journal of Differential Equations*, 350:251–307, 2023.

[11] Jean-Michel Coron. *Control and nonlinearity*. Number 136. American Mathematical Soc., 2007.

[12] Jean-Michel Coron. Stabilization of control systems and nonlinearities. In *Proceedings of the 8th International Congress on Industrial and Applied Mathematics*, pages 17–40. Higher Ed. Press Beijing, 2015.

[13] Jean-Michel Coron, Ludovick Gagnon, and Morgan Morancey. Rapid stabilization of a linearized bilinear 1-D Schrödinger equation. *J. Math. Pures Appl. (9)*, 115:24–73, 2018.

[14] Jean-Michel Coron, Amaury Hayat, Shengquan Xiang, and Christophe Zhang. Stabilization of the linearized water tank system. *Archive for Rational Mechanics and Analysis*, 244(3):1019–1097, 2022.

[15] Jean-Michel Coron, Long Hu, and Guillaume Olive. Finite-time boundary stabilization of general linear hyperbolic balance laws via fredholm backstepping transformation. *Automatica*, 84:95–100, 2017.

[16] Jean-Michel Coron and Qi Lü. Local rapid stabilization for a Korteweg-de Vries equation with a Neumann boundary control on the right. *Journal de Mathématiques Pures et Appliquées*, 102(6):1080–1120, 2014.

[17] Jean-Michel Coron and Qi Lü. Fredholm transform and local rapid stabilization for a Kuramoto-Sivashinsky equation. *Journal of Differential Equations*, 259(8):3683–3729, 2015.

[18] Joachim Deutscher and Jakob Gabriel. Fredholm backstepping control of coupled linear parabolic pdes with input and output delays. *IEEE Transactions on Automatic Control*, 65(7):3128–3135, 2019.

[19] Klaus-Jochen Engel and Rainer Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, volume 194. Springer Science & Business Media, 1999.

[20] Ronald Aylmer Fisher. The wave of advance of advantageous genes. *Annals of Eugenics*, 7(4):353–369, 1937.

[21] Ludovick Gagnon, Amaury Hayat, Shengquan Xiang, and Christophe Zhang. Fredholm transformation on laplacian and rapid stabilization for the heat equations. *Journal of Functional Analysis*, 2022.

[22] Ludovick Gagnon, Amaury Hayat, Shengquan Xiang, and Christophe Zhang. Fredholm backstepping for critical operators and application to rapid stabilization for the linearized water waves. *Annales de l'Institut Fourier*, 2024.

[23] Ludovick Gagnon, Pierre Lissy, and Swann Marx. A fredholm transformation for the rapid stabilization of a degenerate parabolic equation. *SIAM Journal on Control and Optimization*, 59(5):3828–3859, 2021.

[24] Amaury Hayat and Epiphane Loko. Fredholm backstepping and rapid stabilization of general linear systems. *preprint*, 2024. HAL preprint.

[25] Daniel Henry. *Geometric Theory of Semilinear Parabolic Equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag Berlin Heidelberg, 1981.

[26] Long Hu, Florent Di Meglio, Rafael Vazquez, and Miroslav Krstic. Control of homodirectional and general heterodirectional linear coupled hyperbolic pdes. *IEEE Transactions on Automatic Control*, 61(11):3301–3314, 2015.

[27] A. Kolmogorov, I. Petrovskii, and N. Piskunov. A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem. In V. M. Tikhomirov, editor, *Selected Works of A. N. Kolmogorov I*, pages 248–270. Kluwer, 1991. Translated by V. M. Volosov from Bull. Moscow Univ., Math. Mech. 1, 1–25, 1937.

[28] Vilmos Komornik. Rapid boundary stabilization of linear distributed systems. *SIAM journal on control and optimization*, 35(5):1591–1613, 1997.

[29] Miroslav Krstic and Andrey Smyshlyaev. *Boundary Control of PDEs: A Course on Backstepping Designs*, volume 16 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[30] Karl Kunisch, Donato Vásquez-Varas, and Daniel Walter. Learning optimal feedback operators and their sparse polynomial approximations. *Journal of Machine Learning Research*, 24(301):1–38, 2023.

[31] Yoshiki Kuramoto. Diffusion-induced chaos in reaction systems. *Progress of Theoretical Physics Supplement*, 1978.

[32] Irena Lasiecka and Roberto Triggiani. *Control theory for partial differential equations: continuous and approximation theories. 1: Abstract parabolic systems*, volume 74 of *Encycl. Math. Appl.* Cambridge: Cambridge University Press, 2000.

[33] Irena Lasiecka and Roberto Triggiani. *Control theory for partial differential equations: continuous and approximation theories. 2: Abstract hyperbolic-like systems over a finite time horizon*, volume 75 of *Encycl. Math. Appl.* Cambridge: Cambridge University Press, 2000.

[34] Jacques Louis Lions. *Optimal control of systems governed by partial differential equations*, volume 170. Springer, 1971.

[35] Pierre Lissy and Claudia Moreno. Rapid stabilization of a degenerate parabolic equation using a backstepping approach: the case of a boundary control acting at the degeneracy. *Mathematical Control and Related Fields*, pages 0–0, 2023.

[36] Hanbing Liu, Gengsheng Wang, Yashan Xu, and Huaiqiang Yu. Characterizations of complete stabilizability. *SIAM Journal on Control and Optimization*, 60(4):2040–2069, 2022.

[37] Wei-Jiu Liu and Miroslav Krstić. Stability enhancement by boundary control in the kuramoto–sivashinsky equation. *Nonlinear Analysis: Theory, Methods & Applications*, 43(4):485–507, 2001.

[38] Yaxing Ma, Gengsheng Wang, and Huaiqiang Yu. Feedback law to stabilize linear infinite-dimensional systems. *Mathematical Control and Related Fields*, 13(3):1160–1183, 2023.

[39] Anna Maria Micheletti. Perturbazione dello spettro dell'operatore di laplace, in relazione ad una variazione del campo. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, 26(1):151–169, 1972.

[40] Themistocles M. Rassias Mircea Craioveanu, Mircea Puta. *Old and New Aspects in Spectral Geometry.* Springer, 2001.

[41] Hoai-Minh Nguyen. Rapid stabilization and finite time stabilization of the bilinear Schrödinger equation. *arXiv preprint arXiv:2405.10002*, 2024.

[42] Hoai-Minh Nguyen. Stabilization of control systems associated with a strongly continuous group. *arXiv preprint arXiv:2402.07560*, 2024.

[43] Amnon Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[44] Jean-Pierre Raymond. Stabilizability of infinite dimensional systems by finite dimensional control. *Computational Methods in Applied Mathematics*, 19(2):267–282, 2019.

[45] G. I. Sivashinsky. On flame propagation under conditions of stoichiometry. *SIAM Journal on Applied Mathematics*, 1980.

[46] G.I. Sivashinsky. Nonlinear analysis of hydrodynamic instability in laminar flames—i. derivation of basic equations. *Acta Astronautica*, 1977.

[47] Marshall Slemrod. The linear stabilization problem in Hilbert space. *Journal of Functional Analysis*, 11(3):334–345, 1972.

[48] Emmanuel Trélat. Stabilization of semilinear PDEs, and uniform decay under discretization. In *Evolution equations: long time behavior and control*, volume 439, pages 31–76. Cambridge University Press, 2017.

[49] Emmanuel Trélat, Gengsheng Wang, and Yashan Xu. Characterization by observability inequalities of controllability and stabilization properties. *Pure and Applied Analysis*, 2(1):93–122, 2019.
[50] Marius Tucsnak and George Weiss. *Observation and control for operator semigroups*. Springer Science & Business Media, 2009.
[51] Karen Uhlenbeck. Generic properties of eigenfunctions. *American Journal of Mathematics*, 98(4):1059–1078, 1976.
[52] Jose Manuel Urquiza. Rapid exponential feedback stabilization with unbounded control operators. *SIAM journal on control and optimization*, 43(6):2233–2244, 2005.
[53] Ambroise Vest. Rapid stabilization in a semigroup framework. *SIAM Journal on Control and Optimization*, 51(5):4169–4188, 2013.
[54] Shengquan Xiang. Null controllability of a linearized korteweg–de vries equation by backstepping approach. *SIAM Journal on Control and Optimization*, 57(2):1493–1515, 2019.
[55] Shengquan Xiang. Small-time local stabilization of the two-dimensional incompressible navier–stokes equations. *Annales de l'Institut Henri Poincaré C*, 40(6):1487–1511, 2023.
[56] Shengquan Xiang. Quantitative rapid and finite time stabilization of the heat equation. *ESAIM: Control, Optimisation and Calculus of Variations*, 30:40, 2024.
[57] Jerzy Zabczyk. *Mathematical control theory*. Springer, 2020.