Computing risk averse equilibrium in incomplete market

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Uncertainty on electricity market

- Today, wholesale electricity markets takes the form of an auction that matches supply and demand.
- But, the demand cannot be predicted with absolute certainty. These day-ahead markets must be augmented with balancing markets.
- To reduce CO_2 emissions and increase the penetration of renewables, there are increasing amounts of electricity from intermittent sources such as wind and solar.
- That's why equilibrium on the market are set in a stochastic setting.

Multiple equilibrium in a incomplete market

- In Philpott et al. (2013), the authors present a framework for multistage stochastic equilibria.
- They show that equilibrium in risk-neutral market and equilibrium in complete risk averse markets can be found as solution of a global optimization problem.
- What about risk averse equilibrium in incomplete market ?
- We present a toy problem with agreable properties (strong concavity of utility) that displays multiple equilibrium.
- We show that the classical methods used to find equilibrium (PATH solver and tatônnement's algorithms) fail to find all equilibria.

Outline

1. Statement of the problem
   - Social planner problem (Optimization problem)
   - Equilibrium problem
   - Trading risk with Arrow-Debreu securities
2. Optim. and equilibrium problems
   - In the risk neutral case
   - In the risk averse case
3. Multiple risk averse equilibrium
   - Numerical results
   - Analytical results

Ingredients of the problem

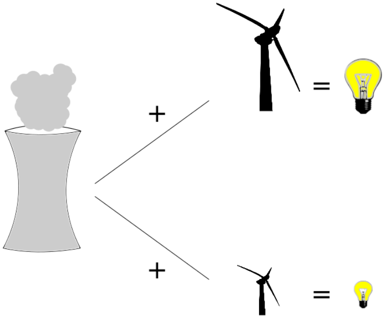

Figure: Illustration of the toy problem

- Two time-step market
- One good traded
- Two agents: producer and consumer
- Finite number of scenario $\omega \in \Omega$
- Consumption on second stage only

Producer and consumer welfare

Producer Welfare

- Step 1: production of x at a marginal cost cx
- Step 2: random production $\mathbf{x}_r$ at uncertain marginal cost $\mathbf{c}_r\mathbf{x}_r$

$$\underbrace{\boldsymbol{W}_p(\omega)}_{\text{producer's welfare}} = -\underbrace{\frac{1}{2}cx^2}_{\text{cost step 1}} - \underbrace{\frac{1}{2}\mathbf{c}_r(\omega)\mathbf{x}_r(\omega)^2}_{\text{cost step 2}}$$

Consumer Welfare

- Step 1: no consumption $\varnothing$
- Step 2: random consumption $\mathbf{y}$ at marginal utility $\mathbf{V} - \mathbf{r}\mathbf{y}$

$$\underbrace{\boldsymbol{W}_c(\omega)}_{\text{consumer's welfare}} = \underbrace{0}_{\text{step 1}} + \underbrace{\mathbf{V}(\omega)\mathbf{y}(\omega) - \frac{1}{2}\mathbf{r}(\omega)\mathbf{y}(\omega)^2}_{\text{consumer's utility at step 2}}$$

Outline

Social planner's welfare

The welfare of the social planner can be defined by

$$\underbrace{\boldsymbol{W}_{sp}(\omega)}_{\text{Social planner's welfare}} = \underbrace{\boldsymbol{W}_{p}(\omega)}_{\text{Producer's welfare}} + \underbrace{\boldsymbol{W}_{c}(\omega)}_{\text{Consumer's welfare}}$$

Where

- $\boldsymbol{W}_p(\omega) = \frac{1}{2}cx^2 - \frac{1}{2}\mathbf{c}_r(\omega)\mathbf{x}_r(\omega)^2$
- $\boldsymbol{W}_c(\omega) = \mathbf{V}(\omega)\mathbf{y}(\omega) - \frac{1}{2}\mathbf{r}(\omega)\mathbf{y}(\omega)^2$

Optimization and uncertainty

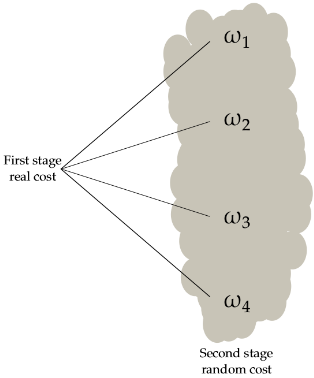

Figure: Aggregating uncertainty with a risk measure to obtain real value

To be able to do optimization, we aggregate uncertainty using:

- the expectation $\mathbb{E}_{\mathbb{P}}$: risk neutral
- a risk measure $\mathbb{F}$: risk averse

Social planner problem

Given a probability distribution $\mathbb{P}$ on Ω, we can define a risk neutral social planner problem

RNSP($\mathbb{P}$):

$$\max_{x,\mathbf{x}_r,\mathbf{y}} \quad \underbrace{\mathbb{E}_{\mathbb{P}}[\boldsymbol{W}_{sp}]}_{\text{expected welfare}}$$
$$\text{s.t.} \quad \underbrace{x + \mathbf{x}_r(\omega)}_{\text{supply}} = \underbrace{\mathbf{y}(\omega)}_{\text{demand}}$$

Given a risk measure $\mathbb{F}$, we can define a risk averse social planner problem

RASP($\mathbb{F}$):

$$\max_{x,\mathbf{x}_r,\mathbf{y}} \quad \underbrace{\mathbb{F}[\boldsymbol{W}_{sp}]}_{\text{risk adjusted welfare}}$$
$$\text{s.t.} \quad \underbrace{x + \mathbf{x}_r(\omega)}_{\text{supply}} = \underbrace{\mathbf{y}(\omega)}_{\text{demand}}$$

Risk measures

- We consider coherent risk measures, with $\mathbb{F}\big[\boldsymbol{Z}\big] = \min_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[\boldsymbol{Z}\big]$.
- If $\mathcal{Q}$ is a polyhedron defined by K extreme points $(\mathbb{Q}_k)_{k\in[\![1;K]\!]}$, then $\mathbb{F}$ is said to be polyhedral with $\mathbb{F}\big[\boldsymbol{Z}\big] = \min_{\mathbb{Q}_1,\dots,\mathbb{Q}_K} \mathbb{E}_{\mathbb{Q}_k}\big[\boldsymbol{Z}\big]$.
- In this case RASP can be written

$$\begin{aligned} \max_{\theta, x, \mathbf{x}_r, \mathbf{y}} \quad & \theta \\ \text{s.t.} \quad & \theta \leq \mathbb{E}_{\mathbb{Q}_k}\big[\boldsymbol{W}_{sp}\big] , \quad \forall k \in [\![1; K]\!] \\ & x + \mathbf{x}_r(\omega) = \mathbf{y}(\omega) , \quad \forall \omega \in \Omega \end{aligned}$$

Remark on non linearity of risk averse social planner

- Linearity of expectation leads to equalities

$$\mathbb{E}_{\mathbb{P}}[\boldsymbol{W}_{sp}] = \underbrace{\mathbb{E}_{\mathbb{P}}[\boldsymbol{W}_p + \boldsymbol{W}_c]}_{\text{expectation of sum}} = \underbrace{\mathbb{E}_{\mathbb{P}}[\boldsymbol{W}_p] + \mathbb{E}_{\mathbb{P}}[\boldsymbol{W}_c]}_{\text{sum of expectations}}$$

- With a general risk measure

$$\mathbb{F}[\boldsymbol{W}_{sp}] = \underbrace{\mathbb{F}[\boldsymbol{W}_p + \boldsymbol{W}_c]}_{\text{risk of sum}} \neq \underbrace{\mathbb{F}[\boldsymbol{W}_p] + \mathbb{F}[\boldsymbol{W}_c]}_{\text{sum of risks}}$$

- There is no natural criterion for a risk averse social planner

Outline

Agent are price takers

Definition

An agent is *price taker* if she acts as if she has no influence on the price.

In the remain of the presentation, we consider that agents are price takers.

Definition risk neutral equilibrium

Definition (Arrow and Debreu (1954))

Given a probability $\mathbb{P}$ on Ω, a risk neutral equilibrium RNEQ($\mathbb{P}$) is a set of prices $\{\boldsymbol{\pi}(\omega) \;,\; \omega \in \Omega\}$ such that there exists a solution to the system

$$\mathrm{RNEQ}(\mathbb{P}): \quad \max_{x, \mathbf{x}_r} \quad \underbrace{\mathbb{E}_{\mathbb{P}}\Big[\boldsymbol{W}_p + \boldsymbol{\pi}\left(x + \mathbf{x}_r\right)\Big]}_{\text{expected profit}}$$

$$\max_{\mathbf{y}} \quad \underbrace{\mathbb{E}_{\mathbb{P}}\big[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}\big]}_{\text{expected utility}}$$

$$\underbrace{0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0}_{\text{market clears}} \;, \quad \forall \omega \in \Omega$$

Remark on complementarity constraints

- Complementarity constraints are defined by

$$0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0\,, \quad \forall \omega \in \Omega$$

- If $\boldsymbol{\pi} > 0$ then supply $=$ demand
- If $\boldsymbol{\pi} = 0$ then supply $\geq$ demand

Consumer is risk insensitive

As the consumer has no first stage decision,
she can optimize each scenario independently

$$\max_{\mathbf{y}} \quad \underbrace{\mathbb{E}_{\mathbb{P}}\big[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}\big]}_{\text{expected utility}}$$

$$\Updownarrow$$

$$\forall \omega \in \Omega\,, \quad \max_{\mathbf{y}(\omega)} \quad \underbrace{\boldsymbol{W}_c(\omega) - \boldsymbol{\pi}(\omega)\mathbf{y}(\omega)}_{\text{scenario independant}}$$

Definition of risk averse equilibrium

Definition

Given two risk measures $\mathbb{F}_p$ and $\mathbb{F}_c$, a risk averse equilibrium $\mathrm{RAEQ}(\mathbb{F}_p, \mathbb{F}_c)$ is a set of prices $\{\boldsymbol{\pi}(\omega) : \omega \in \Omega\}$ such that there exists a solution to the system

$$\mathrm{RAEQ}(\mathbb{F}_p, \mathbb{F}_c): \quad \max_{x, \mathbf{x}_r} \quad \underbrace{\mathbb{F}_p\Big[\boldsymbol{W}_p + \boldsymbol{\pi}\left(x + \mathbf{x}_r\right)\Big]}_{\text{risk adjusted profit}}$$

$$\max_{\mathbf{y}} \quad \underbrace{\mathbb{F}_c\left[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}\right]}_{\text{risk adjusted consumption}}$$

$$\underbrace{0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0}_{\text{market clears}}, \quad \forall \omega \in \Omega$$

Consumer is insensitive to the choice of risk measure

Assuming that the risk measure $\mathbb{F}_c$ of the consumer is monotonic, she can optimize scenario per scenario

$$\max_{\mathbf{y}} \quad \underbrace{\mathbb{F}_c\big[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}\big]}_{\text{risk adjusted consumption}}$$

$$\Updownarrow$$

$$\forall \omega \in \Omega\,, \quad \max_{\mathbf{y}(\omega)} \quad \underbrace{\boldsymbol{W}_c(\omega) - \boldsymbol{\pi}(\omega)\mathbf{y}(\omega)}_{\text{scenario independant}}$$

Risk averse equilibrium with polyhedral risk measure

If the risk measure $\mathbb{F}$ is polyhedral, then RAEQ($\mathbb{F}$) reads

$$\text{RAEQ:} \quad \max_{\theta, x, \mathbf{x}_r} \quad \theta$$
$$\text{s.t.} \quad \theta \leq \mathbb{E}_{\mathbb{Q}_k}\left[\boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r)\right], \quad \forall k \in [\![1; K]\!]$$

$$\max_{\mathbf{y}(\omega)} \quad \boldsymbol{W}_c(\omega) - \boldsymbol{\pi}\mathbf{y}(\omega), \quad \forall \omega \in \Omega$$

$$0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0, \quad \forall \omega \in \Omega$$

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Definition of an Arrow-Debreu security

Definition

An *Arrow-Debreu security* for node $\omega \in \Omega$ is a contract that charges a price $\mu(\omega)$ in the first stage, to receive a payment of 1 in scenario ω.

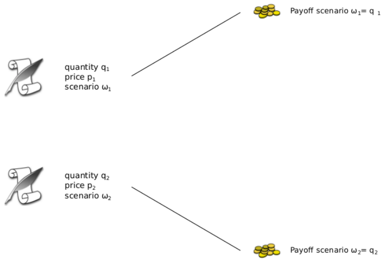


Figure: Representation of two Arrow-Debreu securities with two scenarii

Risk averse equilibrium with risk trading

A *risk trading equilibrium* is sets of prices $\{\boldsymbol{\pi}(\omega)\ ,\ \ \omega \in \Omega\}$ and $\{\boldsymbol{\mu}(\omega)\ ,\ \ \omega \in \Omega\}$ such that there exists a solution to the system:

$$\text{RAEQ-AD:} \quad \max_{\theta, x, \mathbf{x}_r} \quad \theta - \underbrace{\sum_{\omega \in \Omega} \mu(\omega)\mathbf{a}(\omega)}_{\text{cost of contracts purchased}}$$

$$\text{s.t.} \quad \theta \leq \mathbb{E}_{\mathbb{Q}_k}\Big[\boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r) + \mathbf{a}\Big]\ ,\ \ \forall k \in [\![1; K]\!]$$

$$\max_{\phi, \mathbf{y}} \quad \phi - \sum_{\omega \in \Omega} \mu(\omega)\mathbf{b}(\omega)$$

$$\text{s.t.} \quad \phi \leq \mathbb{E}_{\mathbb{Q}_k}\Big[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y} + \mathbf{b}\Big]\ ,\ \ \forall k \in [\![1; K]\!]$$

$$0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0\ ,\ \ \forall \omega \in \Omega$$

$$\underbrace{0 \leq -\mathbf{a}(\omega) - \mathbf{b}(\omega)}_{\text{"supply} \geq \text{demand"}} \perp \boldsymbol{\mu}(\omega) \geq 0\ ,\ \ \forall \omega \in \Omega$$

Conclusion

Until now, we have seen

- social planner's problem in risk neutral and risk averse setting
- equilibrium problem in risk neutral and risk averse setting
- risk trading equilibrium problem in risk averse setting

We will study the link between

- risk neutral social planner and equilibrium problem (RNSP and RNEQ)
- risk averse social planner and risk trading equilibrium (RASP and RAEQ-AD)

Outline

RNSP($\mathbb{P}$) is equivalent to RNEQ($\mathbb{P}$)

Proposition

Let $\mathbb{P}$ be a probability measure over Ω. The elements $(x^\star \mathbf{x}_r^\star, \mathbf{y}_r^\star)$ are optimal solutions to RNSP($\mathbb{P}$) if and only if there exist (non trivial) equilibrium prices $\boldsymbol{\pi}$ for RNEQ($\mathbb{P}$) with associated optimal controls $(x^\star, \mathbf{x}_r^\star, \mathbf{y}^\star)$

Corollary

If producer's criterion and consumer's criterion are strictly concave, then RNSP($\mathbb{P}$) admits a unique solution and RNEQ($\mathbb{P}$) admits a unique equilibrium.

Outline

RAEQ-AD is equivalent to RASP

We adapt a result of Ralph and Smeers (2015)

Proposition

Suppose given equilibrium prices $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$ such that the finite valued vector $(x, \mathbf{x}_r, \mathbf{y}, \mathbf{a}, \mathbf{b}, \theta, \varphi)$ solves RAEQ-AD($\mathbb{F}$). Then $\boldsymbol{\pi}$ are equilibrium price for RNEQ(μ) with optimal value vector $(x, \mathbf{x}_r, \mathbf{y})$. Moreover, $(x, \mathbf{x}_r, \mathbf{y})$ solves RASP($\mathbb{F}$) where μ is the worst case probability.
The reverse holds true

Summing up equivalences

- We have shown two equivalences

$$\mathrm{RNSP}(\mathbb{P}) \Leftrightarrow \mathrm{RNEQ}(\mathbb{P}) \;, \qquad \text{(risk neutral setting)}$$
$$\mathrm{RASP}(\mathbb{F}) \Leftrightarrow \underbrace{\mathrm{RAEQ\text{-}AD}(\mathbb{F})}_{\text{complete market}} \;, \qquad \text{(risk averse setting)}$$

  that lead to result about uniqueness of equilibrium and methods of decomposition

- What can we say about $\underbrace{\mathrm{RAEQ}(\mathbb{F})}_{\text{incomplete market}}$?

Outline

1. Statement of the problem
2. Optim. and equilibrium problems
3. Multiple risk averse equilibrium
   - Numerical results
   - Analytical results

Recall on the problem

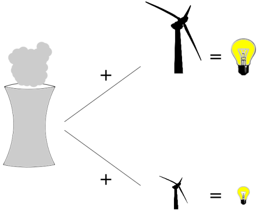


Figure: Illustration of the toy problem

Recall:

- 2 time-step market
- 1 good traded
- 2 agents : 1 consumer, 1 producer
- Consumption on second stage only

We focus on:

- 2 scenarios ω_1 and ω_2
- 2 prices: π_1 and π_2
- 5 controls: x, x_1, x_2, y_1 and y_2
- 2 probabilities $(\underline{p}, 1-\underline{p})$ and $(\bar{p}, 1-\bar{p})$
- $\underline{p} = \frac{1}{4}$, $\bar{p} = \frac{3}{4}$
- prices $0 < \pi_1 < \pi_2$

Computing an equilibrium with GAMS

- GAMS with the solver PATH in the EMP framework (See Britz et al. (2013), Brook et al. (1988), Ferris and Munson (2000) and Ferris et al. (2009))
- different starting points defined by a grid 100×100 over the square $[1.220; 1.255] \times [2.05; 2.18]$
- We find one equilibrium defined by

$$\boldsymbol{\pi} = (\pi_1, \pi_2) = (1.23578; 2.10953)$$

Walras's tâtonnement algorithm

Then we compute the equilibrium using a tâtonnement algorithm.

Data: MAX-ITER, $(\pi_1^0, \pi_2^0), \tau$

Result: A couple $(\pi_1^\star, \pi_2^\star)$ which approximates the equilibrium price $\pi_\sharp$

```
for k from 0 to MAX-ITER do
    Compute an optimal decision for each player given a price :
        x , x_1, x_2 = arg max F[ W_p + π(x + x_r) ];
        y(ω) = arg max F[ W_c − π y ];
    Update the price :
        π_1 = π_1 − τ max {0; y_1 − (x + x_1)};
        π_2 = π_2 − τ max {0; y_2 − (x + x_2)};
end
return (π_1, π_2)
```

$$x\ ,\ \ x_1, x_2 = \arg\max \mathbb{F}\big[\boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r)\big];$$
$$y(\omega) = \arg\max \mathbb{F}[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}];$$
$$\pi_1 = \pi_1 - \tau \max\{0; y_1 - (x + x_1)\};$$
$$\pi_2 = \pi_2 - \tau \max\{0; y_2 - (x + x_2)\};$$

Algorithm 1: Walras' tâtonnement

Computing equilibria with Walras's tâtonnement

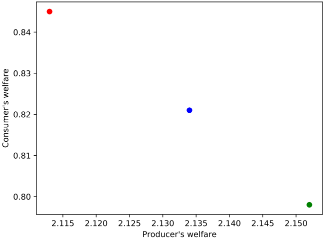
- Running Walras's tâtonnement algorithm starting from (1.25; 2.06), respectively from (1.22; 2.18), with 100 iterations and a step size of 0.1, we find two new equilibria

$$\pi = (1.2256; 2.0698) \text{ and } \pi = (1.2478; 2.1564)$$

- An alternative tatônnement method called FastMarket (see Facchinei and Kanzow (2007)) find the same equilibria

Summing up about computing equilibrium

	Equilibrium prices	Risk adjusted welfares
red (Tâtonnement)	(1.2478; 2.1564)	(2.113; 0.845)
blue (GAMS)	(1.2358; 2.1095)	(2.134; 0.821)
green (Tâtonnement)	(1.2256; 2.0698)	(2.152; 0.798)



- No equilibrium dominates any other

Outline

Optimal control of agents with respect to a price π

There are three regimes

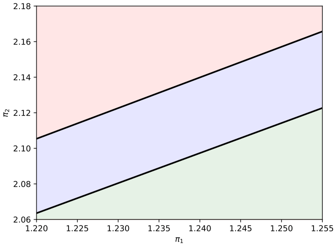


Figure: Illustration of the three regimes

condition	$x^\sharp$	$x_i^\sharp$	$y_i^\sharp$
$x_c \leq \frac{\mathbb{E}_{\bar{p}}[\pi]}{c}$	$\frac{\mathbb{E}_{\bar{p}}[\pi]}{c}$	$\frac{\pi_i}{c_i}$	$\frac{V_i - \pi_i}{r_i}$
$\frac{\mathbb{E}_{\bar{p}}[\pi]}{c} \leq x_c \leq \frac{\mathbb{E}_{\underline{p}}[\pi]}{c}$	x_c	$\frac{\pi_i}{c_i}$	$\frac{V_i - \pi_i}{r_i}$
$\frac{\mathbb{E}_{\underline{p}}[\pi]}{c} \leq x_c$	$\frac{\mathbb{E}_{\underline{p}}[\pi]}{c}$	$\frac{\pi_i}{c_i}$	$\frac{V_i - \pi_i}{r_i}$

Table: Optimal control for producer and consumer problems

$$\text{where } x_c(\pi) = \frac{1}{2(\pi_1 - \pi_2)} \left(\frac{\pi_2^2}{2c_2} - \frac{\pi_1^2}{2c_1} \right)$$

Excess production function

We are now looking for prices (π_1, π_2) such that the complementarity constraints are satisfied

$$z_i(\boldsymbol{\pi}) = \underbrace{x^\sharp(\boldsymbol{\pi}) + x_i^\sharp(\boldsymbol{\pi}) - y_i^\sharp(\boldsymbol{\pi}) = 0}_{\text{market clears for optimal control}}, \qquad i \in \{1, 2\}$$

This excess functions have three regime. In the green and red part the equation is linear, in the blue part the equation is quadratic.

Representation of analytical solutions (scenario 1)

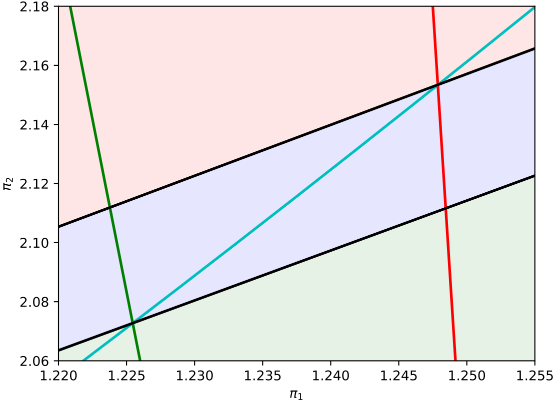


Figure: Null excess function per scenario manifold

Representation of analytical solutions (scenario 2)

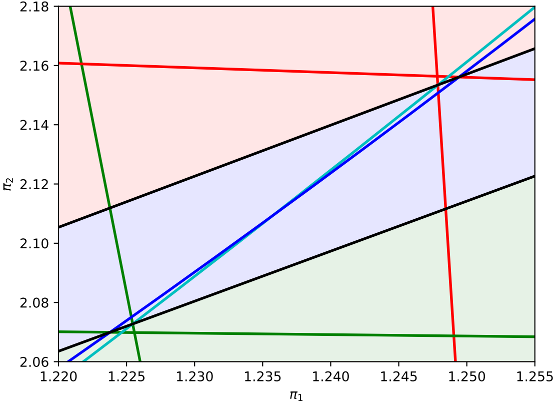


Figure: Null excess function per scenario manifold

Representation of analytical solutions (red equilibrium)

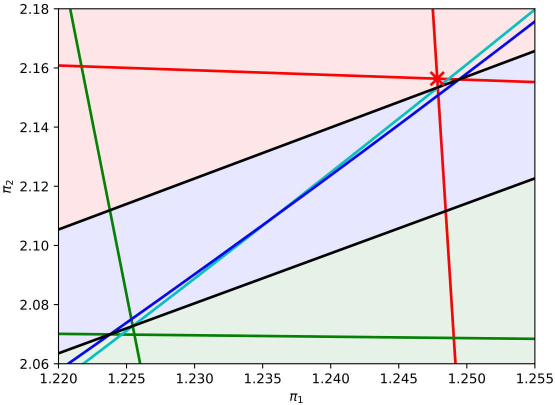


Figure: Null excess function per scenario manifold

Representation of analytical solutions (blue equilibrium)

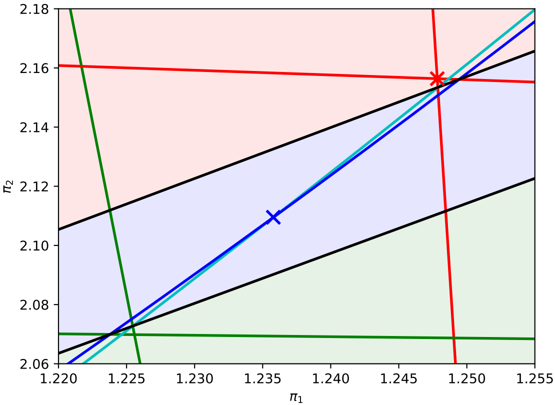


Figure: Null excess function per scenario manifold

Representation of analytical solutions (green equilibrium)

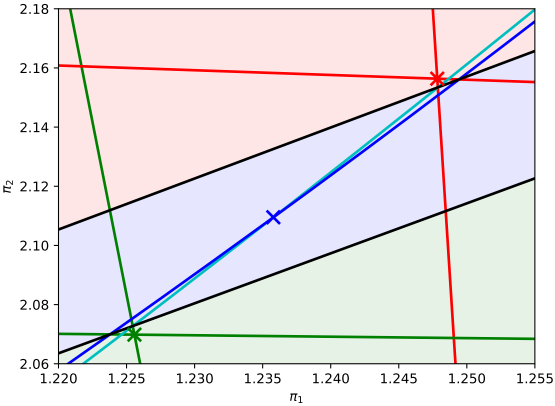


Figure: Null excess function per scenario manifold

Some interesting remarks

Remark

The PATH solver find the blue equilibrium, while the tatônnements methods find equilibrium green and red.
The blue equilibrium is unstable, i.e. $\pi' = z(\pi)$ is unstable around the blue equilibrium.

Remark

There exists a set of non-zero measure of parameters $V_1, V_2, c, c_1, c_2, r_1$, and r_2 (albeit small), that have three distinct equilibrium with the same properties.

Remark

We can show that the blue equilibrium is a convex combination of red and green equilibrium.

Stability of equilibriums (red equilibrium)

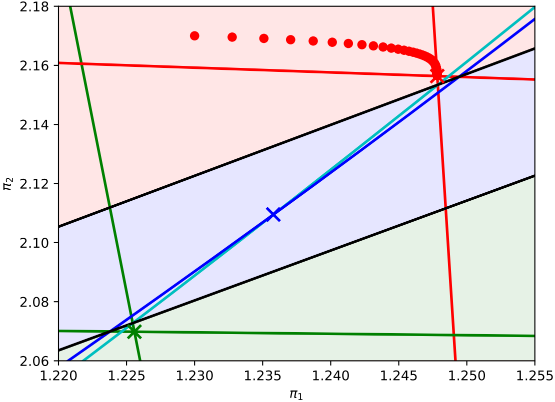


Figure: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Stability of equilibriums (blue equilibrium)

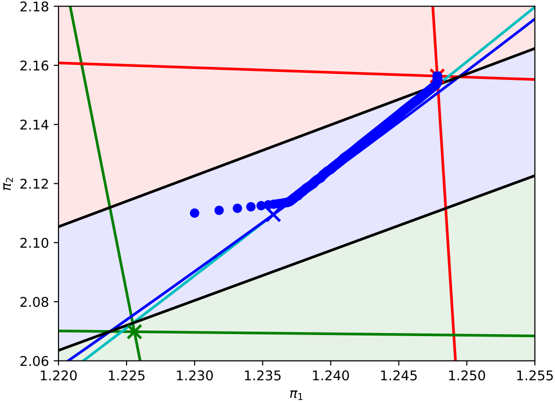


Figure: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Stability of equilibriums (green equilibrium)

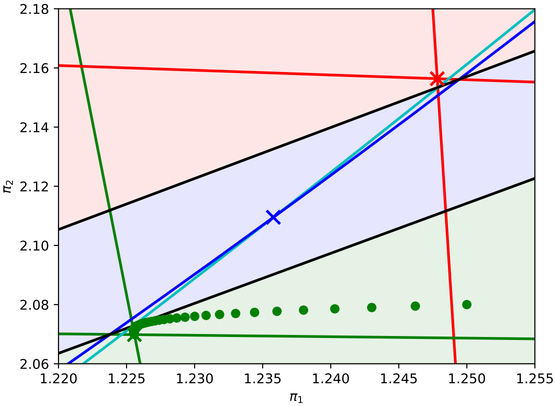


Figure: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Stability of equilibriums (vector field)

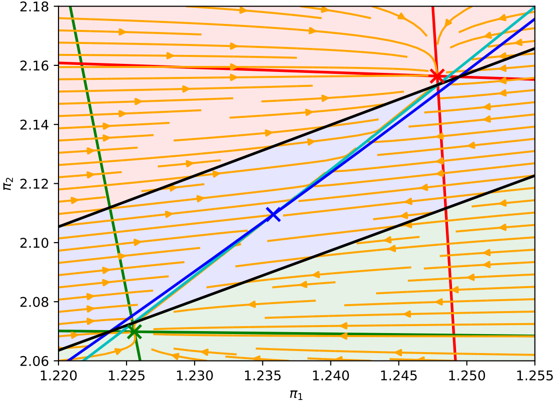


Figure: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Conclusion

In this talk we have

- shown an equivalence between risk averse social planner problem and risk trading equilibrium (respectively risk neutral equivalence)
- given theorems of uniqueness of equilibrium
- shown non uniqueness of equilibrium in risk averse setting without Arrow-Debreu securities

On going work

- Does the counter example extend with multiple agents and scenarios ?
- Do we have uniqueness with bounds on the number of Arrow-Debreu securities exchanged ?

If you want to know more...

Just ask some questions

or have a look at

https://arxiv.org/abs/1706.08398

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