Boundary Stabilization of Star-Shaped Saint-Venant Networks with Combined Subcritical and Supercritical Channels

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ARTICLE INFO

Keywords:
Star-shaped, Saint-Venant equations, Lyapunov approach, Stabilization, Feedback control

ABSTRACT

In this work, we consider the boundary stabilization of a star-shaped water flow network composed by n ($n \geq 3$) channels. Each channel is modeled by Saint-Venant equations with arbitrary friction and slope. Among which, two channels are in supercritical regime, while the remaining $n-2$ channels are in subcritical regime. We show that in this case, one only needs to apply a static feedback control at the inlet of a supercritical channel to achieve the exponential stability of the non-uniform steady-states in the H^2 norm. The main tool we employ is the Lyapunov approach. To validate our theoretical results, a numerical illustration is also given.

1. Introduction

Saint-Venant equations are first order quasilinear hyperbolic systems that describe the flow of shallow water in open channels. They are widely used in hydraulic engineering about rivers, canals, and other waterways (see [11]). In order to improve the authenticity of the model and expand the application scope of the Saint-Venant equations, more and more scientists are interested in problems related to the dynamics of the water flows in networks.

The research about water flows in network usually builds upon the existing results developed in a single channel framework. The initial researches on the static feedback control for the exponential stability of the Saint-Venant equations were based on the simplest model, without considering friction and slope (known as equations without source terms). Among others, one can look at [6, 7, 8] in which Coron et al. discovered a generic H^2-Lyapunov function that can handle feedback control problems in such hyperbolic systems with no source term. Then Bastin et al. in [1] proposed a sufficient condition for the existence of a Lyapunov function for hyperbolic systems with source terms and provided in [3] a new explicit Lyapunov function for the exponential stability of 2×2 density-velocity systems with dissipative source terms representing friction effects, including the Saint-Venant equations without slope. Later on, Hayat et al. in [17] extended the Lyapunov approach to the case of arbitrary slope and friction, and they further apply this theory to more general density-velocity systems in [18], including isentropic Euler equations, traffic flow, etc.

Regarding the boundary stabilization of networks, most of the literature focus on cascade case. In [10], Halleux et al. considered two cascade channels without source terms. Then in [4], Bastin et al. addressed the exponential stability of the linearized Saint-Venant equations for a sloping channel in the L^2 norm. To ensure uniform steady states, they consider a special case where the friction and the slope “compensate" each other and the analysis was also extended to the case of n cascade channels. In the field of PI control, Trinh et al. studied in [23] the output regulation for a cascaded network of Saint-Venant equations without source terms and Hayat et al. in [16] with source terms. For the study of the controllability and stability of star-shaped models, Leugering et al. in [20] developed a star-shaped model, but study the stabilization only for a single channel. Trinh et al. designed in [22] a PI controller for Saint-Venant equations without source terms. Gugat et al. in [13] demonstrated the exponential decay of the flows of the gas networks in the L^2 norm for arbitrarily long pipes. Concerning the boundary controllability of networks, one can refer to [12] and the references therein.

The above works focus on the case where all the channels are in subcritical regime. When subcritical channels and supercritical channels are combined, Gugat et al. constructed in [14] a star-shaped gas pipeline transportation

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model combining supersonic and subsonic flows, and addressed the well-posedness problem with designed coupling conditions at the junction.

In this paper, we explore the use of boundary feedback controls for the stabilization of a star-shaped network with arbitrary slope and friction. This model includes n channels connected by one junction. In this system, two channels are supposed to be in supercritical regime, while the other $n-2$ channels are in subcritical regime. At the junction, there is no control and the boundary conditions are given by the physic of the problem: the conservation of mass and continuity of the water head (see [9, 19]). Surprisingly, in the case where the branches converge to the main stream, the system can be completely stabilized without control at the junction. The only requirement is to apply a single control at the inlet of the supercritical branch. We give explicit conditions on this control. To do so, we derive an efficient Lyapunov function inspired from [1, 17] and obtained the local exponential stability in the sense of H^2 norm.

The structure of the article is the following. In Section 2, we present the problem and our main results. In Section 3, we first deal with the linearized system around any given non-uniform steady states. Here, we construct an efficient Lyapunov function to analyze the global stabilization of the linearized system. Based on this, in Section 4, we explore the local exponential stability for the nonlinear case using a similar Lyapunov function. A numerical illustration is presented to validate our theoretical results in Section 5. Finally, the Appendix contains some technical results and computations

2. The Statement of the Problem and Main Results

We consider the star-shaped network depicted in Figure 1, composed of two channels in the supercritical regime and $n-2$ channels in the subcritical regime. Specifically, Channel 1 and Channel 2 are in supercritical regime, while Channels j (here and hereafter, $j \in \{3, \cdots, n\}$) are in subcritical regime. The $n-1$ branches converge to the main Channel 1. Note that as soon as Channel 2 is in supercritical regime, then Channel 1 has to be in supercritical regime, unless a hydraulic jump (i.e., a shock) occurs at the junction or within a channel (see (2.2)).

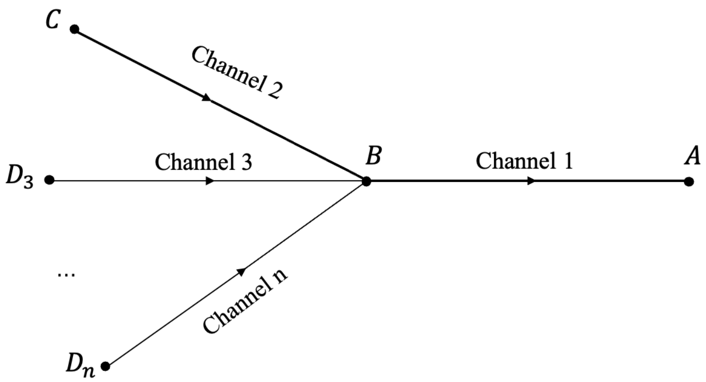

Figure 1: Star-shaped network with all branches converging to the main stream

Each channel is described by Saint-Venant equations with (possibly different) friction and slope defined on $[0,+\infty) \times [0, L_i]$, (here and hereafter, $i \in \{1, \cdots, n\}$) as follows

$$\begin{aligned}
&\partial_t H_i + \partial_x (H_i V_i) = 0, \\
&\partial_t V_i + V_i \partial_x V_i + g \partial_x H_i + g \left(\frac{C_i V_i^2}{H_i} - S_i(x) \right) = 0,
\end{aligned} \tag{2.1}$$

where L_i is the length of Channel i, $H_i = H_i(t,x)$ is the height of the water, $V_i = V_i(t,x)$ is the horizontal water velocity, C_i is the friction coefficient, g is the gravitational acceleration and $S_i(x) \in C^2([0, L_i])$ is the source term corresponding to the slope. The model includes one junction B and n single nodes A, C and D_j. In practice, it is difficult and cumbersome to apply any controls at the junction of a star-shaped network. In this work, we will show that the system (2.1) can still be stabilized in the case where the inflows of D_j are constants and the control is only applied at node C, i.e., the inlet of Channel 2. We consider the following boundary conditions at different nodes

$$\begin{aligned} C &: V_2(t,0) = \mathscr{G}(H_2(t,0)), \quad H_2(t,0) = \mathscr{B}(H_2(t,L_2)),\\ B &: H_2(t,L_2)V_2(t,L_2) + \sum_{j=3}^{n} H_j(t,L_j)V_j(t,L_j) = H_1(t,0)V_1(t,0),\\ &\quad H_1(t,0) = H_2(t,L_2) = H_j(t,L_j),\\ D_j &: H_j(t,0)V_j(t,0) = Q_j, \end{aligned} \tag{2.2}$$

where the functions $\mathscr{G},\mathscr{B}:\mathbb{R}\to\mathbb{R}$ are of class C^2, and Q_j are positive constants. The boundary conditions at junction B are respectively the conservation of mass and the continuity of the water head [9, 19].

Remark 2.1. The first boundary condition at node C: $V_2(t,0) = \mathscr{G}(H_2(t,0))$ is imposed by the physics of the system, for instance the Torricelli law if the water is coming from a larger basin upstream, in which case $\mathscr{G} : x \to \sqrt{2gx}$. In the following we will not assume any knowledge on $\mathscr{G}$. For the practical implementation of the control, we refer to [5].

Remark 2.2. Due to the consideration of supercritical flow in the system, the feedback control $H_2(t,0) = \mathscr{B}(H_2(t,L_2))$ at node C is a non-collocated control. Other boundary conditions at node C would also work, such as

$$V_2(t,0) = \mathscr{B}_1(V_2(t,L_2)),\ H_2(t,0) = \mathscr{B}_2(H_2(t,L_2)). \tag{2.3}$$

However, the first condition would be less physical as it would be nonlocal.

The steady states of this system are a couple of time-invariant non-uniform state $(H_i^*(x), V_i^*(x))$ which satisfy

$$\begin{aligned} &(H_i^* V_i^*)_x = 0,\\ &\left(\frac{V_i^{*2}}{2} + gH_i^*\right)_x + g\left(\frac{C_i V_i^{*2}}{H_i^*} - S_i\right) = 0, \end{aligned} \tag{2.4}$$

together with

$$\begin{aligned} &V_2^*(0) = \mathscr{G}(H_2^*(0)),\\ &H_2^*(L_2)V_2^*(L_2) + \sum_{j=3}^{n} H_j^*(L_j)V_j^*(L_j) = H_1^*(0)V_1^*(0),\\ &H_1^*(0) = H_2^*(L_2) = H_j^*(L_j),\\ &H_j^*(0)V_j^*(0) = Q_j. \end{aligned} \tag{2.5}$$

This can be equivalently reformulated as

$$(V_i^{*2} - g\frac{Q_i}{V_i^*})(V_i^*)_x = -V_i^*(\frac{gC_iV_i^{*3}}{Q_i} - gS_i), \tag{2.6a}$$
$$V_2^*(0) = \mathscr{G}(H_2^*(0)), \tag{2.6b}$$
$$H_1^*(0) = H_2^*(L_2) = H_j^*(L_j), \tag{2.6c}$$
$$H_j^*(x)V_j^*(x) = Q_j, \tag{2.6d}$$
$$H_2^*(x)V_2^*(x) = H_2^*(0)V_2^*(0) =: Q_2, \tag{2.6e}$$
$$H_1^*(x)V_1^*(x) = H_2^*(0)V_2^*(0) + \sum_{j=3}^{n} Q_j =: Q_1. \tag{2.6f}$$

Note that the first equation can be singular if there are points such that $V_i^2 - gQ_i/V_i = 0$ and lead to non-classical solutions (such as hydraulic jumps, see [5]).

In this paper, we focus on physical steady states which satisfy $H_i^*(x) > 0$, $V_i^*(x) > 0$. Here and in the following, j will always refer to the channels 3 to n and l to channels 1 and 2, and we will not repeat. Following these notations, the systems are in the subcritical case for all $x \in [0, L_j]$ and in the supercritical case for $x \in [0, L_l]$, i.e., the following conditions hold

$$V_l^{*2}(x) - gH_l^*(x) > 0, \ \ gH_j^*(x) - V_j^{*2}(x) > 0. \tag{2.7}$$

Subcritical flows (also called fluvial regime) are dominated by gravitational forces and are typically slower with waves that can propagate in both directions, supercritical flows (also called torrential regime) are dominated by inertial forces and behaves as rapid flows, in this case waves can only propagate forward. Supercritical and subcritical flow can be described respectively by the two conditions (2.7). They can equivalently be described by the Froude number (one can refer to the description in [21]). Thanks to the condition (2.7), the equations (2.4)–(2.5) satisfied by the steady states can be rewritten as

$$(V_i^*)_x = \frac{V_i^*(\frac{gC_iV_i^{*3}}{Q_i} - gS_i)}{g\frac{Q_i}{V_i^*} - V_i^{*2}} \tag{2.8}$$

together with (2.6b)–(2.6f).

Remark 2.3. One can note that these equations may have several solutions but, provided with given boundary data $H_2^*(0) = H_{2,0}^* > 0$ and $Q_j > 0$, $j \in \{3, ..., N\}$, there is a unique solution (H_i^*, V_i^*) (that is additionally C^3) defined respectively on $[0, L_i]$ (possibly infinite but when the friction is stronger than the slope, this may imply a limit on the lengths L_i, see [3, 17])). Indeed, from (2.6b), $V_2^*(0)$ is fixed (thus Q_2 is determined by (2.6e)) and using Cauchy-Lipschitz theorem, there is a unique $V_2^* \in C^3([0, L_2])$ (see [15]), as a consequence from (2.6e), one obtains a unique $H_2^* \in C^3([0, L_2])$. From (2.6c), one obtains the value of $H_1^*(0)$ and $H_j^*(L_j)$, which give the value of $V_1^*(0)$ and $V_j^*(L_j)$ from (2.6d) and (2.6f). Solving (2.8) for V_1^* and V_j^*, one can obtain H_1^* and H_j^* from (2.6f) and (2.6d) again. We thus have the existence and uniqueness of the steady state, one can also refer to [24] for more details. We will study the stabilization around an arbitrary steady state in the following.

In order to facilitate our study, using a scaling, we can transfer the system (2.1)–(2.2) to a new one in which the length of each channel is unit. To that end, we introduce for the i th channel the new variable: $\bar{x} := x/L_i$ and denote by $\bar{H}_i(t, \bar{x}) := H_i(t, L_i\bar{x})$, $\bar{V}_i(t, \bar{x}) := V_i(t, L_i\bar{x})$. Without ambiguity, we still denote by x the space variable in the following, then we obtain

$$\begin{aligned} &\partial_t \bar{H}_i + \frac{1}{L_i}\partial_x(\bar{H}_i\bar{V}_i) = 0, \\ &\partial_t \bar{V}_i + \frac{1}{L_i}\bar{V}_i\partial_x\bar{V}_i + \frac{g}{L_i}\partial_x\bar{H}_i + g\left(\frac{C_i\bar{V}_i^2}{\bar{H}_i} - S_i(L_ix)\right) = 0, \end{aligned} \tag{2.9}$$

and the boundary conditions

$$\begin{aligned} C\ &: \bar{V}_2(t, 0) = \mathscr{G}(\bar{H}_2(t, 0)), \quad \bar{H}_2(t, 0) = \mathscr{B}(\bar{H}_2(t, 1)), \\ B\ &: \bar{H}_2(t, 1)\bar{V}_2(t, 1) + \sum_{j=3}^{n} \bar{H}_j(t, 1)\bar{V}_j(t, 1) = \bar{H}_1(t, 0)\bar{V}_1(t, 0), \\ &\quad \bar{H}_1(t, 0) = \bar{H}_2(t, 1) = \bar{H}_j(t, 1), \\ D_j\ &: \bar{H}_j(t, 0)\bar{V}_j(t, 0) = Q_j, \end{aligned} \tag{2.10}$$

Noticing that if we denote by $(\bar{H}_i^*, \bar{V}_i^*)$ the steady states of the new system (2.9)–(2.10), then

$$\bar{H}_i^*(x) = H_i^*(L_ix), \quad \bar{V}_i^*(x) = V_i^*(L_ix), \quad x \in [0, 1]. \tag{2.11}$$

We give the definition of the local exponential stability of the steady states for the H^2 norm. For any given initial data

$$H_i(0, x) = H_i^0(x), \ V_i(0, x) = V_i^0(x), \ x \in [0, L_i] \tag{2.12}$$

with $(H_i^0, V_i^0) \in H^2((0, L_i); \mathbb{R}^2)$, we have

Definition 1. *The steady state $(H_i^*(x), V_i^*(x))$ of the system (2.1), (2.2) and (2.12) is (locally) exponentially stable for the H^2 norm if there exist $\delta > 0$, $\nu > 0$ and $C > 0$ such that, for any $(H_i^0(x), V_i^0(x)) \in H^2((0, L_i); \mathbb{R}^2)$ satisfying*

$$\|H_i^0 - H_i^*\|_{H^2((0,L_i);\mathbb{R})} + \|V_i^0 - V_i^*\|_{H^2((0,L_i);\mathbb{R})} < \delta \tag{2.13}$$

and the first order compatibility conditions associated to the system (2.1)–(2.2) (see [2, Section 4.5.2]), there exists a unique solution $(H_i(t,\cdot), V_i(t,\cdot))$ to the Cauchy problem (2.1), (2.2) and (2.12) and it satisfies

$$\begin{aligned} &\sum_{i=1}^{n}\Big(\|H_i(t,\cdot) - H_i^*\|_{H^2((0,L_i);\mathbb{R})} + \|V_i(t,\cdot) - V_i^*\|_{H^2((0,L_i);\mathbb{R})}\Big) \\ &\qquad\qquad \leq Ce^{-\nu t}\left(\sum_{i=1}^{n}\Big(\|H_i^0 - H_i^*\|_{H^2((0,L_i);\mathbb{R})} + \|V_i^0 - V_i^*\|_{H^2((0,L_i);\mathbb{R})}\Big)\right). \end{aligned} \tag{2.14}$$

We have the main theorem

Theorem 2.1. *For any boundary feedback control $\mathscr{B}$ satisfying*

$$\mathscr{B}'(H_2^*(L_2))^2 < \left(\mathscr{G}'(H_2^*(0))^2 + \frac{g}{H_2^*(0)}\right)^{-1}\frac{2g}{E}, \tag{2.15}$$

where E is a constant and

$$E = H_1^*(0)e^{\int_0^{L_2}\left|\frac{\varphi_2(x)\delta_{12}(x)}{\lambda_{12}(x)} + \frac{\varphi_2^{-1}(x)\gamma_{22}(x)}{\lambda_{22}(x)}\right|ds}\left(e^{-2\int_0^{L_2}\frac{\gamma_{12}(s)}{\lambda_{12}(s)}ds} + e^{-2\int_0^{L_2}\frac{\delta_{22}(s)}{\lambda_{22}(s)}ds}\right), \tag{2.16}$$

here γ_{12}, γ_{22}, δ_{12}, δ_{22}, λ_{12}, λ_{22} and $\varphi_2(x)$ are functions given below by (3.9), (3.10) and (3.11) depending only on the physical parameters of the system for Channel 2, the nonlinear hyperbolic system (2.1), (2.2) and (2.12) is exponentially stable for the H^2 norm .

Remark 2.4 (Single control). Note that for any given physical constraint $\mathscr{G}$, there exists a range of control $\mathscr{B}$ such that condition (2.15) is satisfied.

In the following, we only need to consider the new system (2.9)–(2.10).

3. A Lyapunov Function for the Linearized System

We first focus on the stabilization of the linearization of the new system (2.9)–(2.10). To that end, we define the perturbations h_i and v_i as

$$h_i(t,x) = \bar{H}_i(t,x) - \bar{H}_i^*(x), \quad v_i(t,x) = \bar{V}_i(t,x) - \bar{V}_i^*(x), \quad x \in [0,1]. \tag{3.1}$$

Then the linearization of (2.9) around the steady state is

$$\begin{aligned} &\partial_t h_i + \frac{1}{L_i}\bar{V}_i^*\partial_x h_i + \frac{1}{L_i}\bar{H}_i^*\partial_x v_i + \frac{1}{L_i}(\partial_x \bar{V}_i^*)h_i + \frac{1}{L_i}(\partial_x \bar{H}_i^*)v_i = 0, \\ &\partial_t \bar{v}_i + \frac{1}{L_i}g\partial_x h_i + \frac{1}{L_i}\bar{V}_i^*\partial_x v_i - gC_i\frac{\bar{V}_i^{*2}}{\bar{H}_i^{*2}}h_i + \left(\frac{1}{L_i}\partial_x\bar{V}_i^* + 2gC_i\frac{\bar{V}_i^*}{\bar{H}_i^*}\right)v_i = 0. \end{aligned} \tag{3.2}$$

The boundary conditions (2.2) become

$$
\begin{aligned}
C\ &: v_2(t,0) = k_1 h_2(t,0), \quad h_2(t,0) = k_2 h_2(t,1),\\
B\ &: v_1(t,0) = v_2(t,1) + \sum_{j=3}^{n} v_j(t,1),\\
&\quad h_1(t,0) = h_2(t,1) = h_j(t,1),\\
D_j\ &: h_j(t,0) = -\frac{\bar{H}_j^*(0)}{\bar{V}_j^*(0)} v_j(t,0),
\end{aligned}
\tag{3.3}
$$

where $k_1 = \mathscr{G}'(\bar{H}_2^*(0))$ is imposed, and $k_2 = \mathscr{B}'(\bar{H}_2^*(1))$ is the tuning parameter to be determined. We consider the following initial condition for the linearized system (3.2)–(3.3)

$$
h_i(0,x) = h_i^0(x), \quad v_i(0,x) = v_i^0(x) \tag{3.4}
$$

satisfying

$$
(h_i^0(x), v_i^0(x)) \in L^2((0,1);\mathbb{R}^2). \tag{3.5}
$$

As $(\bar{H}_i^*, \bar{V}_i^*)$ is C^3 in $[0,1]$, the Cauchy problem (3.2)–(3.4) has a unique solution in $C^0([0,+\infty); \prod_{i=1}^{n} L^2((0,1),\mathbb{R}^2))$ (see [2, Appendix A-B]). To further simplify the linearized system (3.2), we transform it into the following Riemann coordinates representation with

$$
\begin{aligned}
\xi_{1i}(t,x) &= L_i\left(v_i(t,x) + h_i(t,x)\sqrt{\frac{g}{\bar{H}_i^*(x)}}\right),\\
\xi_{2i}(t,x) &= L_i\left(v_i(t,x) - h_i(t,x)\sqrt{\frac{g}{\bar{H}_i^*(x)}}\right).
\end{aligned}
\tag{3.6}
$$

which is an invertible transformation with inverse

$$
\begin{aligned}
h_i(t,x) &= \frac{\xi_{1i}(t,x) - \xi_{2i}(t,x)}{2L_i}\sqrt{\frac{\bar{H}_i^*(x)}{g}},\\
v_i(t,x) &= \frac{\xi_{1i}(t,x) + \xi_{2i}(t,x)}{2L_i}.
\end{aligned}
\tag{3.7}
$$

The linearized system (3.2) can be rewritten in the following characteristic form as

$$
\begin{aligned}
&\partial_t \xi_{1l} + \lambda_{1l}(x)\partial_x \xi_{1l} + \gamma_{1l}(x)\xi_{1l} + \delta_{1l}(x)\xi_{2l} = 0,\\
&\partial_t \xi_{2l} + \lambda_{2l}(x)\partial_x \xi_{2l} + \gamma_{2l}(x)\xi_{1l} + \delta_{2l}(x)\xi_{2l} = 0,\\
&\partial_t \xi_{1j} + \lambda_{1j}(x)\partial_x \xi_{1j} + \gamma_{1j}(x)\xi_{1j} + \delta_{1j}(x)\xi_{2j} = 0,\\
&\partial_t \xi_{2j} - \lambda_{2j}(x)\partial_x \xi_{2j} + \gamma_{2j}(x)\xi_{1j} + \delta_{2j}(x)\xi_{2j} = 0,
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
\lambda_{1i}(x) &= \frac{1}{L_i}\left(\bar{V}_i^*(x) + \sqrt{g\bar{H}_i^*(x)}\right) > 0,\\
\lambda_{2l}(x) &= \frac{1}{L_l}\left(\bar{V}_l^*(x) - \sqrt{g\bar{H}_l^*(x)}\right) > 0,\\
\lambda_{2j}(x) &= \frac{1}{L_j}\left(-\bar{V}_j^*(x) + \sqrt{g\bar{H}_j^*(x)}\right) > 0
\end{aligned}
\tag{3.9}
$$

and

$$\begin{aligned}
\gamma_{1i}(x) &= g\frac{C_i\bar{V}_i^{*2}}{\bar{H}_i^*}\left[-\frac{3}{4(\sqrt{g\bar{H}_i^*}+\bar{V}_i^*)}+\frac{1}{\bar{V}_i^*}-\frac{1}{2\sqrt{g\bar{H}_i^*}}\right]+\frac{3g\bar{S}_i}{4(\sqrt{g\bar{H}_i^*}+\bar{V}_i^*)},\\
\delta_{1i}(x) &= g\frac{C_i\bar{V}_i^{*2}}{\bar{H}_i^*}\left[-\frac{1}{4(\sqrt{g\bar{H}_i^*}+\bar{V}_i^*)}+\frac{1}{\bar{V}_i^*}+\frac{1}{2\sqrt{g\bar{H}_i^*}}\right]+\frac{g\bar{S}_i}{4(\sqrt{g\bar{H}_i^*}+\bar{V}_i^*)},\\
\gamma_{2i}(x) &= g\frac{C_i\bar{V}_i^{*2}}{\bar{H}_i^*}\left[\frac{1}{4(\sqrt{g\bar{H}_i^*}-\bar{V}_i^*)}+\frac{1}{\bar{V}_i^*}-\frac{1}{2\sqrt{g\bar{H}_i^*}}\right]-\frac{g\bar{S}_i}{4(\sqrt{g\bar{H}_i^*}-\bar{V}_i^*)},\\
\delta_{2i}(x) &= g\frac{C_i\bar{V}_i^{*2}}{\bar{H}_i^*}\left[\frac{3}{(\sqrt{g\bar{H}_i^*}-\bar{V}_i^*)}+\frac{1}{\bar{V}_i^*}+\frac{1}{2\sqrt{g\bar{H}_i^*}}\right]-\frac{3g\bar{S}_i}{4(\sqrt{g\bar{H}_i^*}-\bar{V}_i^*)}.
\end{aligned} \tag{3.10}$$

Define furthermore

$$\varphi_{1l}(x)=e^{\int_0^x\frac{\gamma_{1l}(s)}{\lambda_{1l}(s)}ds},\ \varphi_{2l}(x)=e^{\int_0^x\frac{\delta_{2l}(s)}{\lambda_{2l}(s)}ds},\ \varphi_l(x)=\frac{\varphi_{1l}(x)}{\varphi_{2l}(x)}=e^{\int_0^x\left(\frac{\gamma_{1l}(s)}{\lambda_{1l}(s)}-\frac{\delta_{2l}(s)}{\lambda_{2l}(s)}\right)ds} \tag{3.11}$$

and

$$\varphi_{1j}(x)=e^{\int_0^x\frac{\gamma_{1j}(s)}{\lambda_{1j}(s)}ds},\ \varphi_{2j}(x)=e^{-\int_0^x\frac{\delta_{2j}(s)}{\lambda_{2j}(s)}ds},\ \varphi_j(x)=\frac{\varphi_{1j}(x)}{\varphi_{2j}(x)}=e^{\int_0^x\left(\frac{\gamma_{1j}(s)}{\lambda_{1j}(s)}+\frac{\delta_{2j}(s)}{\lambda_{2j}(s)}\right)ds}. \tag{3.12}$$

We then introduce the new coordinates

$$\begin{pmatrix} y_{1i}\\ y_{2i}\end{pmatrix}=\begin{pmatrix}\varphi_{1i} & 0\\ 0 & \varphi_{2i}\end{pmatrix}\begin{pmatrix}\xi_{1i}\\ \xi_{2i}\end{pmatrix}. \tag{3.13}$$

The system (3.8) is transformed into the following system expressed in the new coordinates

$$\begin{aligned}
&\partial_t y_{1l}+\lambda_{1l}(x)\partial_x y_{1l}+a_l(x)y_{2l}=0,\\
&\partial_t y_{2l}+\lambda_{2l}(x)\partial_x y_{2l}+b_l(x)y_{1l}=0,\\
&\partial_t y_{1j}+\lambda_{1j}(x)\partial_x y_{1j}+a_j(x)y_{2j}=0,\\
&\partial_t y_{2j}-\lambda_{2j}(x)\partial_x y_{2j}+b_j(x)y_{1j}=0
\end{aligned} \tag{3.14}$$

with

$$a_i(x)=\varphi_i(x)\delta_{1i}(x),\quad b_i(x)=\varphi_i^{-1}(x)\gamma_{2i}(x). \tag{3.15}$$

We give the following definition of the exponential stability for the linearized system in the L^2 norm.

Definition 2. *The system (3.2)–(3.4) is exponentially stable in the L^2 norm if there exist $\nu>0$ and $C>0$ such that for every $(h_i^0, v_i^0)\in L^2((0,1);\mathbb{R}^2)$, the solution to the Cauchy problem (3.2)–(3.4) satisfies*

$$\sum_{i=1}^n ||(h_i(t,\cdot),v_i(t,\cdot)||_{L^2((0,1);\mathbb{R}^2)}\le Ce^{-\nu t}\sum_{i=1}^n ||(h_i^0,v_i^0)||_{L^2((0,1);\mathbb{R}^2)},\ \forall t\in[0,+\infty). \tag{3.16}$$

Then we have

Theorem 3.1. *The system (3.2)–(3.4) is exponentially stable in the L^2 norm provided that k_1 and k_2 satisfy*

$$k_2^2\left(k_1^2+\frac{g}{H_2^*(0)}\right)<\frac{2g}{H_1^*(0)e^{\int_0^1 p(s)ds}\left(e^{-2\int_0^1\frac{\gamma_{12}(s)}{\lambda_{12}(s)}ds}+e^{-2\int_0^1\frac{\delta_{22}(s)}{\lambda_{22}(s)}ds}\right)}, \tag{3.17}$$

where p, γ_{12}, δ_{22}, λ_{12} and λ_{22} are functions given by (3.9), (3.10) and (3.40).

Proof. To show the stabilization for the linearized system (3.2)–(3.4), we use the following Lyapunov function

$$V(t)=\sum_{l=1}^{2}\int_0^1\Big(f_{1l}(x)y_{1l}^2(t,x)+f_{2l}(x)y_{2l}^2(t,x)\Big)dx+\sum_{j=3}^{n}\int_0^1\Big(f_{1j}(x)y_{1j}^2(t,x)+f_{2j}(x)y_{2j}^2(t,x)\Big)dx, \tag{3.18}$$

where we define

$$f_{1l}(x)=\frac{\alpha_l}{\lambda_{1l}\eta_l},\quad f_{2l}(x)=\frac{\alpha_l}{\lambda_{2l}\eta_l} \tag{3.19}$$

and

$$f_{1j}(x)=\frac{\alpha_j}{\lambda_{1j}\eta_j},\quad f_{2j}(x)=\frac{\alpha_j\eta_j}{\lambda_{2j}}, \tag{3.20}$$

where α_l and α_j are positive constants to be chosen and η_l is the solution to

$$\begin{cases}\eta_l'=\left|\dfrac{a_l}{\lambda_{1l}}+\dfrac{b_l}{\lambda_{2l}}\right|\eta_l+\varepsilon,\\ \eta_l(0)=\dfrac{1}{\varepsilon},\end{cases} \tag{3.21}$$

while η_j is the solution to

$$\begin{cases}\eta_j'=\left|\dfrac{a_j}{\lambda_{1j}}+\dfrac{b_j}{\lambda_{2j}}\eta_j^2\right|+\varepsilon,\\ \eta_j(0)=\dfrac{\lambda_{2j}(0)}{\lambda_{1j}(0)}+\varepsilon,\end{cases} \tag{3.22}$$

where the constant $\varepsilon>0$ is sufficiently small and will be determined later on. Thanks to the fact that

$$\eta_j^0=\frac{\lambda_{2j}}{\lambda_{1j}}\varphi_j \tag{3.23}$$

is a solution to (3.22) when $\varepsilon=0$ (see [17, Lemma 5] for the details). Thus, the solution to (3.22) will exist on $[0,1]$ for ε small enough.

Note that $\bar{H}_i^*$ and $\bar{V}_i^*$ are positive and continuous in $[0,1]$, hence are uniformly bounded by below. Thus, the change of coordinates (3.6) and its inverse (3.7) are all bounded transformations from $L^2(0,1)$ to $L^2(0,1)$, the same with the change of variable (3.13) and its inverse. Then it is not difficult to check that there exists a constant $C>0$ such that

$$\frac{1}{C}\sum_{i=1}^{n}\|(h_i(t,\cdot),v_i(t,\cdot))\|_{L^2((0,1);\mathbb{R}^2)}\le V(t)\le C\sum_{i=1}^{n}\|(h_i(t,\cdot),v_i(t,\cdot))\|_{L^2((0,1);\mathbb{R}^2)} \tag{3.24}$$

for any $t\in[0,+\infty)$.

Using similar density argument as in [2, Section 2.1.3], we only need to consider the case where the solutions of (3.2)–(3.4) are C^1 with respect to both t and x. The time derivative of V along the trajectories of (3.14) is

$$
\begin{aligned}
\dot{V}(t) =& \sum_{l=1}^{2} \int_0^1 \left(2f_{1l}y_{1l}(-\lambda_{1l}\partial_x y_{1l} - a_l y_{2l}) + 2f_{2l}y_{2l}(-\lambda_{2l}\partial_x y_{2l} - b_l y_{1l})\right) dx \\
&+ \sum_{j=3}^{n} \int_0^1 \left(2f_{1j}y_{1j}(-\lambda_{1j}\partial_x y_{1j} - a_j y_{2j}) + 2f_{2j}y_{2j}(\lambda_{2j}\partial_x y_{2j} - b_j y_{1j})\right) dx \\
=& - B(t) - \sum_{i=1}^{n} \int_0^1 \begin{pmatrix} y_{1i} \\ y_{2i} \end{pmatrix}^T N_i(x) \begin{pmatrix} y_{1i} \\ y_{2i} \end{pmatrix} dx,
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
B(t) =& \sum_{l=1}^{2} \left(f_{1l}(1)\lambda_{1l}(1)y_{1l}^2(t,1) + f_{2l}(1)\lambda_{2l}(1)y_{2l}^2(t,1) - f_{1l}(0)\lambda_{1l}(0)y_{1l}^2(t,0) - f_{2l}(0)\lambda_{2l}(0)y_{2l}^2(t,0)\right) \\
&+ \sum_{j=3}^{n} \left(f_{1j}(1)\lambda_{1j}(1)y_{1j}^2(t,1) - f_{2j}(1)\lambda_{2j}(1)y_{2j}^2(t,1) - f_{1j}(0)\lambda_{1j}(0)y_{1j}^2(t,0) + f_{2j}(0)\lambda_{2j}(0)y_{2j}^2(t,0)\right)
\end{aligned}
\tag{3.26}
$$

is the term including the information of the boundary condition and

$$
N_l(x) = \begin{pmatrix} -(f_{1l}\lambda_{1l})_x & f_{1l}a_l + f_{2l}b_l \\ f_{1l}a_l + f_{2l}b_l & -(f_{2l}\lambda_{2l})_x \end{pmatrix}, \quad N_j(x) = \begin{pmatrix} -(f_{1j}\lambda_{1j})_x & f_{1j}a_j + f_{2j}b_j \\ f_{1j}a_j + f_{2j}b_j & (f_{2j}\lambda_{2j})_x \end{pmatrix}. \tag{3.27}
$$

One can easily check thanks to (3.19)–(3.22) that

$$
-(f_{1l}\lambda_{1l})_x > 0, \quad -(f_{1j}\lambda_{1j})_x > 0, \quad \det N_i > 0 \tag{3.28}
$$

showing that N_i are positive definite.

Next, we analyze the boundary part $B(t)$. Observe that it is more convenient to use the physical boundary conditions (3.3). We express (3.26) in physical coordinates, using (3.6) and (3.13). We obtain

$$
\begin{aligned}
B(t) =& \sum_{l=1}^{2} \Bigg[f_{1l}(1)\lambda_{1l}(1)\varphi_{1l}^2(1)\left(v_l(t,1) + h_l(t,1)\sqrt{\frac{g}{\bar{H}_l^*(1)}}\right)^2 \\
&+ f_{2l}(1)\lambda_{2l}(1)\varphi_{2l}^2(1)\left(v_l(t,1) - h_l(t,1)\sqrt{\frac{g}{\bar{H}_l^*(1)}}\right)^2 \\
&- f_{1l}(0)\lambda_{1l}(0)\left(v_l(t,0) + h_l(t,0)\sqrt{\frac{g}{\bar{H}_l^*(0)}}\right)^2 - f_{2l}(0)\lambda_{2l}(0)\left(v_l(t,0) - h_l(t,0)\sqrt{\frac{g}{\bar{H}_l^*(0)}}\right)^2 \Bigg] \\
&+ \sum_{j=3}^{n} \Bigg[f_{1j}(1)\lambda_{1j}(1)\varphi_{1j}^2(1)\left(v_j(t,1) + h_j(t,1)\sqrt{\frac{g}{\bar{H}_j^*(1)}}\right)^2 \\
&- f_{2j}(1)\lambda_{2j}(1)\varphi_{2j}^2(1)\left(v_j(t,1) - h_j(t,1)\sqrt{\frac{g}{\bar{H}_j^*(1)}}\right)^2 \\
&- f_{1j}(0)\lambda_{1j}(0)\left(v_j(t,0) + h_j(t,0)\sqrt{\frac{g}{\bar{H}_j^*(0)}}\right)^2 + f_{2j}(0)\lambda_{2j}(0)\left(v_j(t,0) - h_j(t,0)\sqrt{\frac{g}{\bar{H}_j^*(0)}}\right)^2 \Bigg].
\end{aligned}
\tag{3.29}
$$

Denote by

$$
r = \left(v_2(t,1), \quad v_3(t,1), \quad \cdots, \quad v_n(t,1), \quad h_1(t,0)\right)^T. \tag{3.30}
$$

Substituting (3.3) into (3.29), we get

$$B(t) = r^T M(x) r + \sum_{j=3}^{n} F_j v_j^2(t,0) + G(t), \tag{3.31}$$

where M is a $n \times n$ matrix defined in (3.36). It can be checked directly from (3.20) and (3.22) that

$$F_j = \frac{1}{\bar{V}_j^{*2}(0)} \left[\lambda_{2j}(0) f_{2j}(0) \left(\sqrt{g \bar{H}_j^*(0)} + \bar{V}_j^*(0) \right)^2 - \lambda_{1j}(0) f_{1j}(0) \left(\sqrt{g \bar{H}_j^*(0)} - \bar{V}_j^*(0) \right)^2 \right] > 0 \tag{3.32}$$

and

$$\begin{aligned} G(t) =& f_{11}(1)\lambda_{11}(1)\varphi_{11}^2(1) \left(v_1(t,1) + h_1(t,1)\sqrt{\frac{g}{\bar{H}_1^*(1)}} \right)^2 \\ &+ f_{21}(1)\lambda_{21}(1)\varphi_{21}^2(1) \left(v_1(t,1) - h_1(t,1)\sqrt{\frac{g}{\bar{H}_1^*(1)}} \right)^2 > 0. \end{aligned} \tag{3.33}$$

Denote by

$$\begin{aligned} &\lambda_{1i}(x) f_{1i}(x) \varphi_{1i}^2(x) - \lambda_{2i}(x) f_{2i}(x) \varphi_{2i}^2(x) =: Z_i(x) =: \alpha_i \widetilde{Z}_i(x), \\ &\lambda_{1i}(x) f_{1i}(x) \varphi_{1i}^2(x) + \lambda_{2i}(x) f_{2i}(x) \varphi_{2i}^2(x) =: W_i(x) =: \alpha_i \widetilde{W}_i(x). \end{aligned} \tag{3.34}$$

Using (3.19)–(3.22), one has

$$W_l(0) = 2\alpha_l \varepsilon, \quad Z_l(0) = 0. \tag{3.35}$$

Then, noting additionally that $\bar{H}_j^*(1) = \bar{H}_2^*(1) = \bar{H}_1^*(0)$ thanks to the boundary conditions (2.2), we have

$$M = \begin{pmatrix} \theta_1 & -W_1(0) & -W_1(0) & \cdots & -W_1(0) & \frac{\sqrt{g\bar{H}_1^*(0)} Z_2(1)}{\bar{H}_1^*(0)} \\ -W_1(0) & Z_3(1) - W_1(0) & -W_1(0) & \cdots & -W_1(0) & \frac{\sqrt{g\bar{H}_1^*(0)} W_3(1)}{\bar{H}_1^*(0)} \\ -W_1(0) & -W_1(0) & Z_4(1) - W_1(0) & \cdots & -W_1(0) & \frac{\sqrt{g\bar{H}_1^*(0)} W_4(1)}{\bar{H}_1^*(0)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -W_1(0) & -W_1(0) & -W_1(0) & \cdots & Z_n(1) - W_1(0) & \frac{\sqrt{g\bar{H}_1^*(0)} W_n(1)}{\bar{H}_1^*(0)} \\ \frac{\sqrt{g\bar{H}_1^*(0)} Z_2(1)}{\bar{H}_1^*(0)} & \frac{\sqrt{g\bar{H}_1^*(0)} W_3(1)}{\bar{H}_1^*(0)} & \frac{\sqrt{g\bar{H}_1^*(0)} W_4(1)}{\bar{H}_1^*(0)} & \cdots & \frac{\sqrt{g\bar{H}_1^*(0)} W_n(1)}{\bar{H}_1^*(0)} & \theta_2 \end{pmatrix}, \tag{3.36}$$

where

$$\begin{aligned} \theta_1 =& W_2(1) - W_1(0), \\ \theta_2 =& \frac{g}{\bar{H}_1^*(0)} \left(W_2(1) - \frac{\bar{H}_1^*(0)}{g} k_2^2 \left(k_1^2 + \frac{g}{\bar{H}_2^*(0)} \right) W_2(0) - W_1(0) + \sum_{j=3}^{n} Z_j(1) \right). \end{aligned} \tag{3.37}$$

We can check that under condition (3.17), the matrix M is positive definite which is crucial but the proof is quite technical, we thus give it as a lemma below. Above all, we have proved that $B(t) > 0$, this together with (3.24), (3.25) and the positive definiteness of N_i completes the proof of Theorem 3.1. □

Lemma 3.1. *The $n \times n$ matrix M defined in (3.36) is positive definite.*

Proof. Since from (3.19) and (3.34), one has

$$Z_2(1) = \frac{\alpha_2}{\eta_2(1)}(\varphi_{12}^2(1) - \varphi_{22}^2(1)), \quad W_2(1) = \frac{\alpha_2}{\eta_2(1)}(\varphi_{12}^2(1) + \varphi_{22}^2(1)), \tag{3.38}$$

where

$$\eta_2(x) = \frac{1}{\varepsilon} e^{\int_0^x p(t)\,dt} + \varepsilon \int_0^x e^{\int_s^x p(t)\,dt}\,ds \tag{3.39}$$

with

$$p(x) := \left| \frac{a_2(x)}{\lambda_{12}(x)} + \frac{b_2(x)}{\lambda_{22}(x)} \right| > 0 \tag{3.40}$$

is solution to (3.21). It follows from (3.35), (3.38) and (3.39) that

$$\begin{aligned}\theta_1 =& \frac{\alpha_2}{\eta_2(1)}(\varphi_{12}^2(1) + \varphi_{22}^2(1)) - 2\alpha_1\varepsilon \\ =& \frac{\alpha_2(\varphi_{12}^2(1) + \varphi_{22}^2(1)) - 2\alpha_1\left(e^{\int_0^1 p(t)\,dt} + \varepsilon^2 \int_0^1 e^{\int_s^1 p(t)\,dt}\,ds\right)}{\eta_2(1)}.\end{aligned} \tag{3.41}$$

From now on, we fix $\alpha_2 = 1$, which is not very limiting since a Lyapunov function is always defined up to a multiplicative constant. From the expression (3.41) we see that for any $\varepsilon > 0$, there exists δ_1 such that if $\alpha_1 \in (0, \delta_1)$, then $\theta_1 > 0$.

Let us first prove that $\det M > 0$, to that end, we perform sequentially the following five steps to matrix M that will not change the value of $\det M$:

Step1: The rows 2 to $n-1$ of matrix M minus the first row;

Step2: The first row is added sequentially by the product of $\frac{W_1(0)}{Z_j(1)}$ with the rows from 2 to $n-1$ respectively;

Step3: The last row is subtracted sequentially by the product of $\frac{W_j(1)\sqrt{g\bar{H}_1^*(0)}}{Z_j(1)\bar{H}_1^*(0)}$ with the rows from 2 to $n-1$ respectively;

Step4: The first column is added sequentially by the product of $\frac{W_2(1)}{Z_j(1)}$ with the columns from 2 to $n-1$ respectively;

Step5: The last column is subtracted sequentially by the product of $\frac{\left(W_j(1)-Z_2(1)\right)\sqrt{g\bar{H}_1^*(0)}}{Z_j(1)\bar{H}_1^*(0)}$ with the columns from 2 to $n-1$ respectively;

We can obtain the following simplified matrix

$$\widetilde{M} = \begin{pmatrix} \beta_1 & 0 & 0 & \cdots & 0 & \beta_2 \\ 0 & Z_3(1) & 0 & \cdots & 0 & 0 \\ 0 & 0 & Z_4(1) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Z_n(1) & 0 \\ \beta_3 & 0 & 0 & \cdots & 0 & \beta_4 \end{pmatrix}, \tag{3.42}$$

with (recall that $\theta_1 + W_1(0) = W_2(1)$ from (3.37))

$$\begin{aligned}
\beta_1 &= W_2(1) - W_1(0) - W_1(0)W_2(1)\sum_{j=3}^{n}\frac{1}{Z_j(1)},\\
\beta_2 &= \frac{\sqrt{g\bar{H}_1^*(0)}}{\bar{H}_1^*(0)}\left(Z_2(1) + W_1(0)\sum_{j=3}^{n}\frac{W_j(1)-Z_2(1)}{Z_j(1)}\right),\\
\beta_3 &= \frac{\sqrt{g\bar{H}_1^*(0)}}{\bar{H}_1^*(0)}\left(Z_2(1) + W_2(1)\sum_{j=3}^{n}\frac{W_j(1)}{Z_j(1)}\right),\\
\beta_4 &= \frac{g}{\bar{H}_1^*(0)}\left(W_2(1) - \frac{\bar{H}_1^*(0)}{g}k_2^2\left(k_1^2+\frac{g}{\bar{H}_2^*(0)}\right)W_2(0) - W_1(0) + \sum_{j=3}^{n}Z_j(1) - \sum_{j=3}^{n}\frac{(W_j(1)-Z_2(1))W_j(1)}{Z_j(1)}\right).
\end{aligned}\tag{3.43}$$

Noticing the expression of (3.23), we obtain that

$$\frac{\varphi_{1j}^2(1)}{\eta_j^0(1)} - \varphi_{2j}^2(1)\eta_j^0(1) = \frac{\varphi_{1j}(1)\varphi_{2j}(1)(\lambda_{1j}(1)-\lambda_{2j}(1))}{\lambda_{2j}(1)} > 0. \tag{3.44}$$

It then follows from (3.20) and (3.34) that for ε small enough

$$Z_j(1) = \alpha_j\left(\frac{\varphi_{1j}^2(1)}{\eta_j(1)} - \varphi_{2j}^2(1)\eta_j(1)\right) > 0. \tag{3.45}$$

On the other hand, from (3.41)

$$\begin{aligned}
\beta_1 =& \frac{\alpha_2(\varphi_{12}^2(1)+\varphi_{22}^2(1)) - 2\alpha_1\left(e^{\int_0^1 p(t)\,dt} + \varepsilon^2\int_0^1 e^{\int_s^1 p(t)\,dt}\,ds\right)}{\eta_2(1)}\\
&-\sum_{j=3}^{n}\frac{2\alpha_1\alpha_2\varepsilon}{Z_j(1)}\frac{(\varphi_{12}^2(1)+\varphi_{22}^2(1))}{\eta_2(1)}.
\end{aligned}\tag{3.46}$$

Thus, there exists $\delta_2 < \delta_1$ (depending on $\varepsilon > 0$) such that if $\alpha_1 \in (0,\delta_2)$, then $\beta_1 > 0$ is guaranteed. We can now check the determinant of $\widetilde{M}$. Direct computation gives

$$\begin{aligned}
\det\widetilde{M} &= \prod_{j=3}^{n}Z_j(1)(\beta_1\beta_4 - \beta_2\beta_3)\\
&= \prod_{j=3}^{n}Z_j(1)\Bigg(\frac{g}{\bar{H}_1^*(0)}\alpha_2^2\Bigg(\widetilde{W}_2^2(1) - \widetilde{Z}_2^2(1) + \widetilde{W}_2(1)\sum_{j=3}^{n}\frac{\alpha_j}{\alpha_2}\left(\widetilde{Z}_j(1) - \frac{\widetilde{W}_j^{\,2}(1)}{\widetilde{Z}_j(1)}\right)\\
&\quad - \widetilde{W}_2(0)\left(\frac{\bar{H}_1^*(0)}{g}k_2^2\left(k_1^2+\frac{g}{\bar{H}_2^*(0)}\right)\widetilde{W}_2(1)\right)\Bigg) + O(\alpha_1)\Bigg),
\end{aligned}\tag{3.47}$$

where the computation of $\det\widetilde{M}$ as well as the explicit expression of $O(\alpha_1)$ is given in the Appendix. For fixed $\alpha_2 = 1$ and ε, one can check that $O(\alpha_1)$ can be made sufficiently small by requiring α_1 small enough (see the Appendix for the details).

From (3.38), it is easy to check that

$$\widetilde{W}_2^2(1) - \widetilde{Z}_2^2(1) > 0. \tag{3.48}$$

Above all, to ensure that det $\widetilde{M} > 0$, we first need to ensure that for given k_1, there always exists k_2 such that

$$\widetilde{W}_2^2(1) - \widetilde{Z}_2^2(1) - \widetilde{W}_2(0)\left(\frac{\bar{H}_1^*(0)}{g}k_2^2\left(k_1^2 + \frac{g}{\bar{H}_2^*(0)}\right)\widetilde{W}_2(1)\right) > 0. \tag{3.49}$$

Since (3.17) holds and is a strict inequality, there exists $\varepsilon > 0$ sufficiently small such that the following holds

$$\begin{aligned} k_2^2\left(k_1^2 + \frac{g}{\bar{H}_2^*(0)}\right) <& \frac{2g}{\bar{H}_1^*(0)\left(e^{\int_0^1 p(s)ds} + \varepsilon^2\int_0^1 e^{\int_s^1 p(t)dt}ds\right)\left(e^{-2\int_0^1 \frac{\gamma_{12}(s)}{\lambda_{12}(s)}ds} + e^{-2\int_0^1 \frac{\delta_{22}(s)}{\lambda_{22}(s)}ds}\right)} \\ =& \frac{g}{\bar{H}_1^*(0)}\frac{2\varphi_{12}^2(1)\varphi_{22}^2(1)}{\varepsilon\eta_2(1)\left(\varphi_{12}^2(1) + \varphi_{22}^2(1)\right)}. \end{aligned} \tag{3.50}$$

Noticing (3.34) and (3.38), this is exactly condition (3.49). From (3.50), one can see that once the parameter k_1 is imposed, the requirements for the tuning parameter k_2 depends only on the steady state $(\bar{H}_i^*(x), \bar{V}_i^*(x))$, $x \in [0,1]$.

From (3.20) and (3.34), one obtains

$$\widetilde{Z}_j(1) = \frac{\varphi_{1j}^2(1)}{\eta_j(1)} - \varphi_{2j}^2(1)\eta_j(1), \tag{3.51}$$

$$\widetilde{W}_j(1) = \frac{\varphi_{1j}^2(1)}{\eta_j(1)} + \varphi_{2j}^2(1)\eta_j(1), \tag{3.52}$$

thus are bounded. Then, given that the remaining term of det $\widetilde{M}$ depends on α_j/α_2, there exists $\delta_3 < \delta_2$ (potentially depending on ε) such that for any $\alpha_1, \alpha_j \in (0, \delta_3)$, det $\widetilde{M} > 0$ and therefore det $M > 0$.

Next, since we have proved that the first order principal minor determinant of M, i.e., $\theta_1 > 0$ and all the remaining k-th ($k = 2, \cdots, n-1$) order principal minor determinant det M_k of M can be computed using the transformations Step 1 and Step 2 as for M and expressed as

$$\begin{aligned} \det M_k =& \prod_{j=3}^{k+1} Z_j(1)\left(W_2(1) - W_1(0) - W_1(0)W_2(1)\sum_{j=3}^{k+1}\frac{1}{Z_j(1)}\right) \\ =& \prod_{j=3}^{k+1} Z_j(1)\left(\alpha_2\widetilde{W}_2(1) - \alpha_1\widetilde{W}_1(0) - \alpha_2\widetilde{W}_1(0)\widetilde{W}_2(1)\sum_{j=3}^{k+1}\frac{\alpha_1}{\alpha_j}\frac{1}{\widetilde{Z}_j(1)}\right). \end{aligned} \tag{3.53}$$

Similarly, there exists $\delta_4 < \delta_3$ (depending on ε) such that if we let α_1 and $\alpha_1/\alpha_j \in (0, \delta_4)$ sequentially, then det $M_k > 0$ ($k = 2, \cdots, n-1$) can be guaranteed. Above all, when α_1, α_j and $\alpha_1/\alpha_j \in (0, \delta_4)$, the matrix M is positive definite.

□

4. Proof for Theorem 2.1

Now we consider the nonlinear system (2.9), we use the same series of changes of variables as (3.6) and (3.13) on (h_i, v_i) that are still defined by (3.1) so that we keep the nonlinear terms of the equations. We obtain

$$\partial_t \mathbf{y} + A(\mathbf{y}, x)\partial_x \mathbf{y} + B(\mathbf{y}, x) = 0, \quad x \in [0,1] \tag{4.1}$$

where

$$\mathbf{y} = (y_{11}, y_{21}, y_{12}, y_{22}, ..., y_{1n}, y_{2n})^T \tag{4.2}$$

and

$$A(\mathbf{y},x)=\mathbf{diag}(\Lambda_1(y_{11},y_{21},x),...,\Lambda_n(y_{1n},y_{2n},x)) \tag{4.3}$$

with

$$\Lambda_l(0,0,x)=\mathbf{diag}\{\lambda_{1l}(x),\lambda_{2l}(x)\}_{l\in\{1,2\}} \quad \text{and} \quad \Lambda_j(0,0,x)=\mathbf{diag}\{\lambda_{1j}(x),-\lambda_{2j}(x)\}_{j\in\{3,...,n\}}. \tag{4.4}$$

At the same time, we have

$$B(\mathbf{0},x)=0,\ \ \frac{\partial B}{\partial \mathbf{y}}(\mathbf{0},x)=\mathbf{diag}\left(B_i(x)\right)_{i\in\{1,2,\cdots,n\}},\quad B_i(x)=\begin{pmatrix}0 & a_i(x)\\ b_i(x) & 0\end{pmatrix}. \tag{4.5}$$

Denote by

$$\begin{aligned}\mathbf{y}^{in}(t)&=(y_{11}(t,0),y_{21}(t,0),y_{12}(t,0),y_{22}(t,0),y_{13}(t,0),y_{23}(t,1),...,y_{1n}(t,0),y_{2n}(t,1)),\\ \mathbf{y}^{out}(t)&=(y_{11}(t,1),y_{12}(t,1),y_{21}(t,1),y_{22}(t,1),y_{13}(t,1),y_{23}(t,0),...,y_{1n}(t,1),y_{2n}(t,0)),\end{aligned} \tag{4.6}$$

the nonlinear boundary conditions (2.10) can be written in the following form

$$\mathbf{y}^{in}(t)=\mathcal{H}(\mathbf{y}^{out}(t)) \tag{4.7}$$

with

$$\mathcal{H}(\mathbf{0})=\mathbf{0}. \tag{4.8}$$

For the sake of simplicity, we omit the explicit expression $\mathcal{H}$ here, which follows directly from the change of variables (3.6) and (3.13). Very similar to dealing with the linearized system in Section 3, we use the following Lyapunov function to analyze the local exponential stability of the nonlinear system:

$$V=\sum_{k=0}^{2}\sum_{i=1}^{n}\int_0^1\begin{pmatrix}\partial_t^k y_{1i}\\ \partial_t^k y_{2i}\end{pmatrix}^T E(y_{1i},y_{2i},x)^T\begin{pmatrix}f_{1i}(x) & 0\\ 0 & f_{2i}(x)\end{pmatrix}E(y_{1i},y_{2i},x)\begin{pmatrix}\partial_t^k y_{1i}\\ \partial_t^k y_{2i}\end{pmatrix}dx, \tag{4.9}$$

where $E(y_{1i},y_{2i},x)$ is such that

$$E(y_{1i},y_{2i},x)\Lambda_i(y_{1i},y_{2i},x)E(y_{1i},y_{2i},x)^{-1}\text{ is diagonal,} \tag{4.10}$$

and consequently $E(0,0,x)=I_d$, where I_d is the identity matrix.

Noticing the invertibility of transformations (3.6) and (3.13), to prove Theorem 2.1, we only need to prove the following theorem

Theorem 4.1. *Denote $k_1=\mathscr{G}'(\bar{H}_2^*(0))$ and $k_2=\mathscr{B}'(\bar{H}_2^*(1))$. For any k_1 and k_2 satisfying (3.17), the nonlinear system (4.1) and (4.7) is (locally) exponentially stable for the H^2 norm, i.e., there exist $\delta>0$, $\nu>0$ and $C_0>0$ such that, for every initial condition $(y_{1i}(0,x),y_{2i}(0,x))\in H^2((0,1);\mathbb{R}^2)$, $i\in\{1,\cdots,n\}$ satisfying $\|(y_{1i}(0,x),y_{2i}(0,x))\|_{H^2((0,1);\mathbb{R}^2)}<\delta$ and the first-order compatibility conditions associated to (4.1) (see [2]), there exists a unique solution to the Cauchy problem (4.1) and (4.7) defined on $[0,+\infty)\times[0,1]$ and it satisfies*

$$\sum_{i=1}^{n}\|(y_{1i}(t,\cdot),y_{2i}(t,\cdot))\|_{H^2((0,1);\mathbb{R}^2)}\le C_0e^{-\nu t}\left(\sum_{i=1}^{n}\|(y_{1i}(0,x),y_{2i}(0,x))\|_{H^2((0,1);\mathbb{R}^2)}\right),\ \forall t\in[0,+\infty). \tag{4.11}$$

The proof is essentially the same as in the linearized case, thanks to the robustness of basic Lyapunov functions such as (4.9). We can use for instance [2, Theorem 6.10] which states that Theorem 4.1 holds provided that

- The matrix

$$-(\Lambda(\mathbf{0},x)Q(x))'+Q(x)M_1^T(\mathbf{0},x)+M_1(\mathbf{0},x)Q(x) \tag{4.12}$$

  is positive definite, where $Q=\mathrm{diag}(f_{1i}(x),f_{2i}(x))_{i\in\{1,2,\cdots,n\}}$, $\Lambda(\mathbf{0},x)=\mathbf{diag}(\Lambda_1(0,0,x),...,\Lambda_n(0,0,x))$ and $M_1(\mathbf{0},x)=\frac{\partial B}{\partial \mathbf{y}}(\mathbf{0},x)$.

- The matrices M and $F = \mathrm{diag}(F_j)$, given respectively by (3.36), (3.32) and the symmetric matrix

$$\begin{pmatrix} f_{11}(1)\lambda_{11}(1)\varphi_{11}^2(1) + f_{21}(1)\lambda_{21}(1)\varphi_{21}^2(1) & \sqrt{\frac{g}{\bar{H}_1^*(1)}}\left[f_{11}(1)\lambda_{11}(1)\varphi_{11}^2(1) - f_{21}(1)\lambda_{21}(1)\varphi_{21}^2(1)\right] \\ \star & \frac{g}{\bar{H}_1^*(1)}\left[f_{11}(1)\lambda_{11}(1)\varphi_{11}^2(1) + f_{21}(1)\lambda_{21}(1)\varphi_{21}^2(1)\right] \end{pmatrix}$$

  are all positive definite thanks to (3.33).

Remark 4.1. The first condition is a consequence of the definition of the f_{1i} and f_{2i} given by (3.19)–(3.20) (see (3.27)). Notice that for simplicity, we give directly the sufficient conditions for the boundary part in the framework of physical coordinates.

Proof of Theorem 4.1. For clarity, we give here a sketch of proof similar to the one used in [2, Theorem 6.10]. Differentiating V along the smooth solutions of (4.1), then we get

$$\frac{dV}{dt} = -\widetilde{B}(t) - I(t) \tag{4.13}$$

with

$$\begin{aligned} \widetilde{B}(t) = {} & \sum_{k=0}^{2} \partial_t^k r^T M(x)\partial_t^k r + \left[\sum_{i=1}^{n}\left(\sum_{k=0}^{2} |\partial_t^k y_i(t,1)| + |\partial_t^k y_i(t,0)|\right)^2\right] O\left(\sum_{i=1}^{n} \|y_i(t,\cdot)\|_{C^1([0,1];\mathbb{R}^2)}\right) \\ & + \sum_{k=0}^{2}\sum_{j=3}^{n}\left[F_j\left(\partial_t^k v_j(t,0)\right)^2\right] + \sum_{k=0}^{2}\left[f_{11}(L_1)\lambda_{11}(1)(\partial_t^k y_{11}(t,1))^2 + f_{21}(1)\lambda_{21}(1)(\partial_t^k y_{21}(t,1))^2\right], \end{aligned} \tag{4.14}$$

$$I(t) = \sum_{k=0}^{2}\sum_{i=1}^{n}\int_0^1 \partial_t^k y_i^T N_i(x)\partial_t^k y_i dx + O\left(\sum_{i=1}^{n}\|y_i(t,\cdot)\|^3_{H^2((0,1);\mathbb{R}^2)}\right),$$

where r is still defined as in (3.30) and

$$y_i = (y_{1i}, y_{2i})^T$$

and $O(x)$ refers to a function such that $O(x)/|x|$ is bounded when $|x| \to 0$.

In (4.14), N_i, M are defined as in (3.27) and (3.36) respectively that are all positive definite and $F_j > 0$ by (3.32).

From Sobolev inequality one has $\|y_i(t,\cdot)\|_{C^1([0,1];\mathbb{R}^2)} \leq C\|y_i(t,\cdot)\|^2_{H^2((0,1);\mathbb{R}^2)}$. Hence, there exists $\mu > 0$ and $\delta_0 > 0$ independent of $(y_i)_i$ such that if $\sum_{i=1}^{n} \|y_i(0,\cdot)\|_{H^2((0,1);\mathbb{R}^2)} < \delta_0$ then

$$\frac{dV}{dt} \leq -\mu V + O\left(\sum_{i=1}^{n}\|y_i(t,\cdot)\|^3_{H^2((0,1);\mathbb{R}^2)}\right).$$

Since V is equivalent to $\sum_{i=1}^{n}\|y_i(t,\cdot)\|^2_{H^2((0,1);\mathbb{R}^2)}$ when $\sum_{i=1}^{n}\|y_i(0,\cdot)\|_{H^2((0,1);\mathbb{R}^2)}$ is sufficiently small (see (4.9)), there exists $\delta > 0$ such that if $\|y_i(0,\cdot)\|_{H^2((0,1);\mathbb{R}^2)} < \delta$, then

$$\frac{dV}{dt} \leq -\frac{\mu}{2}V. \tag{4.15}$$

The proof of Theorem 4.1 is complete.

□

5. Numerical Illustration

In this section, we illustrate Theorem 2.1 by providing numerical simulations of the H^2 norm of the solutions to the nonlinear system (2.1)–(2.2) (see Figure 2). Here, we consider three channels where Channel 1 and Channel 2 are in supercritical regime and Channel 3 is in subcritical regime (see Figure 1). We also provide the evolution of the velocity at the end point of each channel (see Figure 3). The steady states are chosen with initial condition $Q_3 = 1m^3.s^{-1}$ and $H_2^*(0) = 2m$ (see Remark 2.3). The friction coefficients are chosen as $C_1 = 0.3$, $C_2 = 0.2$, $C_3 = 0.01$; the lengths of the three channels are $L_1 = 1500m$, $L_2 = 1100m$, $L_3 = 1000m$; the control parameters are chosen as $k_1 = 0.1$, $k_2 = 0.01$; the slope functions are $S_i(x) = 1.2\frac{C_i V_i^{*2}(0)}{H_i^*(0)}$(here $i \in \{1, 2, 3\}$) and the acceleration of gravity is $9.81m/s^2$.

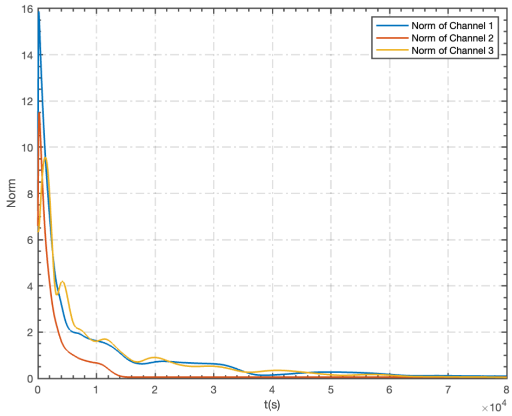

Figure 2: The variations of the norm over time

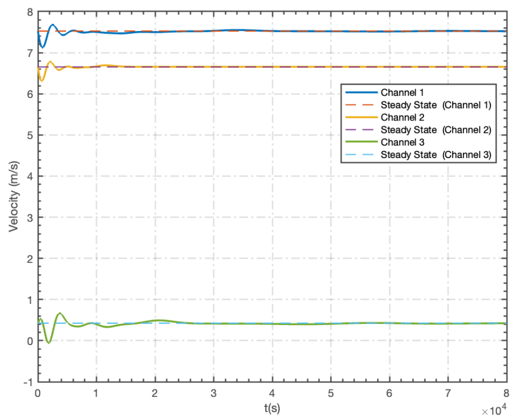

Figure 3: The evolution of the value for the velocity at the end of each channel

6. Conclusion

In this paper, our main contribution is to exhibit an explicit Lyapunov function for studying the exponential stability of the Saint-Venant system in a star-shaped network. The network is composed of both subcritical and supercritical channels connected by one junction. Surprisingly enough, we found that without applying any control at the junction but only a control at the inlet of the supercritical branch, the stability can be achieved. An intriguing question would be to consider the case where the network is only composed of subcritical channels, and to know whether the star-shaped model still achieves stability without any control at the junction.

7. Acknowledgement

The authors would like to thank the National Natural Science Foundation of China (No. 12171368 and 11971119) and Fundamental Research Funds for the Central Universities, China (2023-3-ZD-04) as well as China Scholarship Council (No. 202206260108).

Appendix A The specific expression for $O(\alpha_1)$

From (3.43), we have

$$\begin{aligned}
&\beta_1\beta_4-\beta_2\beta_3\\
=&\frac{g}{\bar{H}_1^*(0)}\left[(W_2(1)-W_1(0)-s_1W_1(0)W_2(1))\right]\left[W_2(1)-\frac{\bar{H}_1^*(0)}{g}k_2^2\left(k_1^2+\frac{g}{\bar{H}_2^*(0)}\right)W_2(0)-W_1(0)+s_3+s_2Z_2(1)\right]\\
&-\frac{g}{\bar{H}_1^*(0)}\left(Z_2(1)+(s_2-s_1Z_2(1))W_1(0)\right)\times\left(Z_2(1)+s_2W_2(1)\right)\\
=&\frac{g}{\bar{H}_1^*(0)}\Bigg(\alpha_2^2\Bigg(\widetilde{W}_2^2(1)-\widetilde{Z}_2^2(1)+\widetilde{W}_2(1)\sum_{j=3}^{n}\frac{\alpha_j}{\alpha_2}\left(\widetilde{Z}_j-\frac{\widetilde{W}_j^{\,2}(1)}{\widetilde{Z}_j(1)}\right)\\
&-\widetilde{W}_2(0)\left(\frac{\bar{H}_1^*(0)}{g}k_2^2(k_1^2+\frac{g}{\bar{H}_2^*(0)})\widetilde{W}_2(1)\right)\Bigg)+O(\alpha_1)\Bigg),
\end{aligned}\tag{A.1}$$

where

$$s_1=\sum_{j=3}^{n}\frac{1}{Z_j(1)},\quad s_2=\sum_{j=3}^{n}\frac{W_j(1)}{Z_j(1)},\quad s_3=\sum_{j=3}^{n}\frac{Z_j^2(1)-W_j^2(1)}{Z_j(1)},\tag{A.2}$$

and

$$\begin{aligned}
O(\alpha_1)=&W_1^2(0)(1+s_1W_2(1))+W_1(0)W_2(0)\left(\frac{\bar{H}_1^*(0)}{g}k_2^2\left(k_1^2+\frac{g}{\bar{H}_2^*(0)}\right)(1+s_1W_2(1))\right)\\
&-W_1(0)\Big(s_3+2s_2Z_2(1)+(2+s_2^2+s_1s_3+s_1W_2(1))W_2(1)-s_1Z_2^2(1)\Big).
\end{aligned}\tag{A.3}$$

By (3.35), we obtain

$$O(\alpha_1)=\alpha_1g_1(\alpha_2,\varepsilon)+\alpha_1^2g_2(\alpha_2,\varepsilon)\tag{A.4}$$

where

$$\begin{aligned}
g_1(\alpha_2,\varepsilon)=&-2\varepsilon\Big(s_3+2s_2Z_2(1)+(2+s_2^2+s_1s_3+s_1W_2(1))W_2(1)-s_1Z_2^2(1)\Big)\\
&+4\varepsilon^2\alpha_2\left(\frac{\bar{H}_1^*(0)}{g}k_2^2\left(k_1^2+\frac{g}{\bar{H}_2^*(0)}\right)(1+s_1W_2(1))\right),\\
g_2(\alpha_2,\varepsilon)=&4\varepsilon^2(1+s_1W_2(1)).
\end{aligned}\tag{A.5}$$

Thus, from (A.4)–(A.5), for fixed $\alpha_2=1$ and fixed ε, we can always choose α_1 (depending on ε) small enough to guarantee that $O(\alpha_1)$ is sufficiently small.

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