Data Driven Robust Optimization Exam

22/03/2020

The exam is made of two independant parts. If necessary, you can admit the results of previous questions.

Send your work (french or english) in pdf by tuesday 8:00 by mail to vincent.leclere@enpc.fr. Please make it easily readable (bonus point for latex). Contact me in case of difficulty.

Some usefull recalls.

1. An SOCP constraint take the form $a^T x + b + \|c^T x + d\|_2 \leq 0$.
2. We have

$$
\begin{aligned}
\sup_{\mathbb{Q}\in\mathbb{B}_\varepsilon(\hat{\mathbb{P}}_N)} \mathbb{E}_{\mathbb{Q}}[\ell(\boldsymbol{\xi})] \quad = \quad & \inf \quad \lambda\varepsilon + \frac{1}{N}\sum_{i=1}^{N} s_i \\
& \text{s.t.} \quad [-\ell_k]^\star(z_{ik} - \nu_{ik}) + \delta^\star(\nu_{ik}, \Xi) - \left\langle z_{ik}\, , \hat{\xi}_i \right\rangle \leq s_i \qquad \forall i \ \in [N], \forall k \in [K] \\
& \qquad\ \ \|z_{ik}\|_\star \leq \lambda
\end{aligned}
$$

   where

   - $\hat{\mathbb{P}}_N$ is the uniform measure on $\{\hat{\xi}_i\}_{i\in[N]}$, which are iid datapoints sampled from $\boldsymbol{\xi}$, such that $\text{supp}(\boldsymbol{\xi}) \subset \Xi$;
   - $\mathbb{B}_\varepsilon(\hat{\mathbb{P}}_N)$ is the Wasserstein ball of radius ε, for norm $\|\cdot\|$, centered in $\hat{\mathbb{P}}_N$;
   - $\ell := \max_{k\in[K]} \ell_k$, where each $-\ell_k$ is a proper convex function

3. A random variable $\boldsymbol{\xi} \in \mathbb{R}^m$, is light-tailed under $\mathbb{P}$ if there exists $a > 1$ such that, $\mathbb{E}_{\mathbb{P}}[\|\boldsymbol{\xi}\|^a] < +\infty$.
4. If $\boldsymbol{\xi} \in \mathbb{R}^m$, $m \geq 3$, is light-tailed under $\mathbb{P}$, then we have, for $\varepsilon < 1$,

$$\mathbb{P}_S(d_W(\mathbb{P}, \hat{\mathbb{P}}_N) \geq \varepsilon) \leq c_1 e^{-c_2 N \varepsilon^m}.$$

1) Data-Driven SOCP constraint

Consider the following problem :

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & q^\top x && (1)\\
s.t. \quad & \|Ax + b\|_2 \ \leq (c + \Delta_c u)^\top x + d && (2)\\
& \|x\|_\infty \leq 1 && (3)
\end{aligned}
$$

where $u \in \mathbb{R}^m$ and $\Delta_c \in M_{n,m}(\mathbb{R})$.

1. (2 points) Let $f(u, x) := \| \ Ax + b\|_2 \ - (c + \Delta_c u)^\top x - d \leq 0$. Compute $f_\star(v, x) := \inf_{u\in\mathbb{R}^m} v^\top u - f(u, x)$.
2. (3 points) We assume that $u \in \mathbb{R}$ (i.e $m = 1$), and that we have 100 independant realizations of $\tilde{u}$. Leveraging the CS test write an SOCP problem, with the same objective function (see section 8.1 of DDRO), whose solution satisfies (2) with probability 95% with confidence 90% in the data.
3. (2 points) We now assume that $\tilde{u} \geq 0$. Refine your previous SOCP using this assumption. What can you say about the relation between both SOCP ?

2) Chance constraint

Consider a random vector $\boldsymbol{\xi}$, of unknown law, taking values in $\mathbb{R}^n$, $n \geq 3$. We assume that we have an iid sample $\{\hat{\xi}_i\}_{i\in[N]}$. We assume that $\text{supp}(\boldsymbol{\xi}) \subset \Xi$, with $\Xi = \{\xi \in \mathbb{R}^m \mid C\xi \leq d\ \}$ a polytope (bounded polyhedron).

Let $A = \{\xi \in \mathbb{R}^m \mid a_k^\top \xi < b_k,\ \forall k \in [K]\}$, such that $A \cap C \neq \emptyset$. We are interested in the following problem

$$\bar{p} := \sup_{\mathbb{Q}\in\mathbb{B}_\varepsilon(\hat{\mathbb{P}}_N)} \quad \mathbb{Q}(\boldsymbol{\xi} \notin A)$$

Define $\ell_k(\xi) = \begin{cases} 1 & \text{if } a_k^\top \xi \geq b_k \\ -\infty & \text{otherwise.} \end{cases}$

1. (2 points) Show that $\mathbb{Q}(\boldsymbol{\xi} \in A) = \mathbb{E}_{\mathbb{Q}}(\ell(\boldsymbol{\xi}))$, for a function ℓ to be defined.
2. (1 point) Compute $[-\ell_k]^\star(z)$
3. (1 point) Compute $\delta^\star(z|\Xi)$
4. (2 points) Give a convex minimization problem with a finite number of constraints whose value is equal to $\bar{p}$.
5. (2 points) Specify the above problem in the case where the Wasserstein norm chosen is $\|\cdot\| = \|\cdot\|_\infty$. What is the simplest class of optimization problem it belongs to ?
6. (2 points) Same question for $\|\cdot\| = \|\cdot\|_2$. Is it numerically simpler, more complex or impossible to say, than the previous one ?
7. (4 points) We are now interested in the following chance constrained problem

$$\begin{aligned} (CCP) \quad \min_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & \mathbb{P}\big(a_k^\top \xi \leq x_k,\ \forall k \in [K]\big) \geq 1 - \varepsilon \end{aligned}$$

   $\mathbb{P}$ is unknown, but we have iid realisations of $\boldsymbol{\xi}$ under $\mathbb{P}$. Give a robust formulation of (CCP) implying a probabilistic guarantee of level $1-\varepsilon$ with confidence $1-\alpha$.