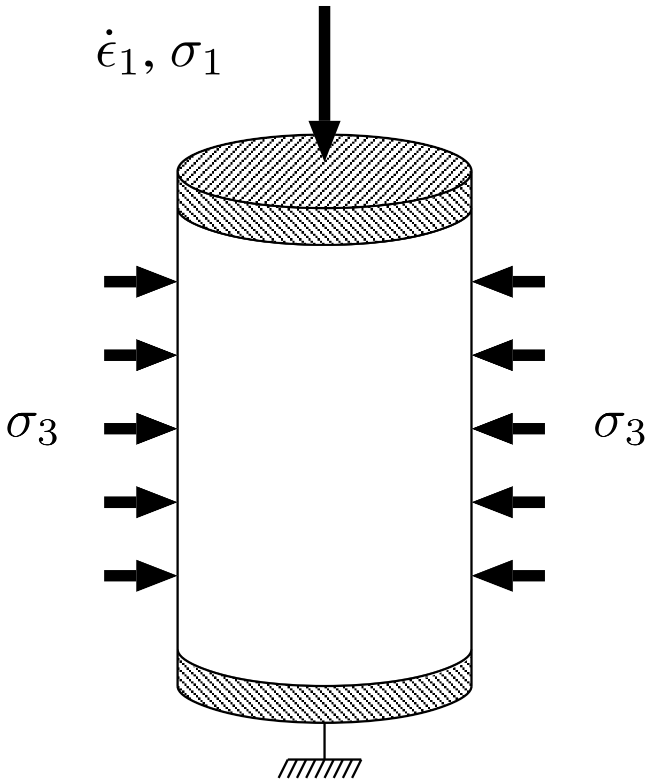
Granular packings:
internal states, quasi-static rheology

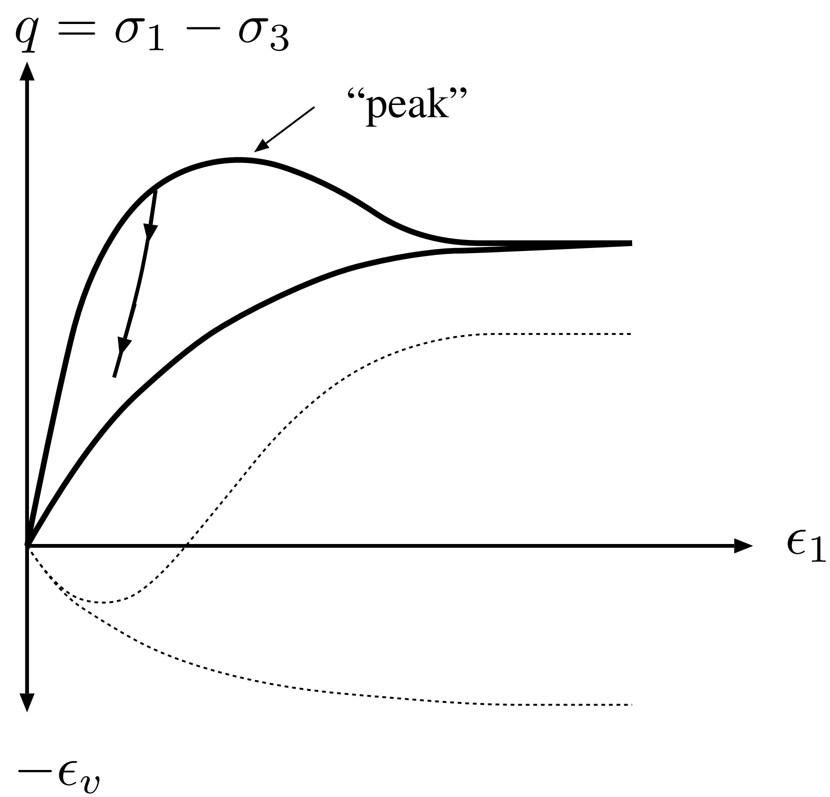
Main tool : grain-level numerical simulation...

- **... of assemblies of spherical grains (3D)...**
  - comparisons with experiments on glass beads
  - geometry of bead packs = traditional research field (should be connected to mechanics nowadays !)
- **... or circular ones (2D) !**
  - investigation of basic rheophysical phenomena
  - treatment of more difficult cases (such as loose cohesive assemblies)

Macroscopic mechanical behaviour: triaxial compression.

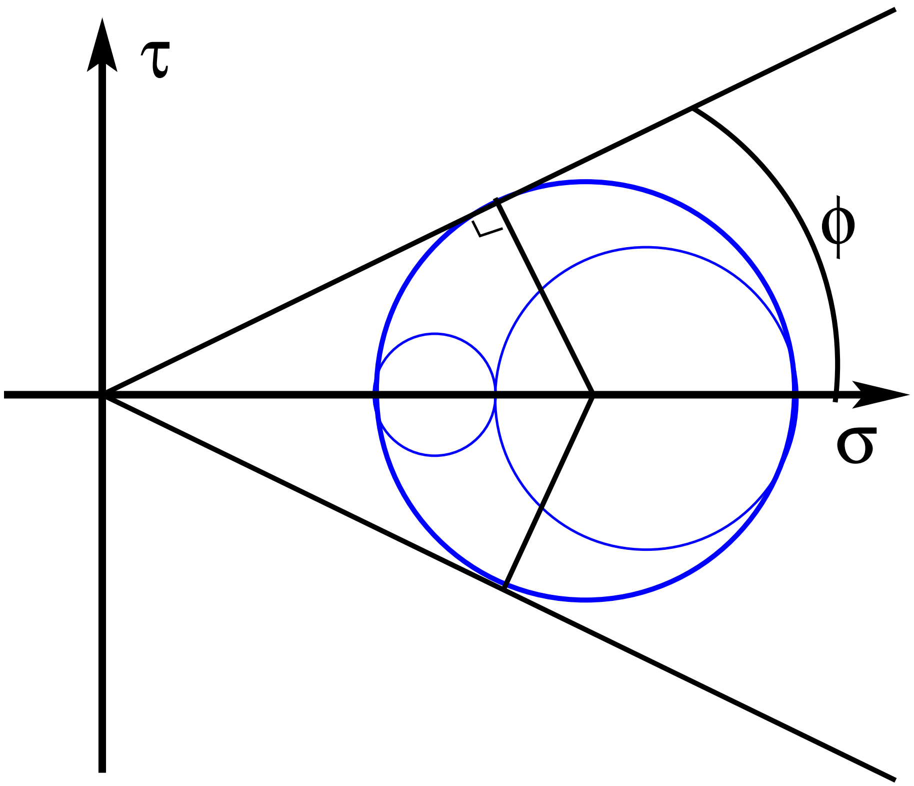


- $\underline{\underline{\sigma}}, \underline{\underline{\epsilon}} \sim$ homogeneous
- $\sigma_2 = \sigma_3$ (pressure of a fluid)
- typically $\sigma \sim 10 - 1000$ kPa and $\epsilon \sim 10^{-2}$
- influence of density
- $\sigma_1/\sigma_3 \leq$ maximum



- fixed principal directions, symmetry of revolution
- most accurate devices measure $\epsilon \sim 10^{-6}$
- stress deviator $q = \sigma_1 - \sigma_3$; volumetric strain $-\epsilon_v = -\epsilon_1 - \epsilon_2 - \epsilon_3$; $\sigma_1, \epsilon_1 = \epsilon_a =$ *axial* stress and strain

Triaxial compression and internal friction

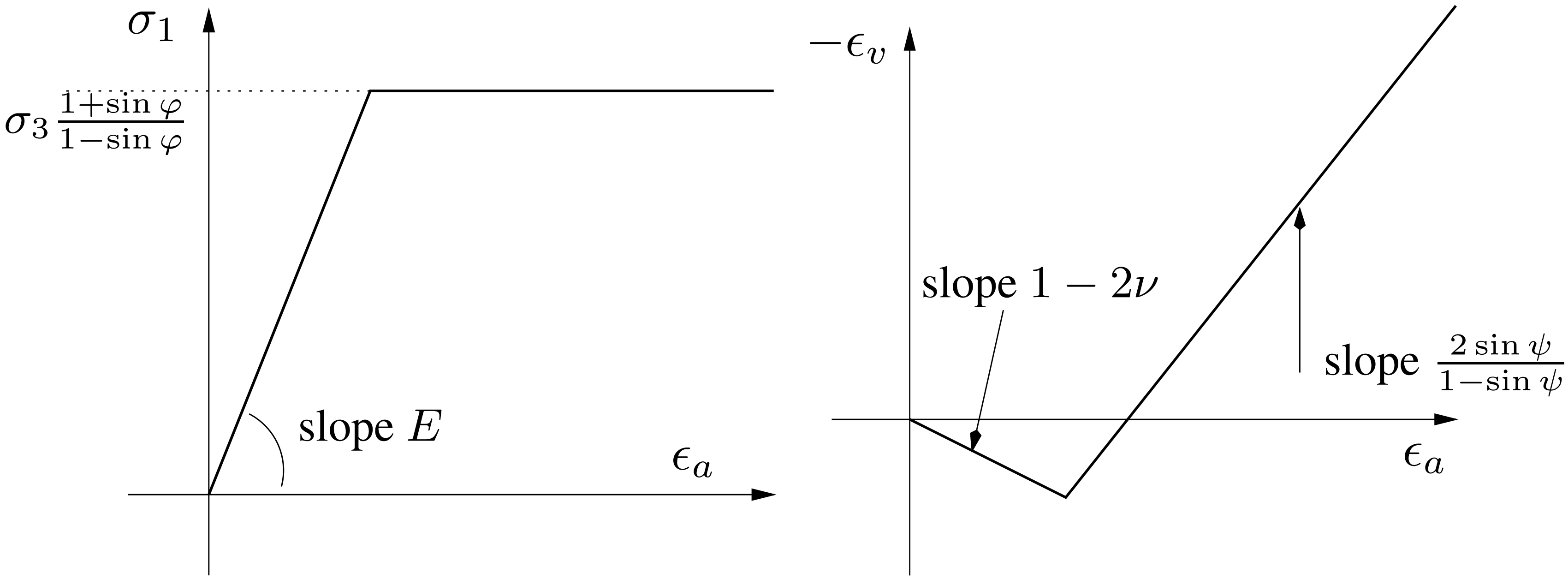


- Mohr's circles = change of coordinates for $\underline{\underline{\sigma}}$
- Coulomb's condition sets maximum value for principal stress ratios

$$\frac{\sigma_1}{\sigma_3} = \frac{1 + \sin\varphi}{1 - \sin\varphi}$$

- Condition reached on planes inclined at $\pm(\pi/4 - \varphi/2)$ w.r.t. direction 1

A simple (oversimplified) macroscopic model



- linear isotropic elasticity + Mohr-Coulomb plasticity criterion + constant "dilatancy angle" ψ (flow rule)
- $E \sim 10MPa$, $\varphi \sim 40°$, $\psi = 10 - 15°$ for sands ($\sigma_3 \sim$ 10–100 kPa
- φ, $\psi \searrow$ when $p \nearrow$ ...
- More accurate models have hardening, anisotropy...

An example of elastoplastic law

- With $\sigma_1 \geq \sigma_2 \geq \sigma_3$ the principal stresses, $f(\underline{\underline{\sigma}}) = |\sigma_1 - \sigma_3| - (\sigma_1 + \sigma_3)\sin\varphi$ is the Mohr-Coulomb **plastic criterion**
- $g(\underline{\underline{\sigma}}) = |\sigma_1 - \sigma_3| - (\sigma_1 + \sigma_3)\sin\psi$, involving the dilatancy angle, is the **plastic potential**, which sets the **flow rule** as

$$\dot{\epsilon}_p = \lambda \frac{\partial g}{\partial \underline{\underline{\sigma}}}$$

- A **hardening rule** would specify how the criterion depends on some other internal variable(s) α, and how α evolves with plastic strains...
- ... thus avoiding the unphysical assumption of elastic behaviour up to deviator peak.

Basic features of macroscopic mechanical behaviour

- dilatant dense states, contractant loose states ; dilatancy = $\boxed{\mathcal{D} = -\frac{d\epsilon_v}{d\epsilon_a}}$

  large strain $\Rightarrow$ critical state, independent of initial conditions

- internal friction angle φ: at peak deviator, at critical plateau

$$\boxed{\frac{\sigma_1}{\sigma_3} = \frac{1+\sin\varphi}{1-\sin\varphi}}$$

- Elasticity: for small stress and strain increments ($\Delta\epsilon \sim 10^{-5}$) static and dynamical measurements coincide. Sound velocities (isotropic case): $\boxed{V_P = \sqrt{\frac{B+\frac{4}{3}G}{\rho_m}}}$ and $\boxed{V_S = \sqrt{\frac{G}{\rho_m}}}$ (B, G = bulk, shear moduli

Classically, internal state = density, or $\boxed{\text{solid fraction } \Phi}$ (or void index $e = (1-\Phi)/\Phi$).

"Random close packing", "random loose packing" with spherical beads ?

III. Microscopic origin of macroscopic behaviour of model granular materials

1. Some general properties of granular packings
2. Assembling process, geometric characterisation (under low stress), elastic properties
   (geometry and initial response)
3. Quasi-static rheology, internal evolution

Comparisons with experiments ? Role of micromechanical parameters ?

Dimensionless control parameters

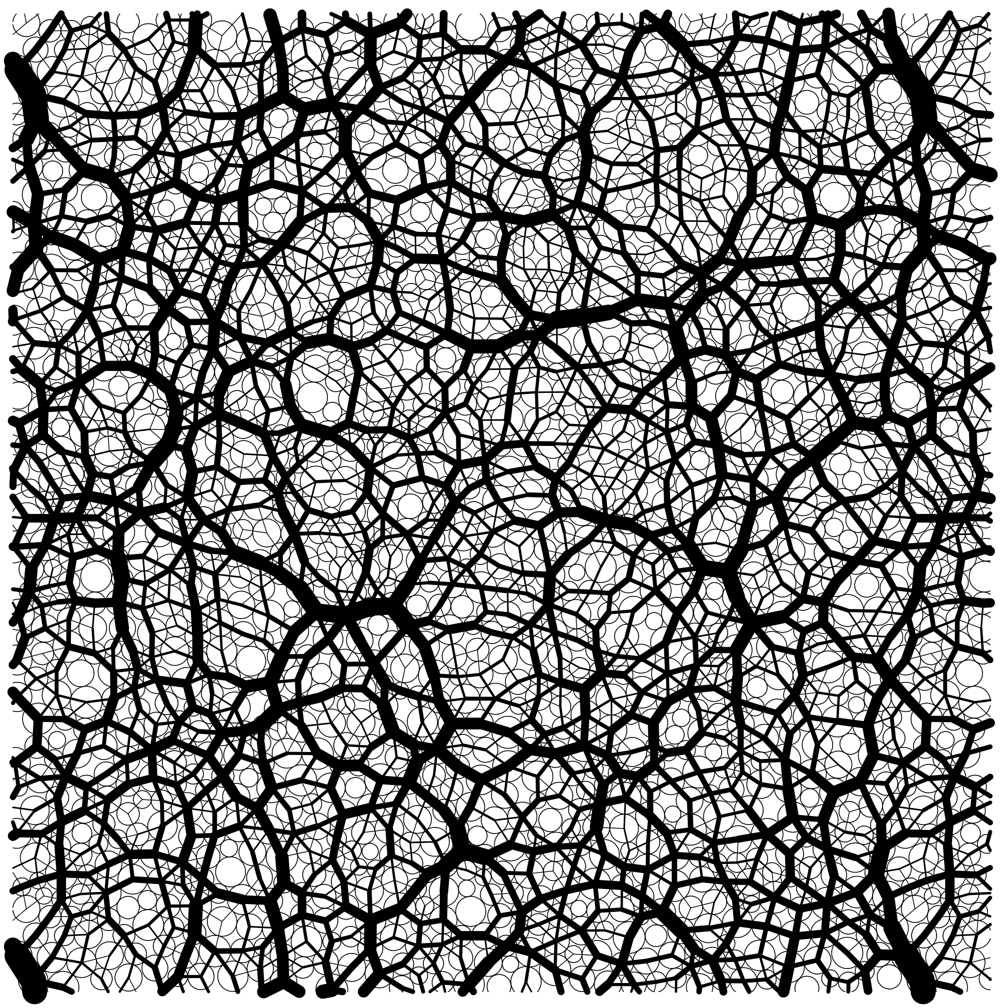
Material parameters + confining pressure P, strain rate $\dot{\epsilon}$,

- Reduced stiffness κ. "Interpenetration" (= contact deflection) $h/a \sim \kappa^{-1}$: $\kappa = (E/(1-\nu^2)P)^{2/3}$ for Hertzian contacts in 3D, $K_N/a^{d-2}P$ for linear law with in d dimensions (a = diameter)
  Glass beads, 100 kPa $\Rightarrow \kappa \sim 8400$ if $E = 70\text{GPa}$, $\nu = 0.3$
- Friction coefficient μ (0.2, $\boxed{0.3}$ ... 1 ??)
- Viscous damping level α (often large in numerical practice)
- Reduced strain rate or inertia number $I = \dot{\epsilon}\sqrt{m/aP}$.
  Quasi-static lab. experiments $\Rightarrow I \sim 10^{-9}$
  Numerically: $I = 10^{-5}$ already very slow and cautious!
  $I =$ important parameter for dense flows
  (da Cruz, GdR Midi, Pouliquen...)

Important limits to be investigated

- **Quasistatic limit:** $I \to 0$ (or $\Delta q/\sigma_2 \to 0$ if applied deviator stepwise increased)
  Is I or $\Delta q/\sigma_2$ small enough ? Do dynamical parameters become irrelevant ? (inertia, viscous forces)
- **Rigid limit:** $\kappa \to +\infty$. Stiffness level irrelevant ? Rigid contact model possible ?
- **Large system limit:** $n \to +\infty$.

Geometric and micromechanical features

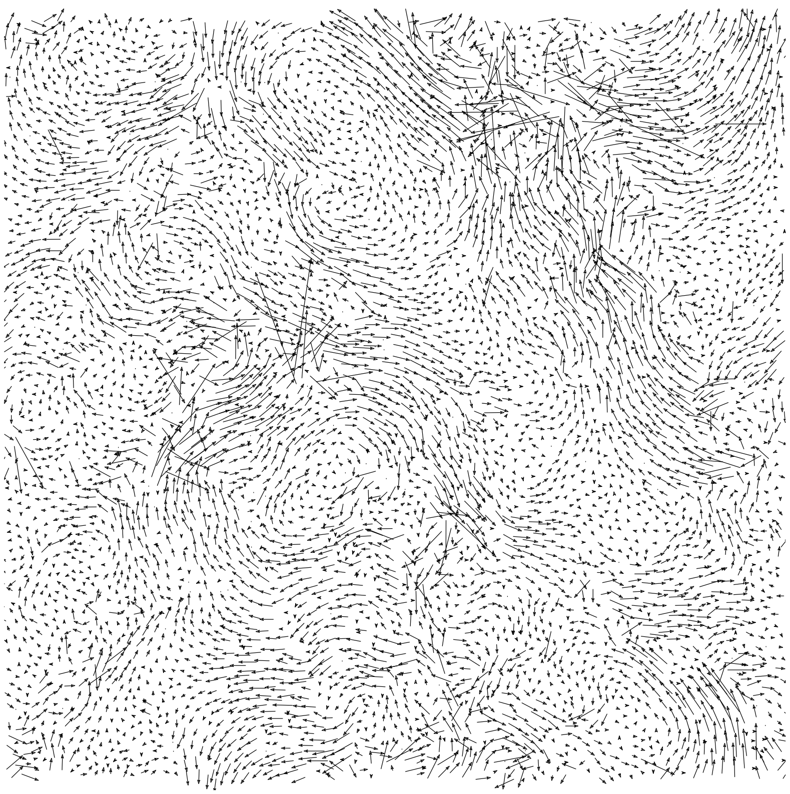


- Note periodic boundary conditions
- Force disorder (force chains, wide force distribution)
- Coordination number $z = 2N_C/n$ (n grains, N_c force-carrying contacts)
- Rattlers – fraction x_0 of grain number – carry no force
- Backbone = force-carrying network of non-rattler grains
- Backbone coordination number = $z^* = \dfrac{z}{1 - x_0}$

Geometric and micromechanical features

- Force disorder related to paucity of contacts: for $\kappa \to \infty$, $z^* \leq 6$ (spheres, 3D) or $z^* \leq 4$ (disks, 2D), due to absence of force indeterminacy on regarding contacts as frictionless

- In addition to Φ, z, x_0, force distribution, friction mobilization, introduce fabric or distribution of contact orientations



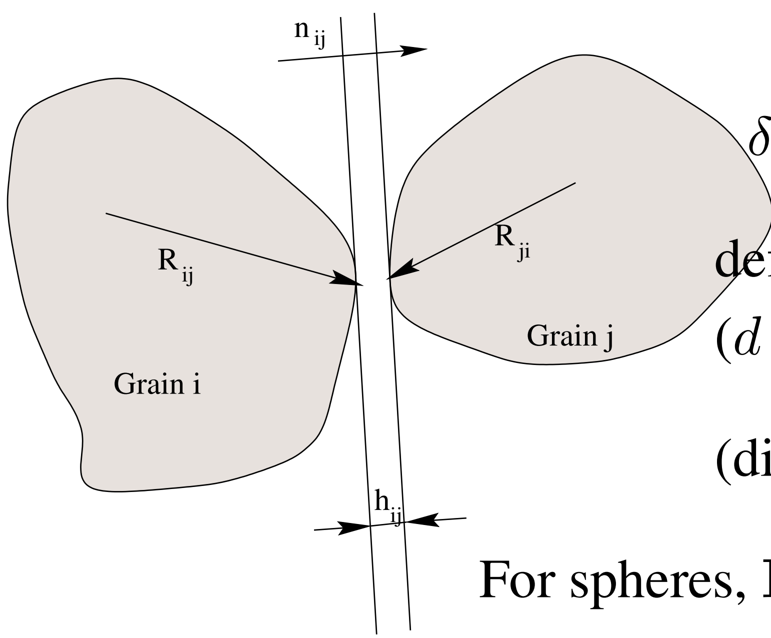
Displacement field $\tilde{\mathbf{u}}_i$ corresponding to small strains ϵ_1, ϵ_2, effect of global strain subtracted:

$$\tilde{\mathbf{u}}_i = \mathbf{u}_i + \underline{\underline{\epsilon}} \cdot \mathbf{r}_i$$

$$\Delta^2 = \frac{1}{n^* ||\underline{\underline{\epsilon}}||^2} \sum_{i=1}^{n^*} ||\tilde{\mathbf{u}}_i||^2$$

to characterize displacement fluctuations. Δ^2 sometimes large (~ 100)... Correlation length ?

Some properties of discrete structures
Relative displacements, rigidity matrix



$$\delta \mathbf{U}_{ij} = \mathbf{u}_i - \mathbf{u}_j + \delta\theta_i \wedge \mathbf{R}_{ij} - \delta\theta_j \wedge \mathbf{R}_{ji}$$

defines the rigidity matrix $\underline{\underline{\mathbf{G}}}$
($d \times N_c$ rows in dimension d, N_f columns)

(dim . N_f) $\mathbf{U} \mapsto \underline{\underline{G}} \cdot \mathbf{U} = \delta \mathbf{U}$ (dim. $3N_c$ in 3D)

For spheres, $\mathbf{R}_{ij} = R_i \mathbf{n}_{ij}$, $\mathbf{R}_{ji} = -R_j \mathbf{n}_{ij}$ and

$$\delta \mathbf{U}_{ij} = \mathbf{u}_i - \mathbf{u}_j + (R_i \delta\theta_i + R_j \delta\theta_j) \wedge \mathbf{n}_{ij}$$

Properties of rigidity matrices

- "Mechanism" motions: $\mathbf{U}$ such that $\underline{\underline{G}} \cdot \mathbf{U} = 0$. $\rightarrow$ k-dimensional space, k=degree of displacement indeterminacy. Includes global rigid-body motions
- Compatibility of relative displacements : $\delta\mathbf{U}$ corresponds to displacement vector $\mathbf{U}$ by $\underline{\underline{G}}$

Equilibrium condition = linear relation between contact forces and external load

$$\mathbf{F}_i^{ext} = \sum_{j \neq i} \mathbf{F}_{ij} \qquad (\mathbf{F}_{ij} = \text{force exerted by } i \text{ on } j \text{ at contact})$$

$$\mathbf{\Gamma}_i^{ext} = \sum_{j \neq i} \mathbf{F}_{ij} \wedge \mathbf{R}_{ij} \qquad \text{(moments of contact forces)}$$

If $\mathbf{f}$ is the vector of contact forces, $\mathbf{F}^{ext}$ the applied load, then

$$\mathbf{F}^{ext} = \underline{\underline{H}} \cdot \mathbf{f}$$

Properties of rigidity matrices

- Self-balanced contact forces: $\mathbf{f}$ such that $\underline{\underline{H}} \cdot \mathbf{f} = 0$. $\rightarrow$ space of dimension h, degree of force indeterminacy.
- Supportable loading vector = $\mathbf{F}^{ext}$ corresponding to some $\mathbf{f}$ by $\underline{\underline{H}}$

We use an assumption of small displacements (**ASD**)
($\mathbf{n}_{ij}$, $\mathbf{R}_{ij}$ constant, displacements delat with as infinitesimal, or like velocities)

For $\mathbf{f}$ and $\delta\mathbf{U}$, distinguish **normal and tangential parts**
With frictionless contacts **ignore tangential components**

Theorem of virtual work

$$\underline{\underline{H}} = \underline{\underline{G}}^T$$

If $\mathbf{f}$, a set of contact forces, balances load $\mathbf{F}^{ext}$
If $\mathbf{U}$, displacement vector, corresponds to relative displacements $\delta\mathbf{U}$, then (ASD)

$$\mathbf{f} \cdot \delta\mathbf{U} = \mathbf{F}^{ext} \cdot \mathbf{U}$$

Consequences: (exploit relation between rank and kernel dimension, and also that the range of $\underline{\underline{G}}^T$ is the orthogonal of the kernel of $\underline{\underline{G}}$)

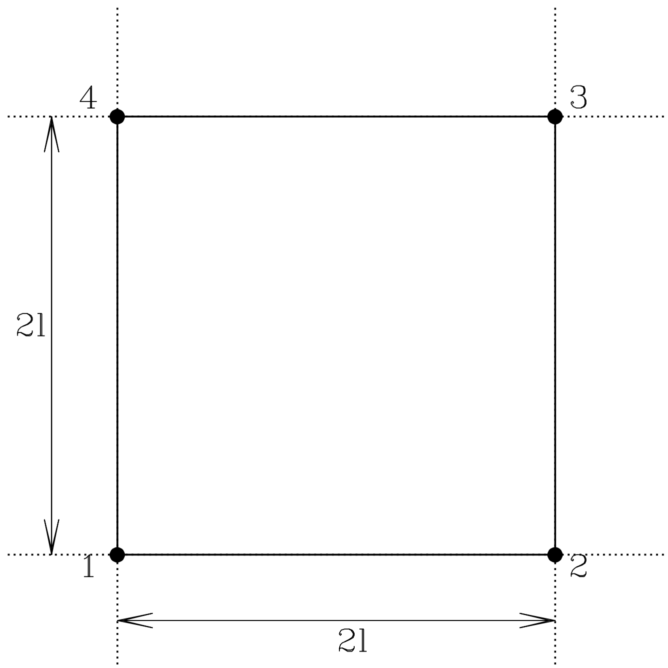
- Criterion of compatibility of relative des displacements (orthogonality to self-balanced forces), criterion for loads to be supportable (orthogonality to mechanisms)
- relation $N_f + h = dN_c + k$ between force and displacement degrees of indeterminacy. Without friction $N_f + h = N_c + k$

In a large system, $N_c = zn/2$ ($n =$ nb of grains).
For frictionless disks or spheres, $k \geq n$ (2D) or $k \geq 3n$ (3D)

Isostaticity properties

- If grains are rigid and frictionless, then, generically, $h = 0$
  $\Rightarrow$ **upper bound to coordination number**
  $z \leq 12$ (3D, general case) ; $z \leq 6$ (spheres) ; $z \leq 10$ (objects with axis of revolution) $z \leq 6$ (2D, general case) ; $z \leq 4$ (disks)
- With friction, z is in general lower, and there is relatively little force indeterminacy. Hence the importance of geometry in determination of force values
- Heterogeneous aspect, with force chains and wide distribution of force values
- Importance of inequalities to be satisfied by forces
- with cohesionless spheres, one has $k = 0$ **on the backbone** (= force-carrying structure), *i.e.* **isostaticity** (regular invertible rigidity matrix), apart from possible global rigid body motions (mechanisms would cause instabilities)
  $z^* = 6$ (3D), $z^* = 4$ (2D)

Why are four-legged tables wobbly ?



Square table: $N_f = 6$, $N_c = 4$.

Assuption: no friction. $k = 3 \Rightarrow h = 1$

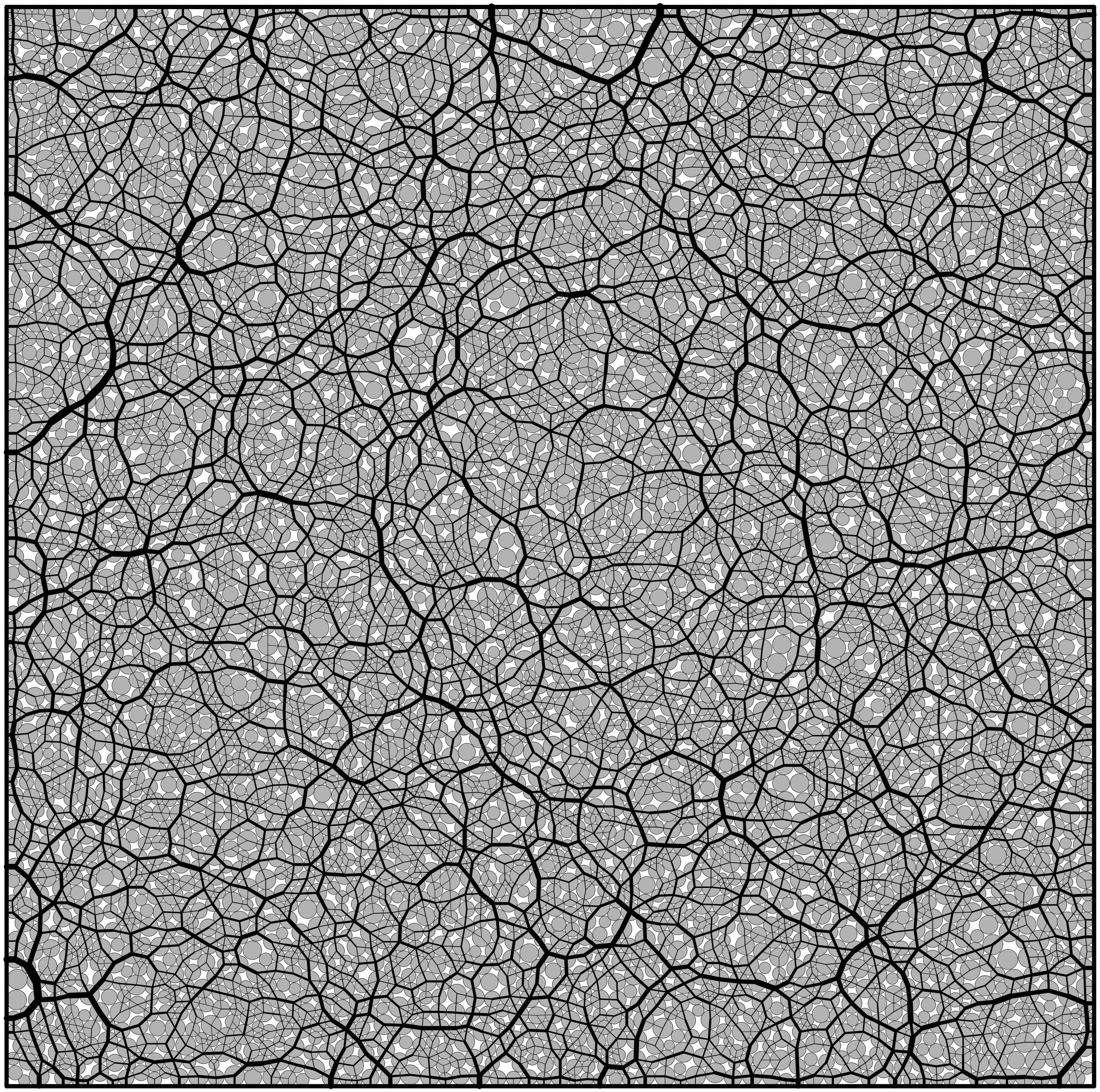
Self-balanced forces: $F_1 = -F_2 = F_3 = -F_4$

Length of legs : $L + \delta_i,\ 1 \leq i \leq 4$

Theorem of virtual work $\Rightarrow \displaystyle\sum_{i=1}^{4} F_i \delta_i = 0$ with 4 simultaneous contacts

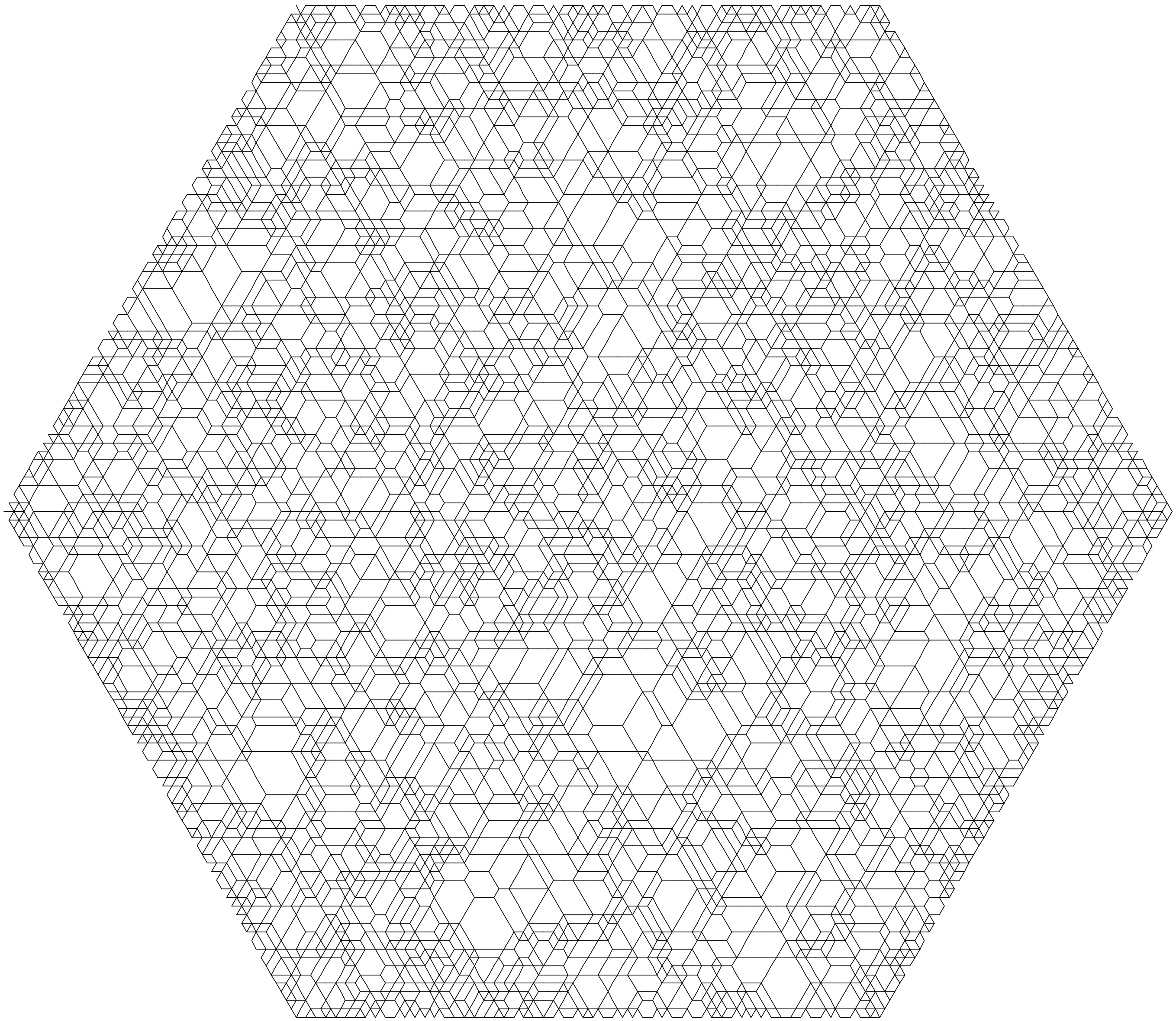
Whence $\delta_1 + \delta_3 = \delta_2 + \delta_4$, an occurrence of zero probability... Equivalent to condition of leg extremities being within same plane:

$$\begin{vmatrix} 2l & 2l & 0 \\ 0 & 2l & 2l \\ \delta_2 - \delta_1 & \delta_3 - \delta_1 & \delta_4 - \delta_1 \end{vmatrix} = 0$$



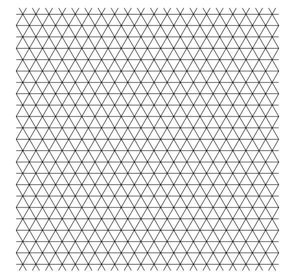
$n = 4900$ disks,
$n^* = 4633$ are active
2 mobile walls
$N_f = 9802$,
$h = 0$,
$k = 534$ (rattlers)
isostatic force-carrying structure with 9268 contacts
JUST ENOUGH FORCES FOR EQUILIBRIUM !

EQUILIBRIUM, RIGID, FRICTIONLESS CONTACTS, ISOTROPIC LOAD

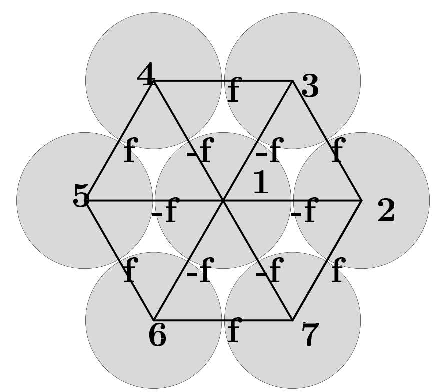


$\neq$

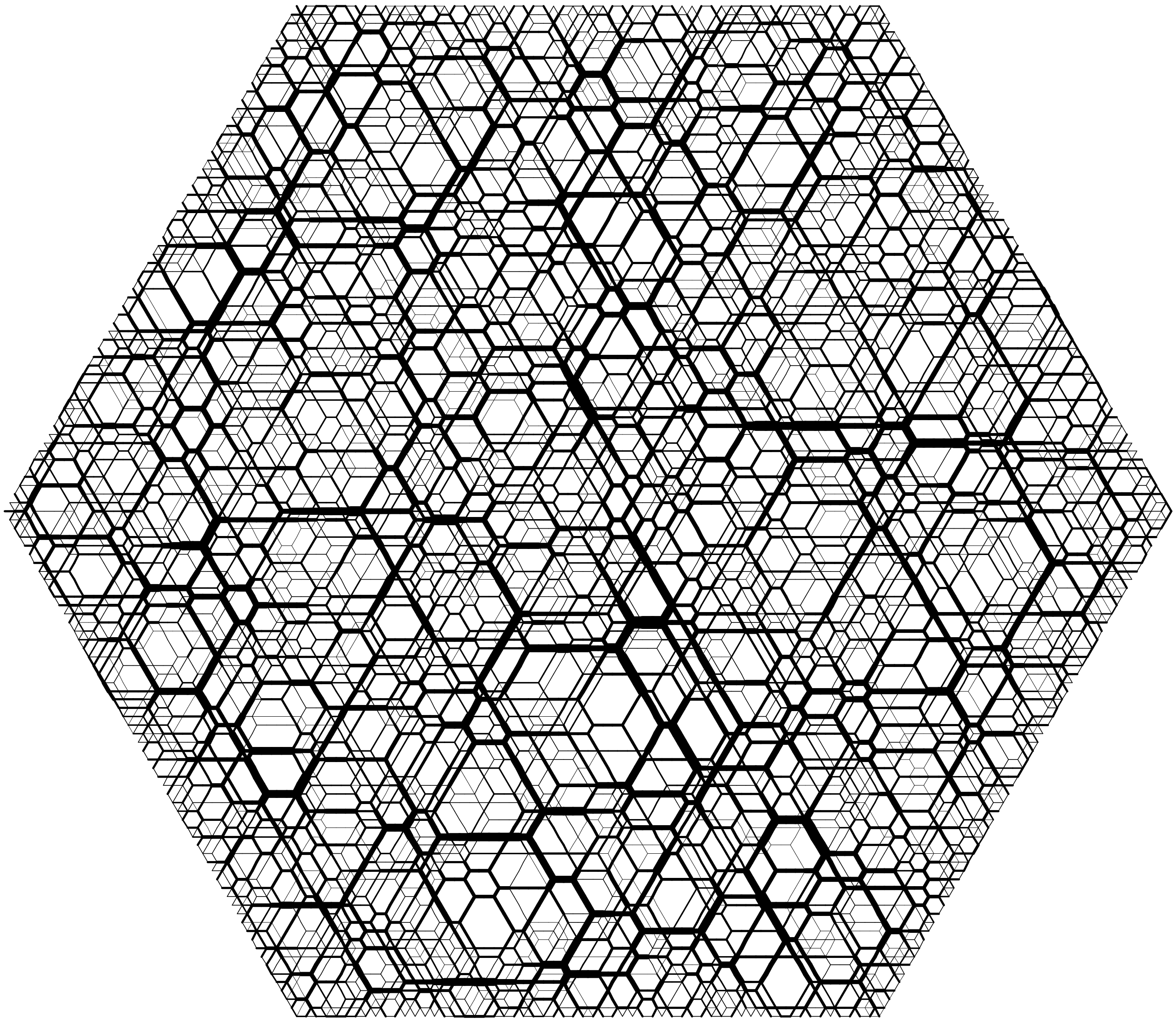
complete lattice !



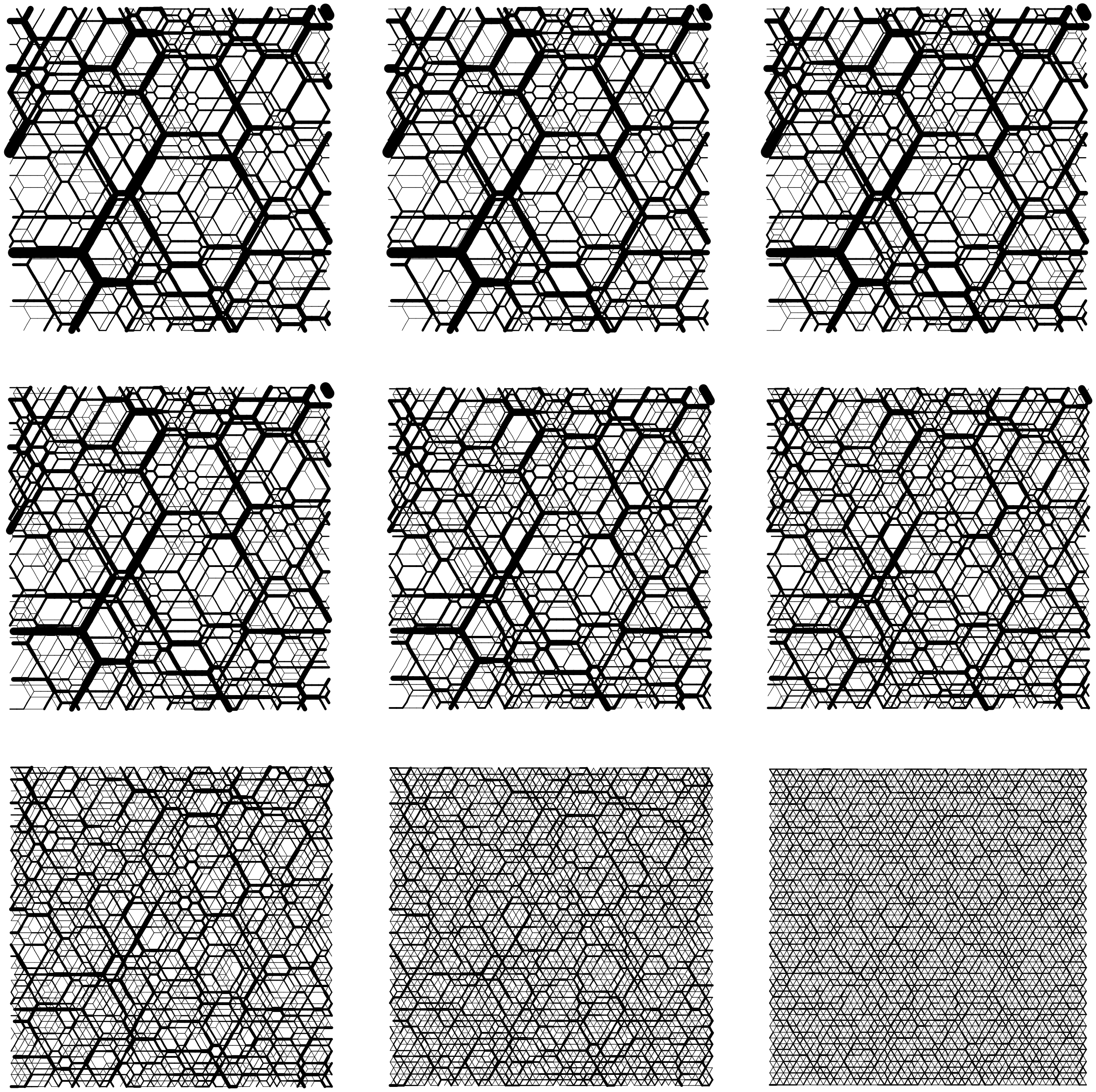
No such pattern:
(hyperstatic)



Regular lattice, but small polydispersity: active contacts



Same structure with force intensity encoded as line thickness

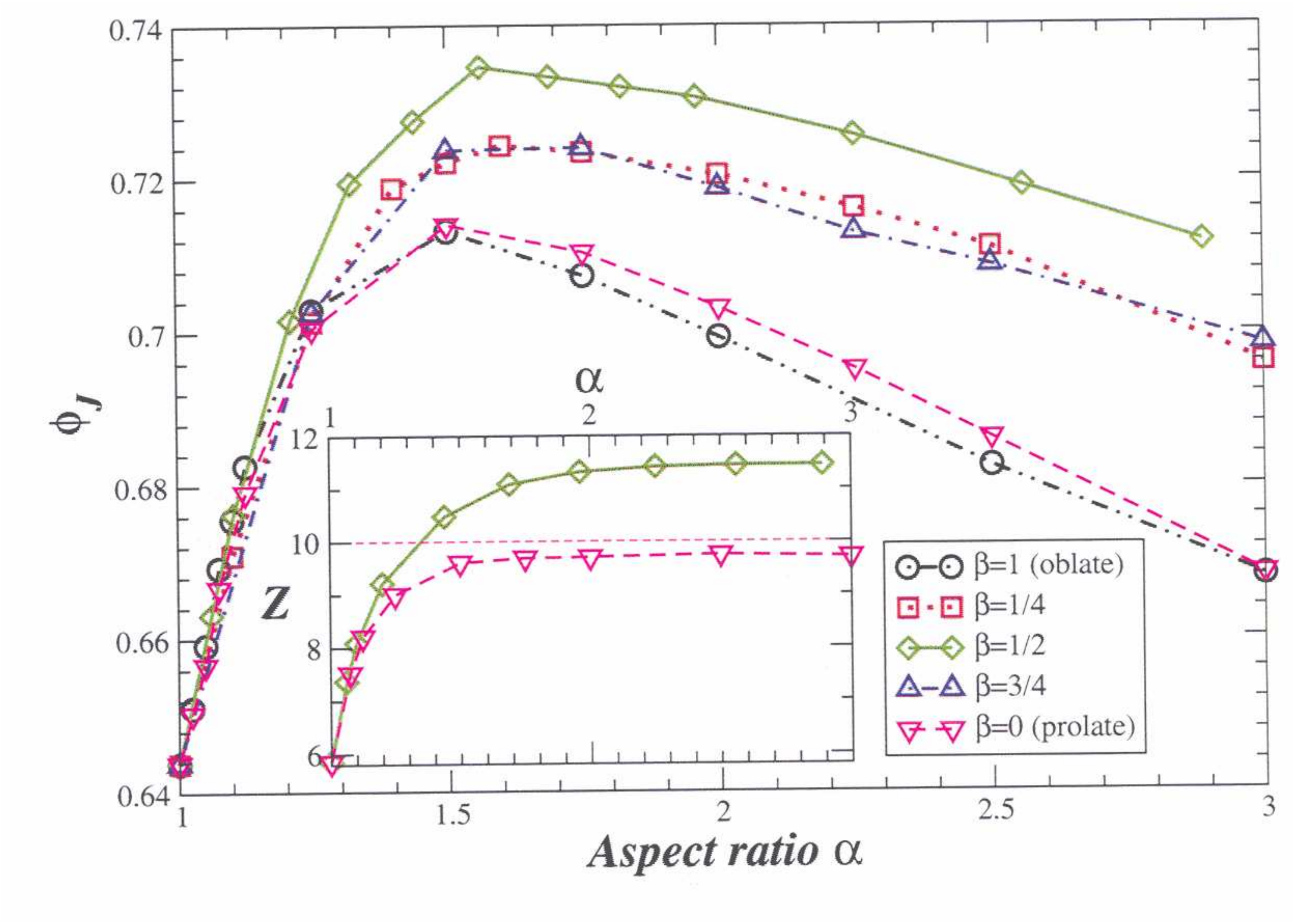


Effect of pressure increase – or lesser contact stiffness. In general, larger coordination numbers are obtained with softer contacts – and tables cease to wobble once on a rug

Why assemble frictionless grains ?

- Contact law irrelevant in rigid limit because no indeterminacy !
  (Unfortunately, this wonderful property is lost with friction)
- Frictionless, rigid grains under isotropic pressure stabilize in configuration of minimum volume, subject to steric exclusion
- $\Rightarrow$ interesting limit, extreme case of contact scarcity
- $\Rightarrow$ effects of perturbations on contact network ?
- Numerically, obtention of remarkable random close packing state, with $\Phi \simeq 0.639$ (identical spheres), **unique** unless traces of crystallization are induced by enduring agitation

Frictionless, rigid, non-spherical particles



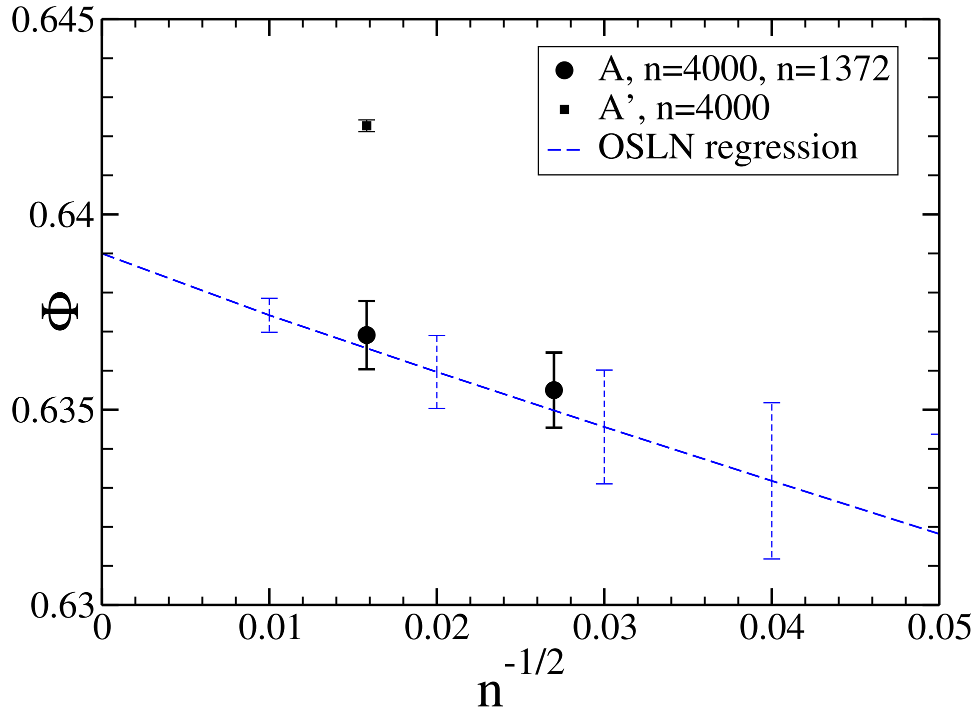
(Donev *et al.*, Phys. Rev. E, 2007).

Φ_{RCP} and z for ellipsoids, axes 1, α^{β}, α. $\sim$ no rattler. Note $k > 0$.

Sample assembling procedures

- In the lab or in numerical simulations, assembling stage partly determines final mechanical properties
- Dense configurations are obtained on circumventing influence of friction: lubrication, vibration
- Cohesion can make packings very loose (there is no contact law-independent definition of a low Φ limit)
- laboratory methods include controlled pluviation and layerwise tamping
- numerically, possible to use lower μ on preparing equilibrium configuration
- with friction *Φ and z^* independent for isotropic states*
  vibration procedure $\rightarrow$ low coordination in final equilibrated state

The random close packing state



A = fast compression, frictionless. A' = longer agitation (Lubachevsky-Stillinger algorithm)

OSLN = results by O'Hern *et al.*, 2003, different simulation process

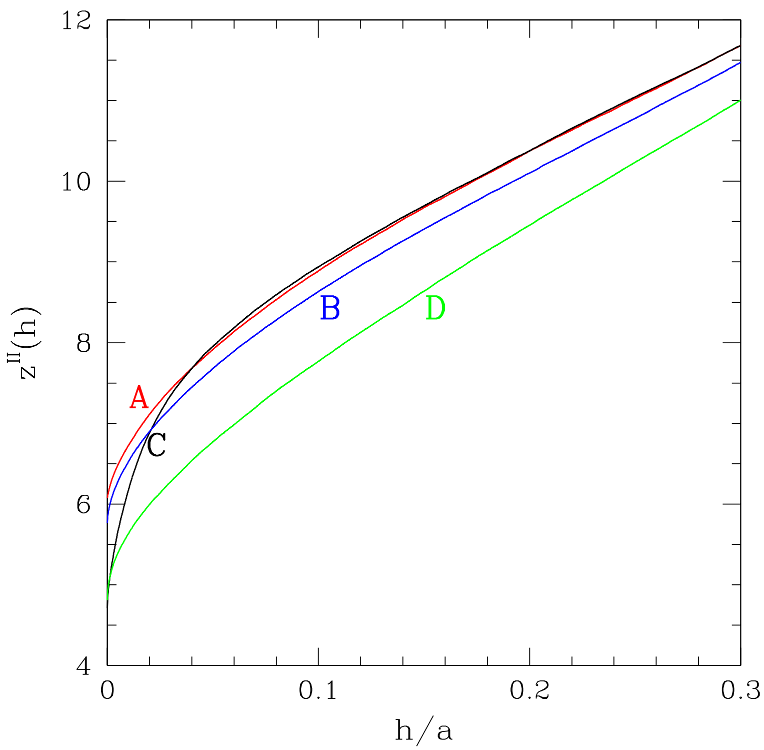
A' more ordered than A.

With bidisperse systems: separation rather than crystallisation.

Geometry of sphere assemblies: interstices

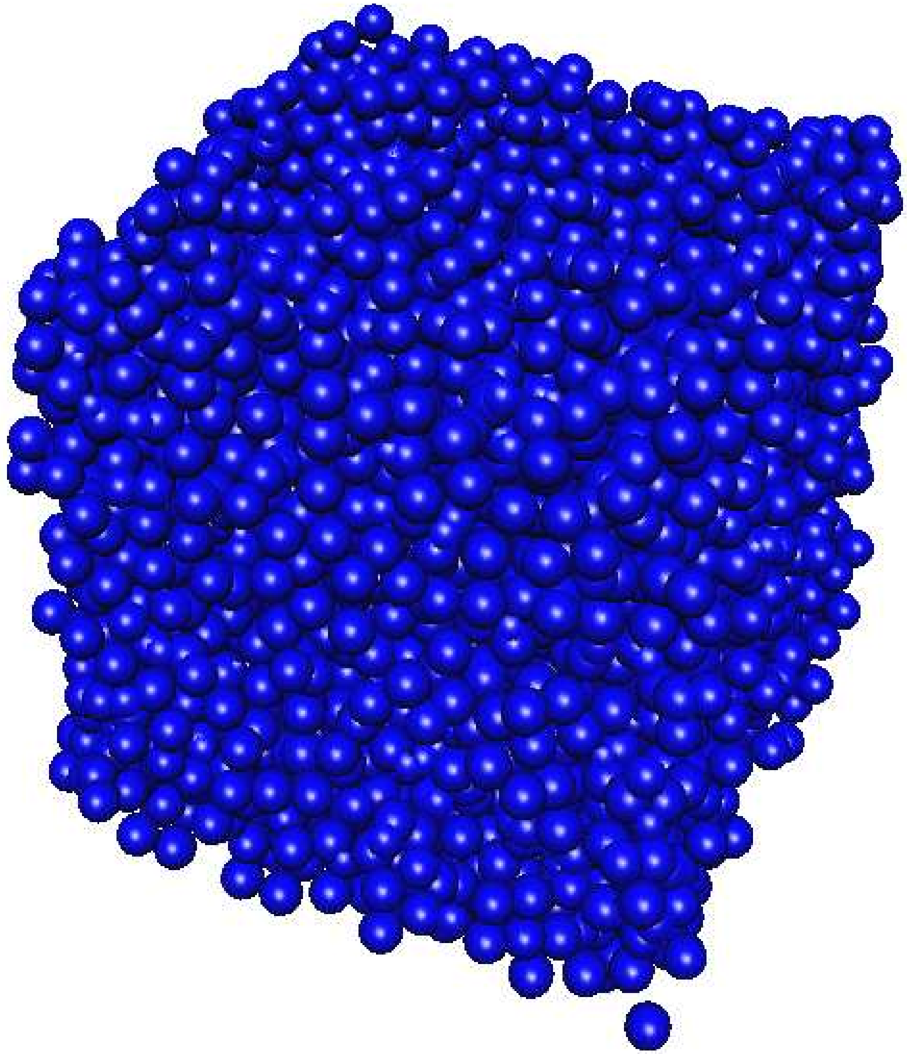
Four different isotropic packing structures, with solid fraction and coordination number varying *independently*

$$\Phi_A \simeq \Phi_C > \Phi_B > \Phi_D, \text{ but } z_A > z_B > z_C \simeq z_D$$



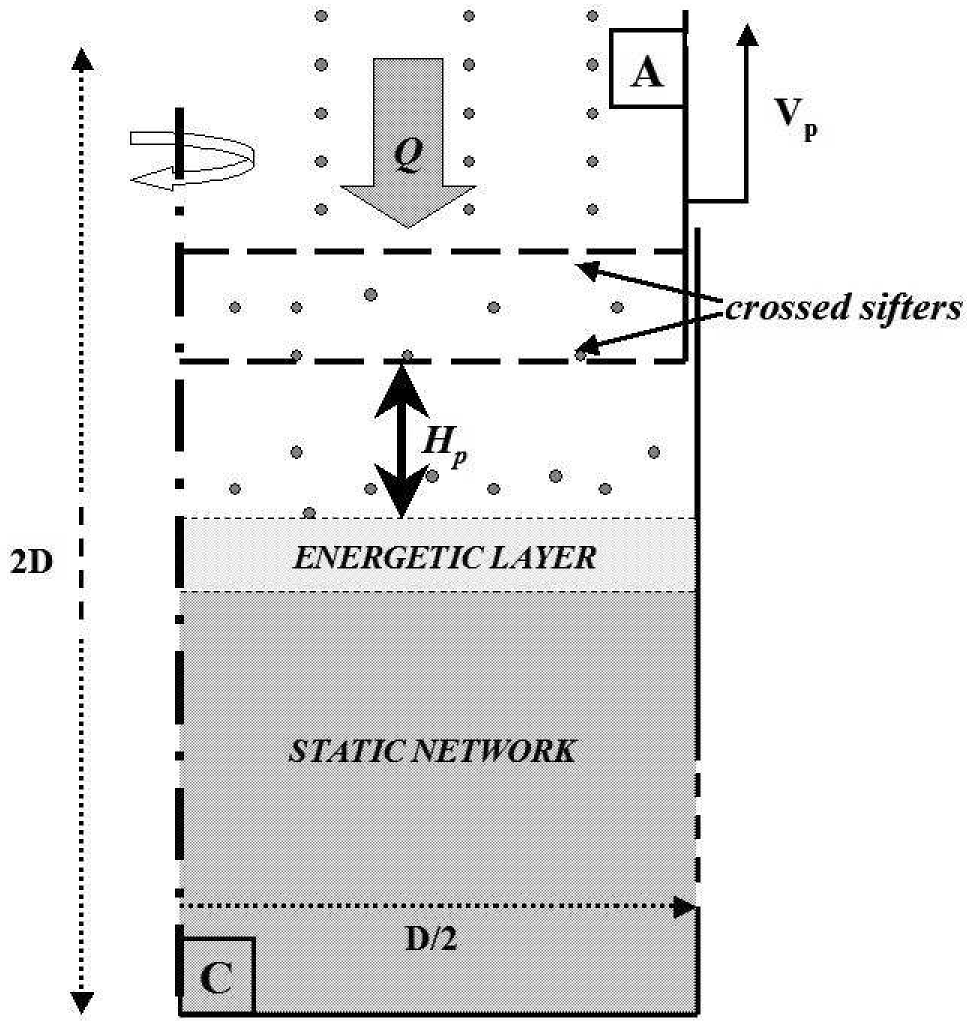
Gap-dependent coordination number: number of neighbors at distance $\leq h$. Here rattlers have been "stuck" to backbone to get a fully defined packing geometry

Results for $h/a \leq 0.04$ not determined by density, still inaccessible to direct measurements (X-ray tomography, Aste *et al.* 2004, 2005 : accuracy of $\sim 0.05 \times a$)



Typical numerical samples are made of 4000 or 5000 beads. Check for reproducibility and sample to sample fluctuations

Pluviation : principle, control parameters

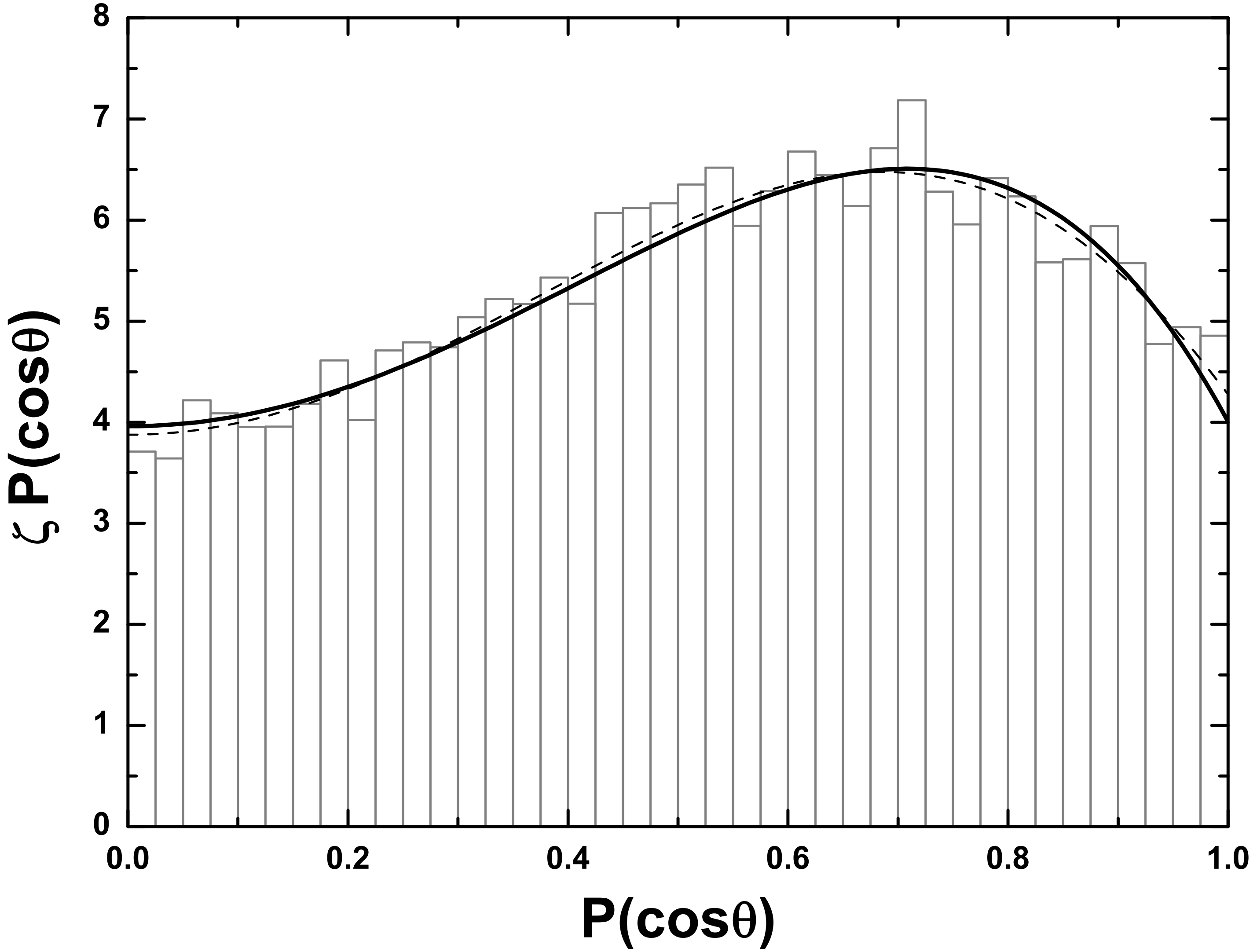


- Constant height of free fall $H_p \Rightarrow$ dimensionless ratio $\boxed{H_p^* = \frac{H_p}{a}}$
- mass flow rate per unit area Q, controlled from upper reservoir outlet
  $\Rightarrow$ reduced flow rate
  $$\boxed{Q^* = \frac{Q}{\rho_p \sqrt{ag}}}$$
- agitation in superficial layer, approach to equilibrium below
- Final density $\nearrow$ as $H_p^* \nearrow$ and as $Q^* \searrow$

Simulating the pluviation process: results

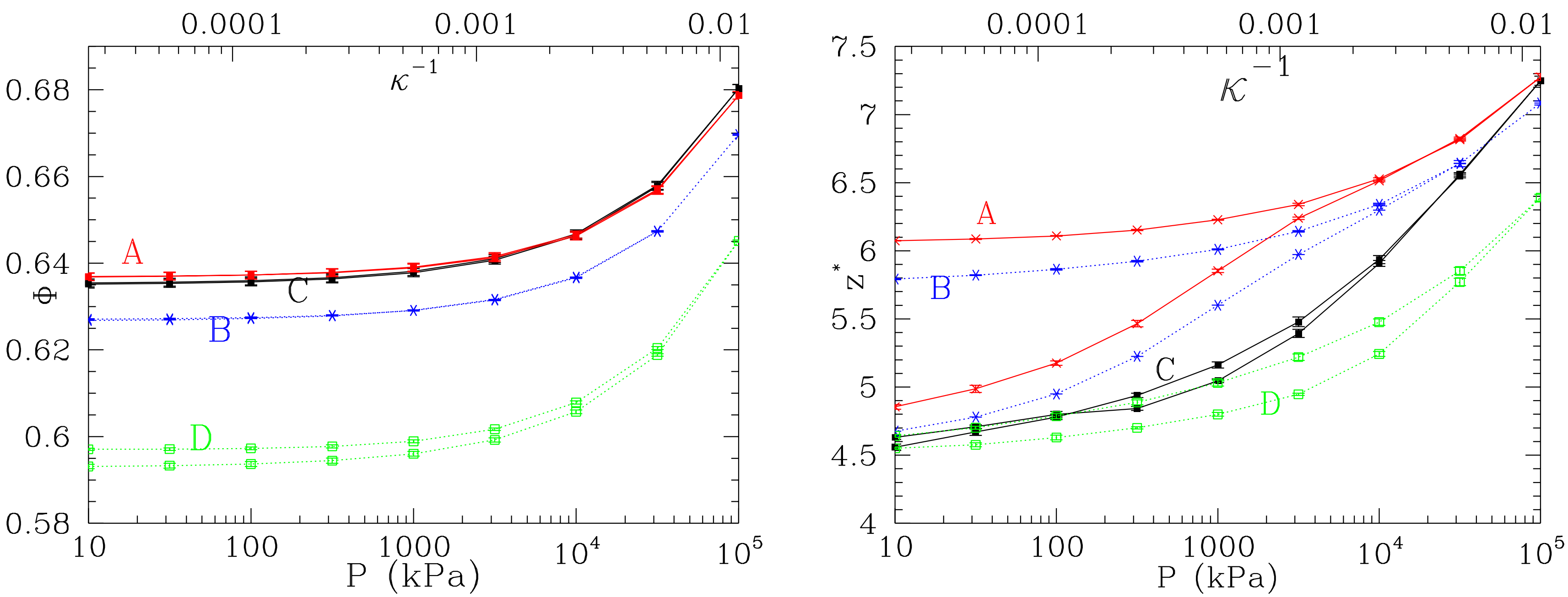
- anisotropic states, characterised by distribution of $\cos\theta$, $\theta =$ angle between normal to contact and vertical direction
- Homogeneity: same state, apart from stress level, except near bottom or top
  Wrong if H_p not constant !
- Under agitated upper layer, nearly quasistatic oedometric compression
- Influence of viscous damping (bad news !)
- Difficult to compare with experiment (damping + shape/size of beads) $\Rightarrow$ compare mechanical properties !
- Coordination and fabric conserved on isotropically compressing
- Moderate fabric anisotropy and rather large coordination number (closer to A than C in dense states) with “reasonable” choices of damping parameters

Final state (simulations): contact orientations



$P(\cos\theta)$ well fitted by its development to order 4 (2 coeff.) in Legendre polynomials $\rightarrow$ solid line (order 6 = dotted line). Here $\zeta =$ coordination number

Solid fraction and coordination number in isotropic pressure cycle



Initially isotropic states A, B, C, D. Very nearly reversible for Φ, not reversible for z^*, which decreases if initially high.

Similar in systems assembled by pluviation. Preparation process include compression stage in practice

A microscopic expression of the stress tensor

Plane surface S, of equation $z = z_0$, area A within material, unit normal vector$\mathbf{n}$ (towards growing z). a= grain diameter

$\mathbf{J}(z_0)$ = momentum transmitted from $z < z_0$ to $z > z_0$ in unit time= (kinetic contribution +) contribution of forces $\mathbf{J}_f(z_0)$.

In equilibrium $\mathbf{J} = \mathbf{J}_f$

Then $\mathbf{J}(z_0) = A\underline{\underline{\sigma}} \cdot \mathbf{n}$, or $J^\alpha(z_0) = A\sigma_{\alpha z}$ for coordinate α

$$\mathbf{J}(z_0) = \sum_{i \,|\, z_i < z_0 ,\, j \,|\, z_j > z_0} \mathbf{F}_{ij}$$

Macroscopic stresses are assumed to vary on scale $L \gg a \Rightarrow$ possible to average on position z_0 ($a \ll l \ll L$)

$$
\begin{aligned}
A\underline{\underline{\sigma}} \cdot \mathbf{n} &= \frac{1}{l} \int_{z_0-l/2}^{z_0+l/2} \mathbf{J}(z) dz \\
&= \frac{1}{2l} \sum_{|z_i-z_0|<l/2, |z_j-z_0|<l/2} \mathbf{F}_{ij}(z_j - z_i) \\
&= \frac{1}{2l} \sum_{|z_i-z_0|<l/2, |z_j-z_0|<l/2} \mathbf{F}_{ij} \left[(\mathbf{r}_j - \mathbf{r}_i) \cdot \mathbf{n}\right]
\end{aligned}
$$

whence for a sample of volume V where stresses are uniform :

$$
\underline{\underline{\sigma}} = \frac{1}{V} \sum_{i=1}^{N} \frac{1}{2} \left(\sum_{j,\, j \neq i} \mathbf{F}_{ij} \otimes \mathbf{r}_{ij} \right),
$$

with $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$, or, in another form

$$
\underline{\underline{\sigma}} = \frac{1}{V} \sum_{1 \leq i < j \leq N} \mathbf{F}_{ij} \otimes \mathbf{r}_{ij}.
$$

$$
\sigma_{\alpha\beta} = \frac{1}{V} \sum_{i<j} F_{ij}^{(\alpha)} r_{ij}^{(\beta)}
$$

Application : relation between pressure and average force

With spheres $\mathbf{r}_{ij}$ and $\mathbf{n}_{ij}$ are parallel

$$P = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3V}\sum_{i<j} F_{ij}^N (R_i + R_j)$$

N_c contacts. Diameter a $\Rightarrow P = \dfrac{aN_c}{3V}\langle F^N \rangle$

Contact density N_c/V also reads $z\Phi/(2v)$ with v=volume of one grain

Hence

$$P = \frac{z\Phi}{\pi a^2}\langle F^N \rangle$$

Other derivation *via* theorem of virtual work

Impose some *homogeneous strain* $\underline{\underline{\epsilon}}$ on moving peripheral grains:

$$\mathbf{u}_i = -\underline{\underline{\epsilon}} \cdot \mathbf{r}_i \quad \text{if } i \text{ belongs to the boundary}$$

Then the work of external forces is, by definition :

$$\delta W = V \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}.$$

Taking equilibrated internal (contact) forces corresponding to $\underline{\underline{\sigma}}$, and displacements as

$$\mathbf{u}_i = -\underline{\underline{\epsilon}} \cdot \mathbf{r}_i \quad \textit{for all } i$$

one also has:

$$\delta W = \sum_{i<j} \mathbf{F}_{ij} \cdot \left(\underline{\underline{\epsilon}} \cdot \mathbf{r}_{ij}\right)$$

and (as $\underline{\underline{\epsilon}}$ is arbitrary) :

$$\underline{\underline{\sigma}} = \frac{1}{V} \sum_{i<j} \mathbf{F}_{ij} \otimes \mathbf{r}_{ij}$$

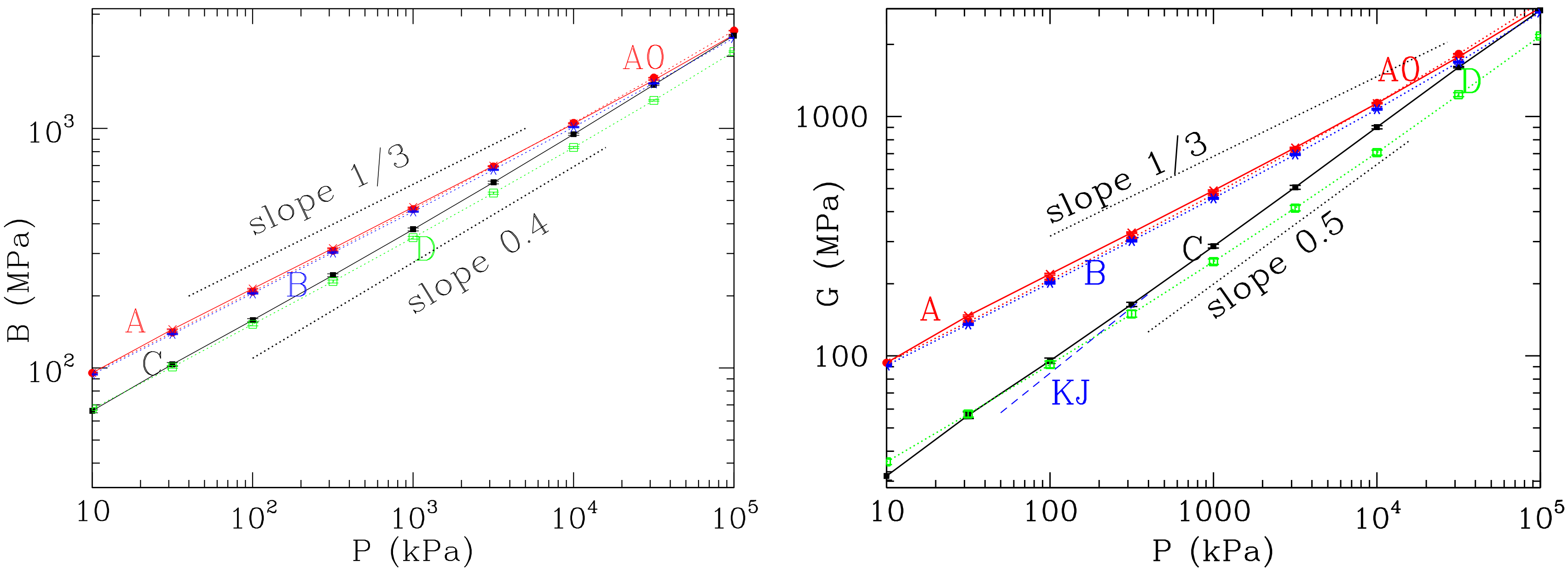
Elastic moduli (under isotropic pressure)

- B and G to be evaluated with very low strains or stress increments. Their very definition implies (accurate) approximations
- Method: dynamical simulation or use of stiffness matrix.
- Average contact stiffnesses scale as $P^{1/3}$ because of Hertz's law
- Voigt-like (for B et G), Reuss-like (for B) bounds available, knowing Φ, z, moments of force distribution ($Z(\alpha) = \langle F_N^\alpha \rangle / \langle F_N \rangle^\alpha$)

$$\frac{1}{2}\left(\frac{z\Phi\tilde{E}}{3\pi}\right)^{2/3}\frac{P^{1/3}}{\tilde{Z}(5/3)} = B^{\text{Reuss}} \leq B \leq B^{\text{Voigt}} = \frac{1}{2}\left(\frac{z\Phi\tilde{E}}{3\pi}\right)^{2/3} P^{1/3} Z(1/3)$$

$$G \leq G^{\text{Voigt}} = \frac{6 + 9\beta_T}{10} B^{\text{Voigt}}$$

Elastic moduli in isotropic systems

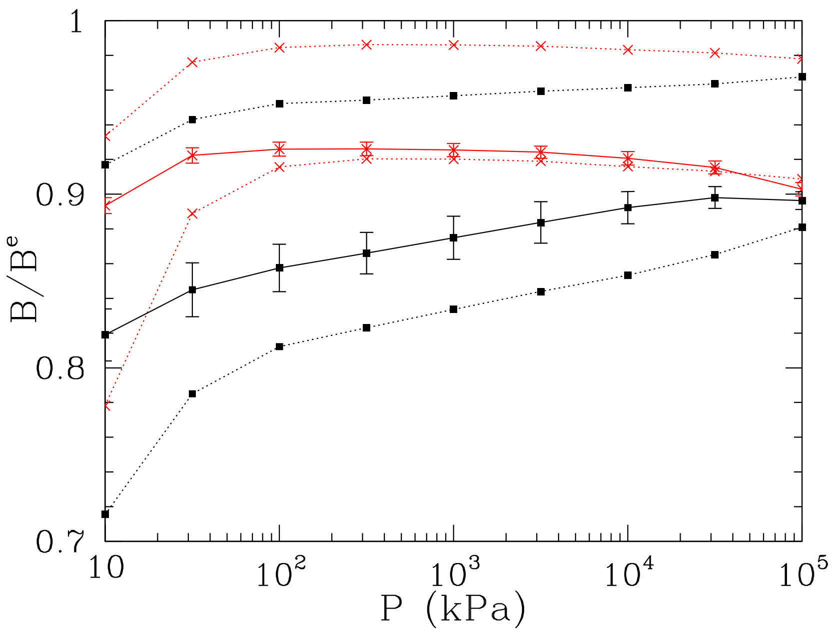


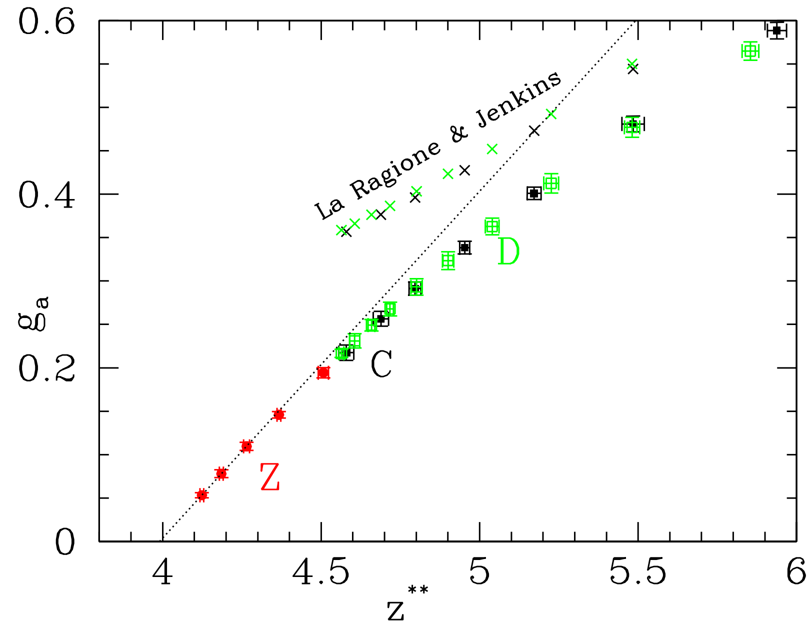
A et B : high z ($\sim$ 6 under small P) ; C et D : low z ($\sim$ 4.1 under small P)

$\Rightarrow$ elastic moduli provide access to coordination numbers

"KJ" = experimental results, loose packing of glass beads

Predictions of moduli ?



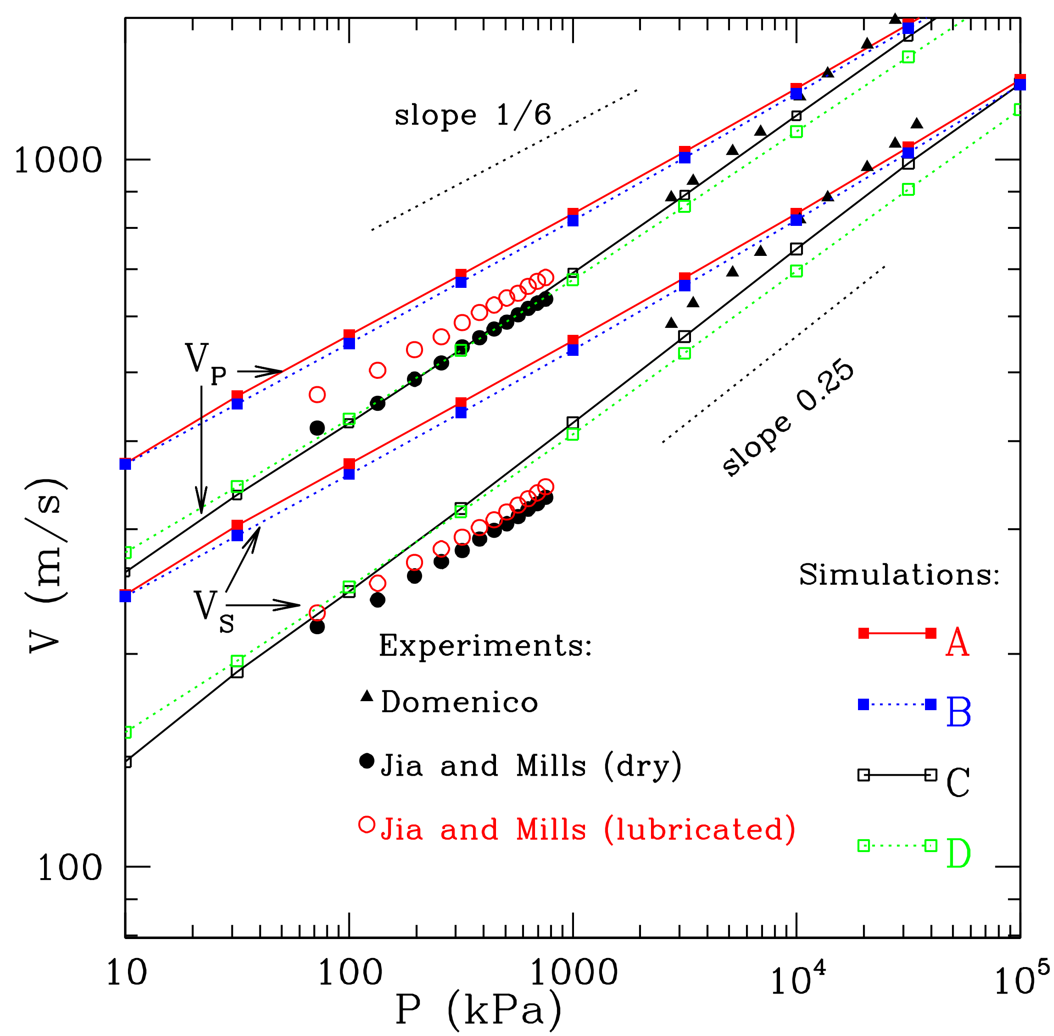


Shown : amplitudes, normalized by average stiffness.

B accurately bracketed by Voigt and Reuss bounds ; G difficult to estimate, especially in poorly coordinated systems, even with sophisticated schemes (La Ragione-Jenkins)

G anomalous, proportional to degree of force indeterminacy when it is small

Comparisons with experimental results: speed of sound

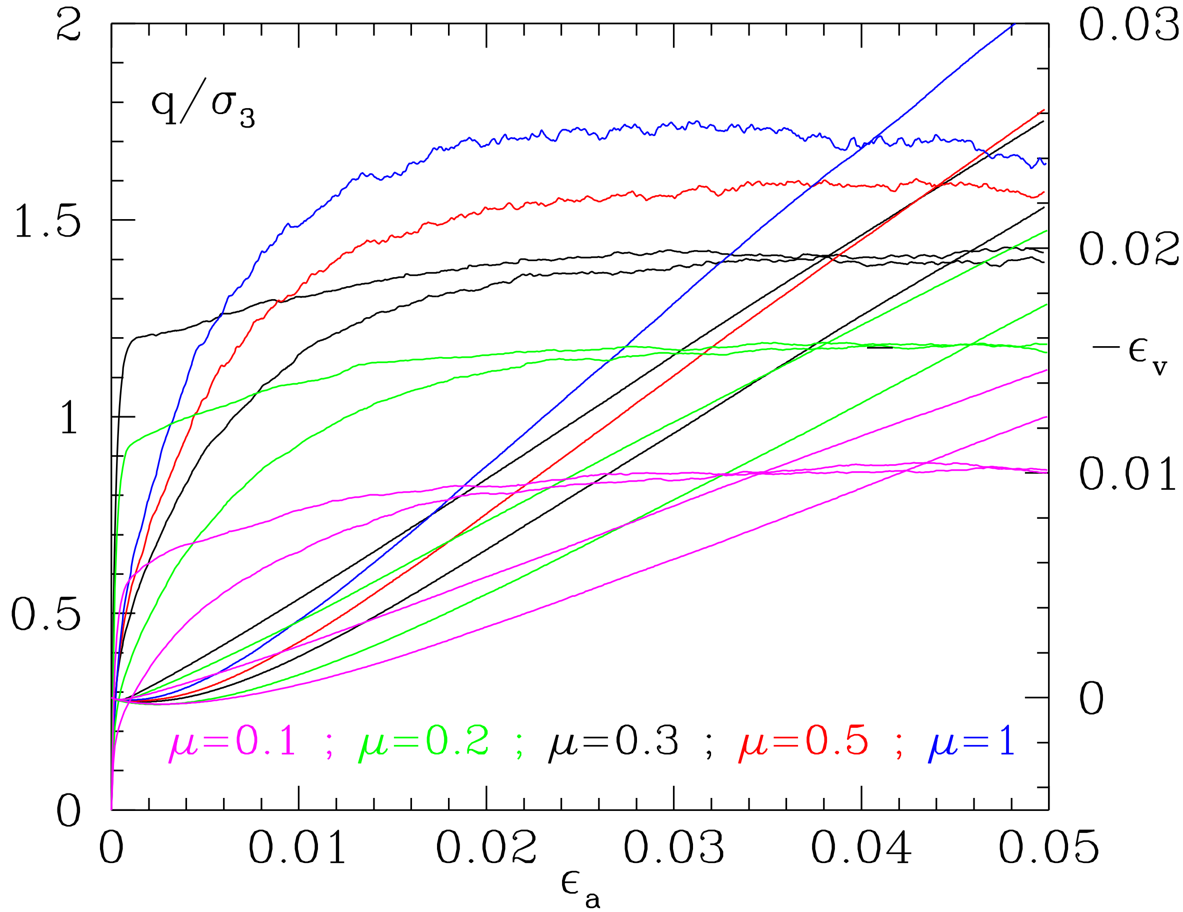


C better model for dry grains. Effects of lubrication in experiment (Φ decreasing from 0.64 to 0.62) similar to B versus C in simulations . Anisotropy ?

Some conclusions on sample preparation and resulting elastic moduli

- Density alone not enough to classify packings: coordination number may change a lot for dense samples (Not recognised yet ! And study of assembling process still neglected...) Extreme cases obtained with perfect lubrication, with vibration
- Compacting = avoiding the effects of friction
- Moderate anisotropy in simulations of pluviation, obtained states closer to partially lubricated ones
- Confrontations with experiment: best with elastic moduli, which indirectly determine coordination
- Needed:
  - more experimental results on elastic moduli (full anisotropic data, 5 moduli in samples obtained by pluviation)
  - Better-characterized experimental assembling procedure (than “tapping”, “mixing with a lubricant”...)
  - Better model for viscous dissipation in contacts

Triaxial compression from isotropic states A (large z) and C (small z) importance of coordination number



Internal friction at peak + dilatancy related to density

strain to peak related to coordination number

Triaxial tests on frictionless spheres

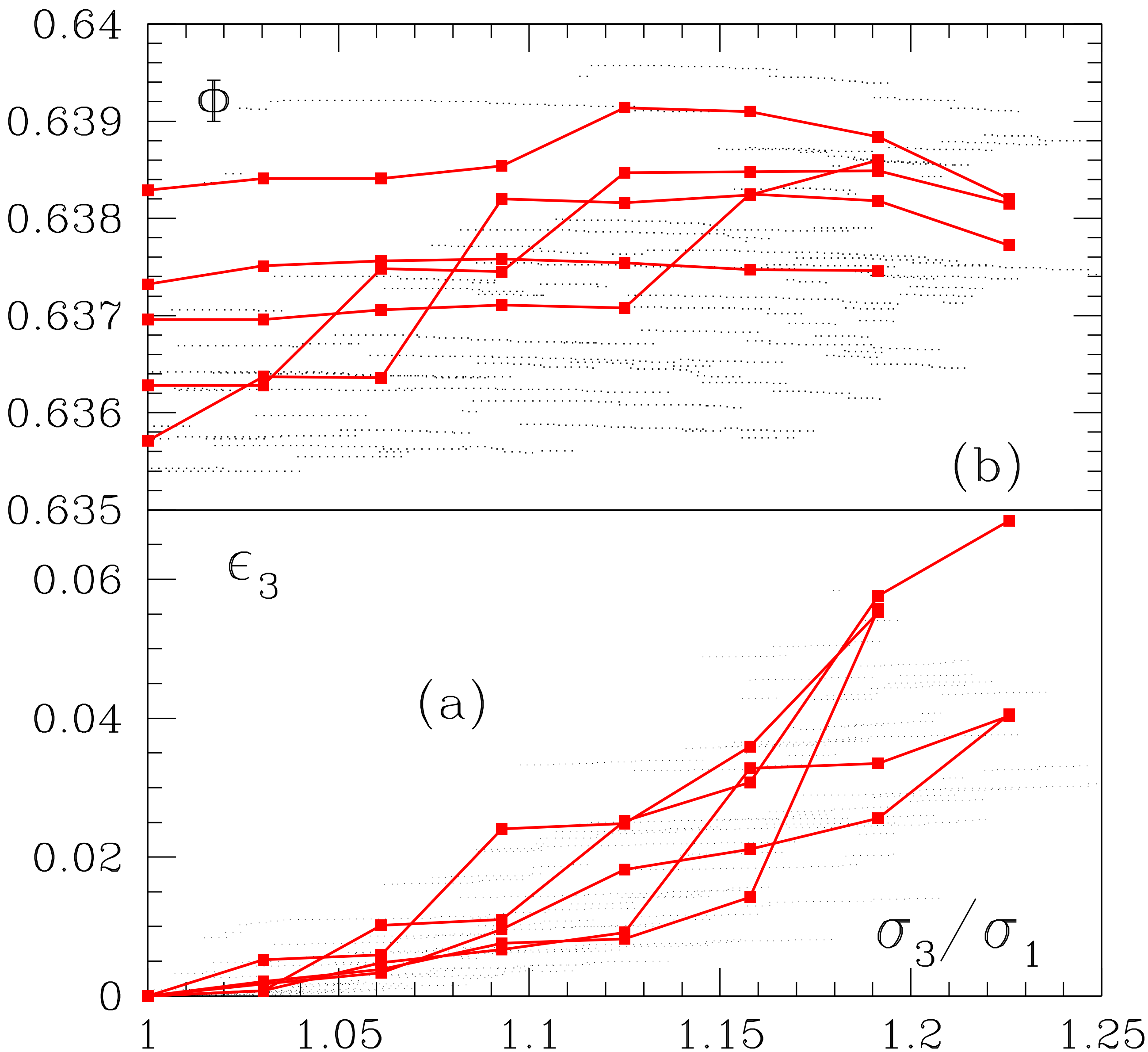
From initial isotropic state, apply:

$$\begin{cases} \sigma_1 & = p - q/2 \\ \sigma_2 & = p - q/2 \\ \sigma_3 & = p + q \end{cases}$$

increasing stepwise q/p by 0.02, waiting for equilibrium

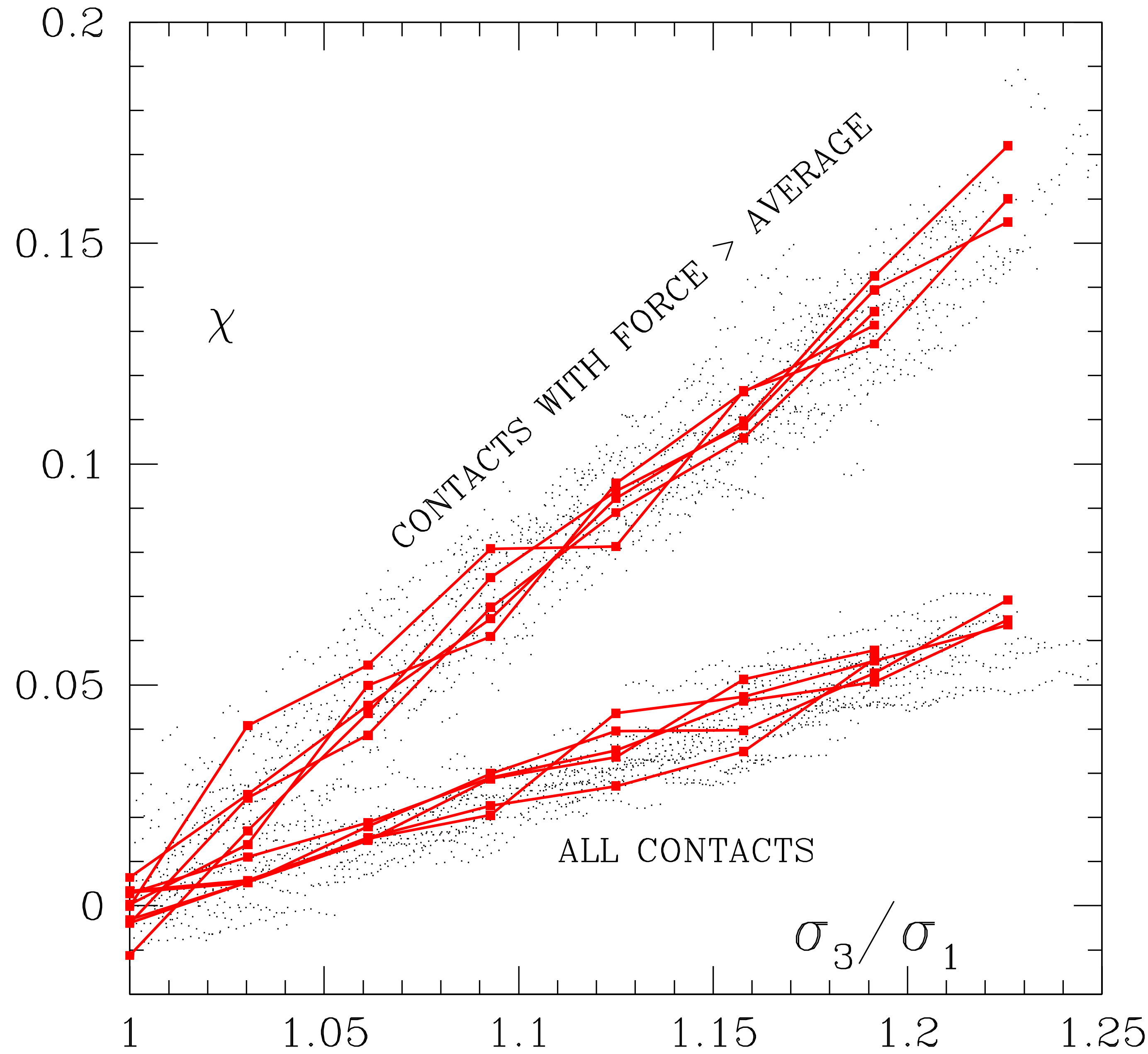
Triaxial tests on frictionless spheres

Packing fraction Φ and axial strain ϵ_3 vs. principal stress ratio. $n = 1372$ (small symbols), $n = 4000$ (connected dots)



Triaxial tests on frictionless spheres

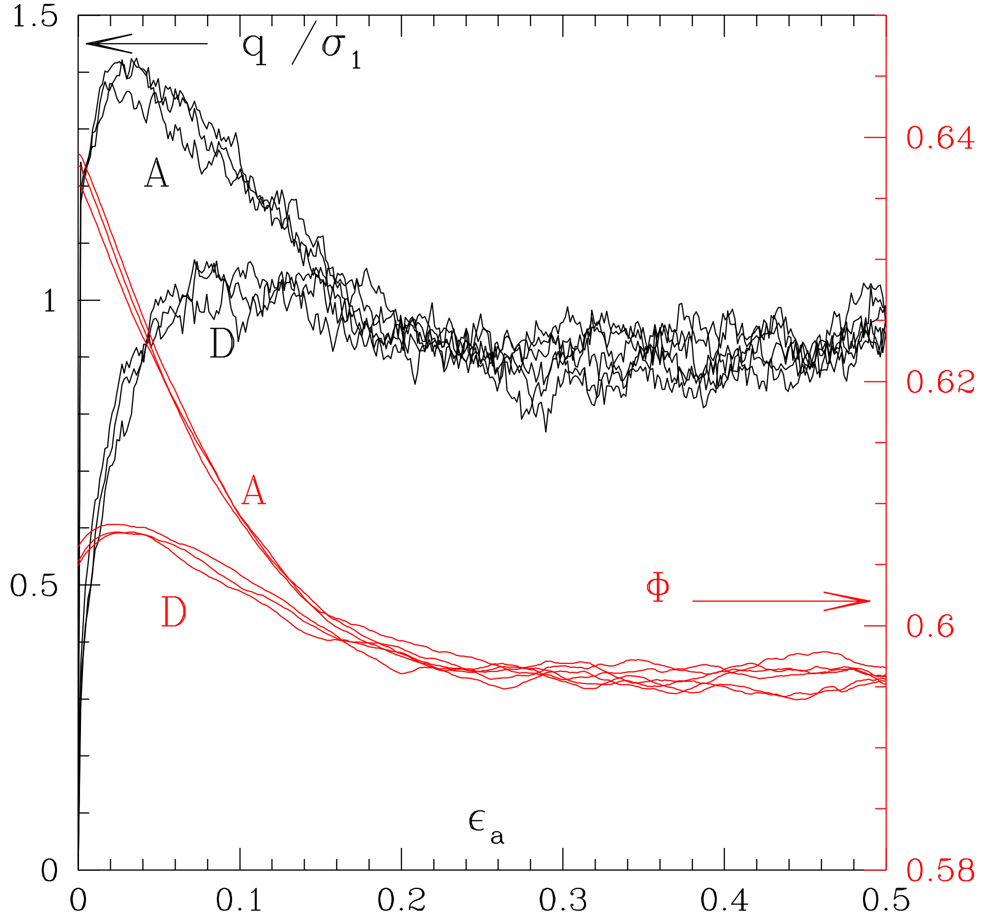
Fabric parameter $\chi = 3\langle n_z^2 \rangle - 1$ versus principal stress ratio.



Triaxial tests on frictionless spheres: conclusions

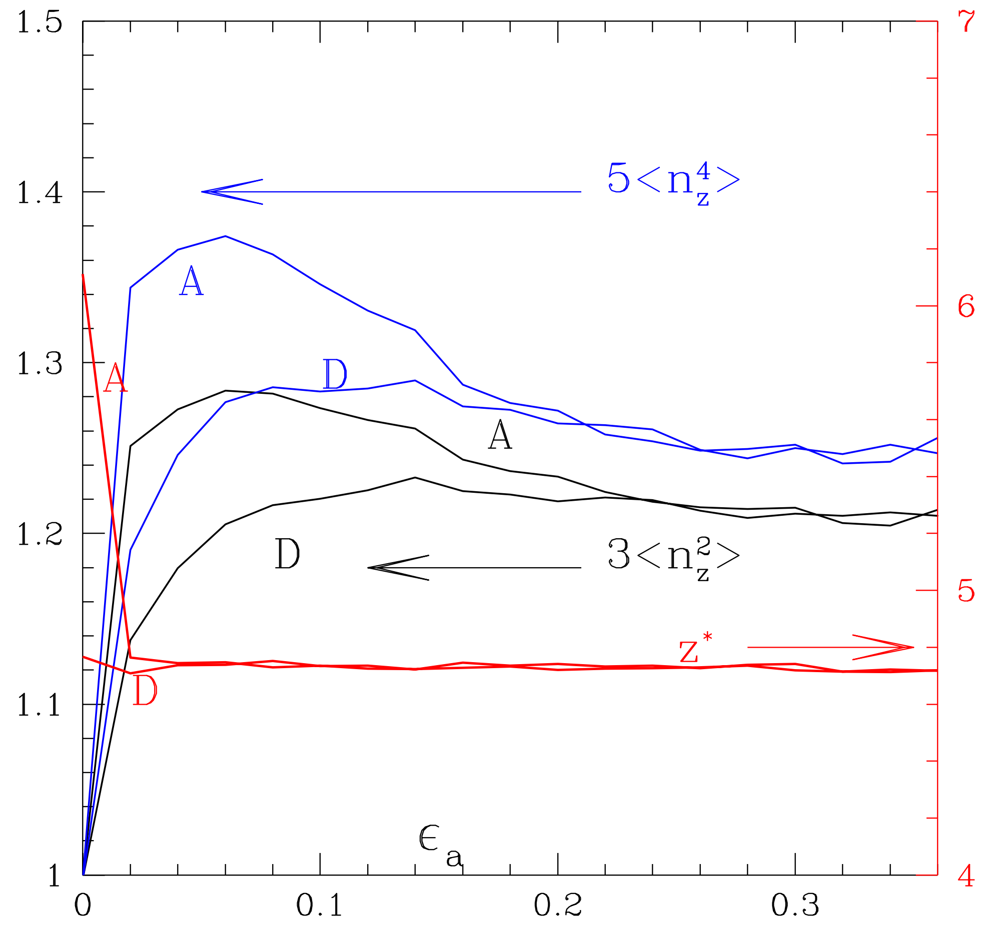
- Apparently, no clear approach to stress-strain curve (it was concluded before that no such curve existed, Combe 2000)
- evidence for a fabric/stress ratio relationship
- internal friction angle $\sim$ 5 or 6 degrees
- no dilatancy, RCP density for different stress states
- Contradicts “stress-dilatancy” idea that internal friction combines intergranular friction and dilatancy effects

Simulated behaviour for large strain: approach to critical state



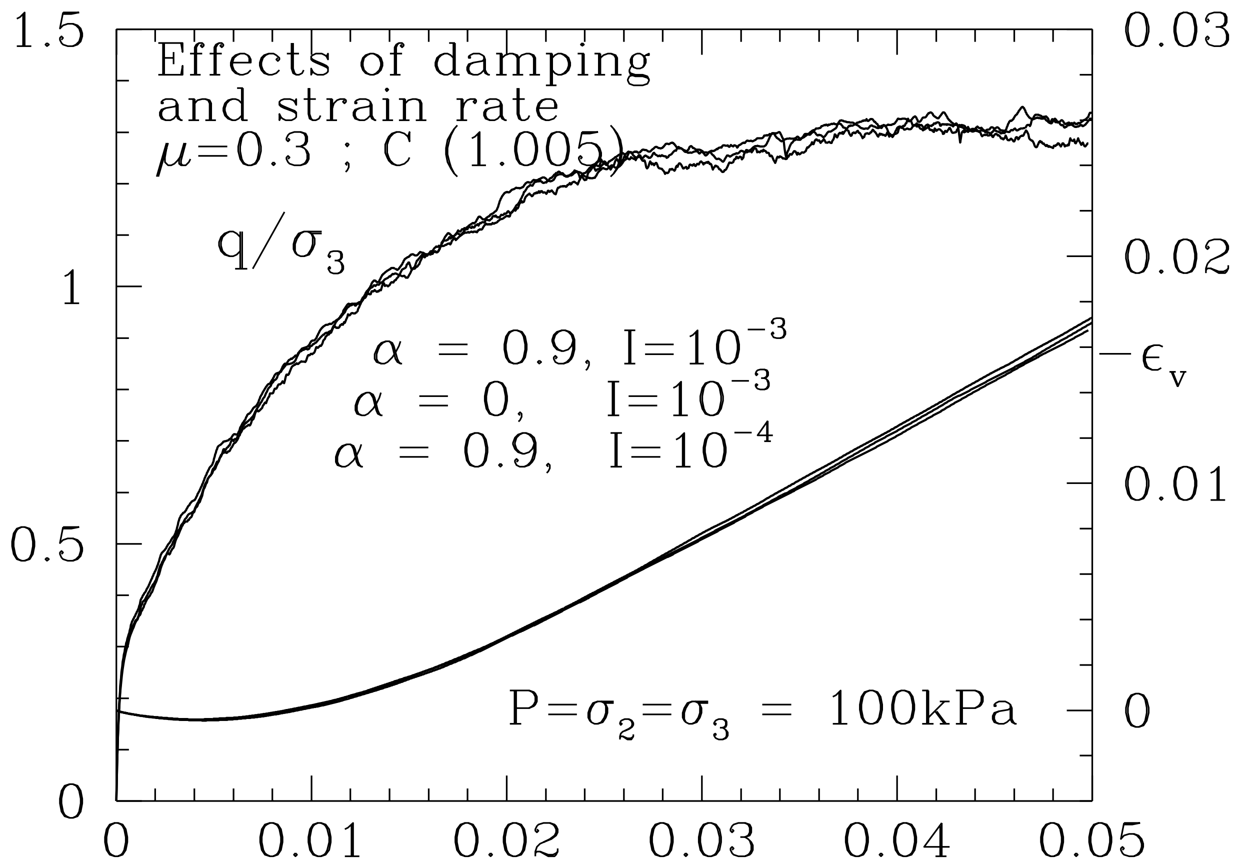
A plastic plateau independent of initial state appears for large strains, and solid fraction approaches “critical” value

Internal state variables on approaching critical state



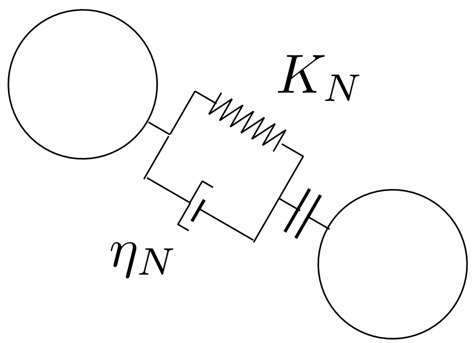
Internal variables like moments of unit vector coordinate distribution and coordination also approach “critical” values independent of initial state

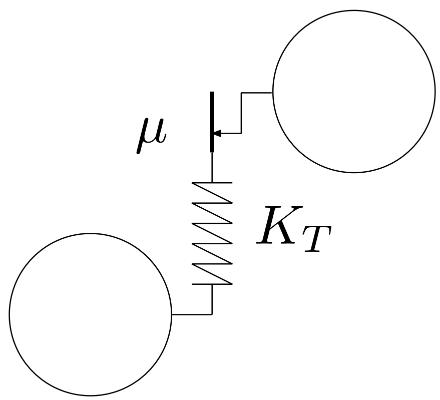
Do we reach the quasistatic limit ?



No influence of dynamical parameters !

Granular packing = contact network = network of rheological elements ?

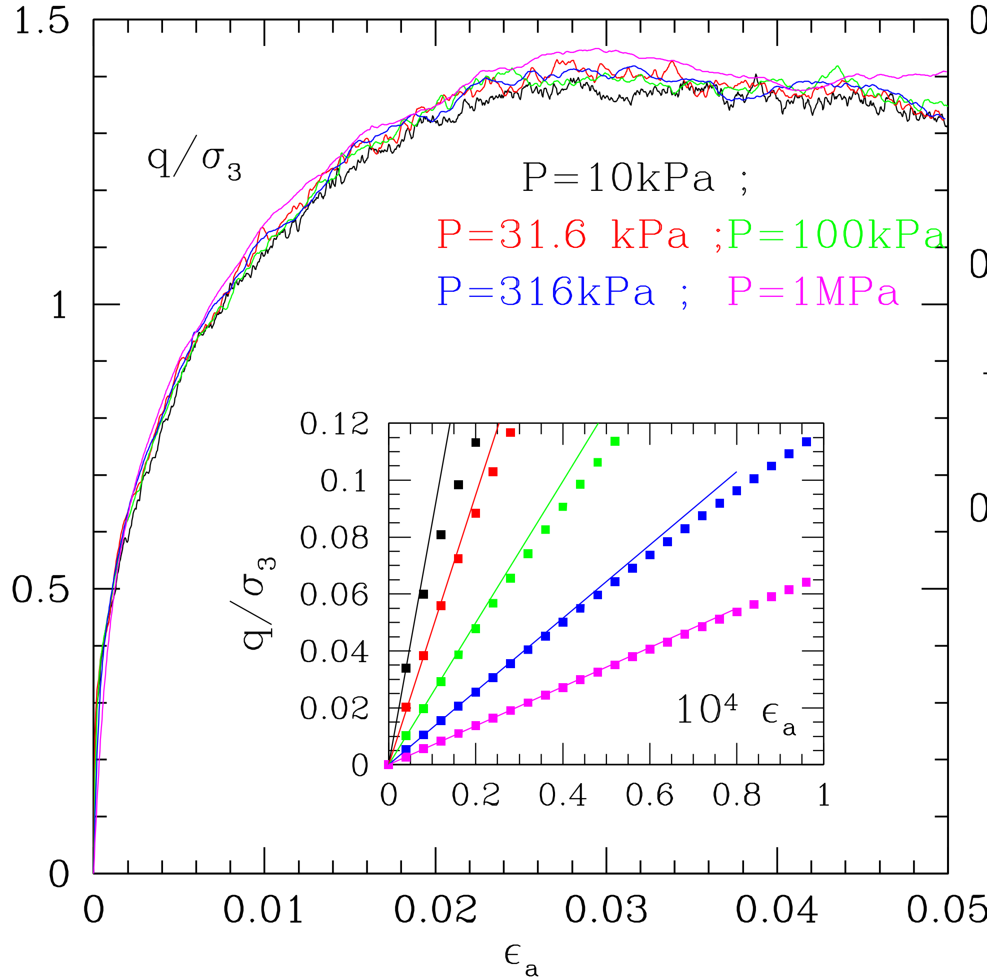


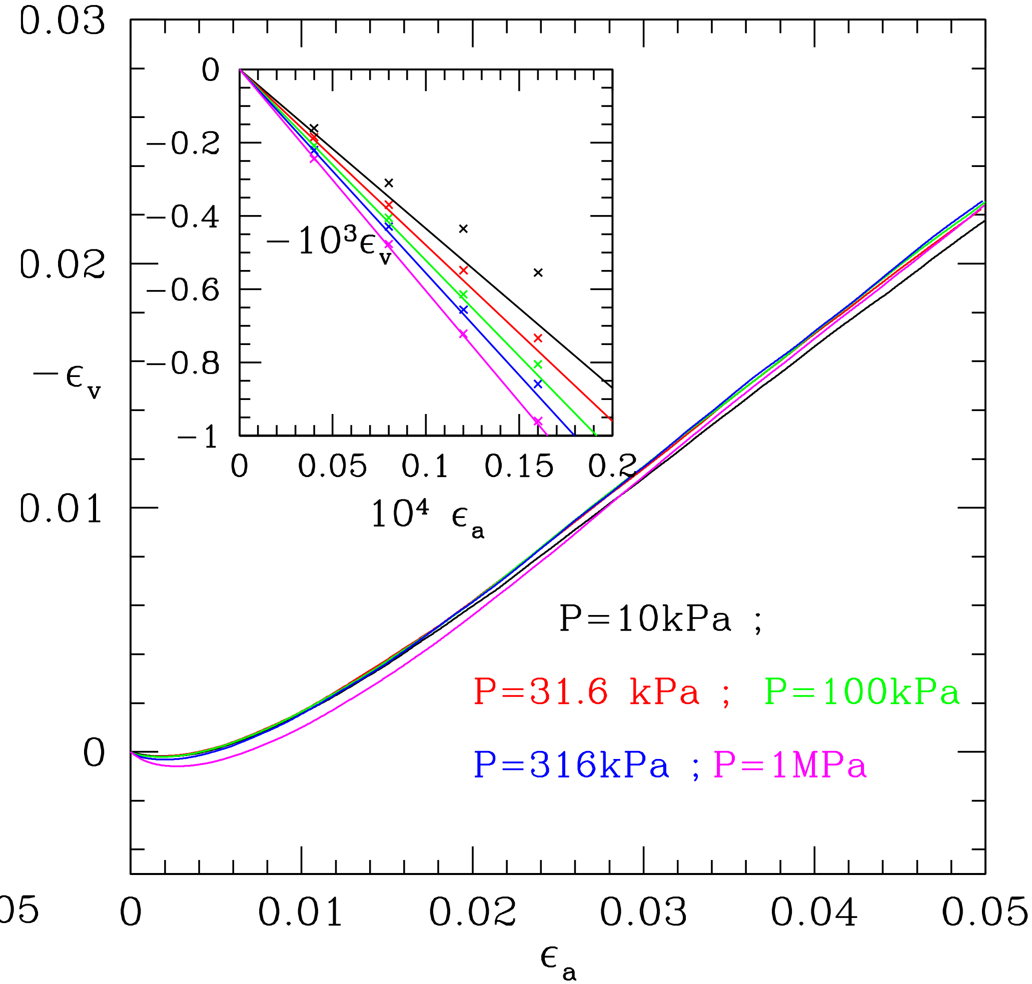


K_N,K_T, η_N depend on elastic forces F_N, F_T

Network of such elements : strains inversely proportional to stiffness under given stresses... but networks may break !

Triaxial compression, influence of κ, few contacts initially

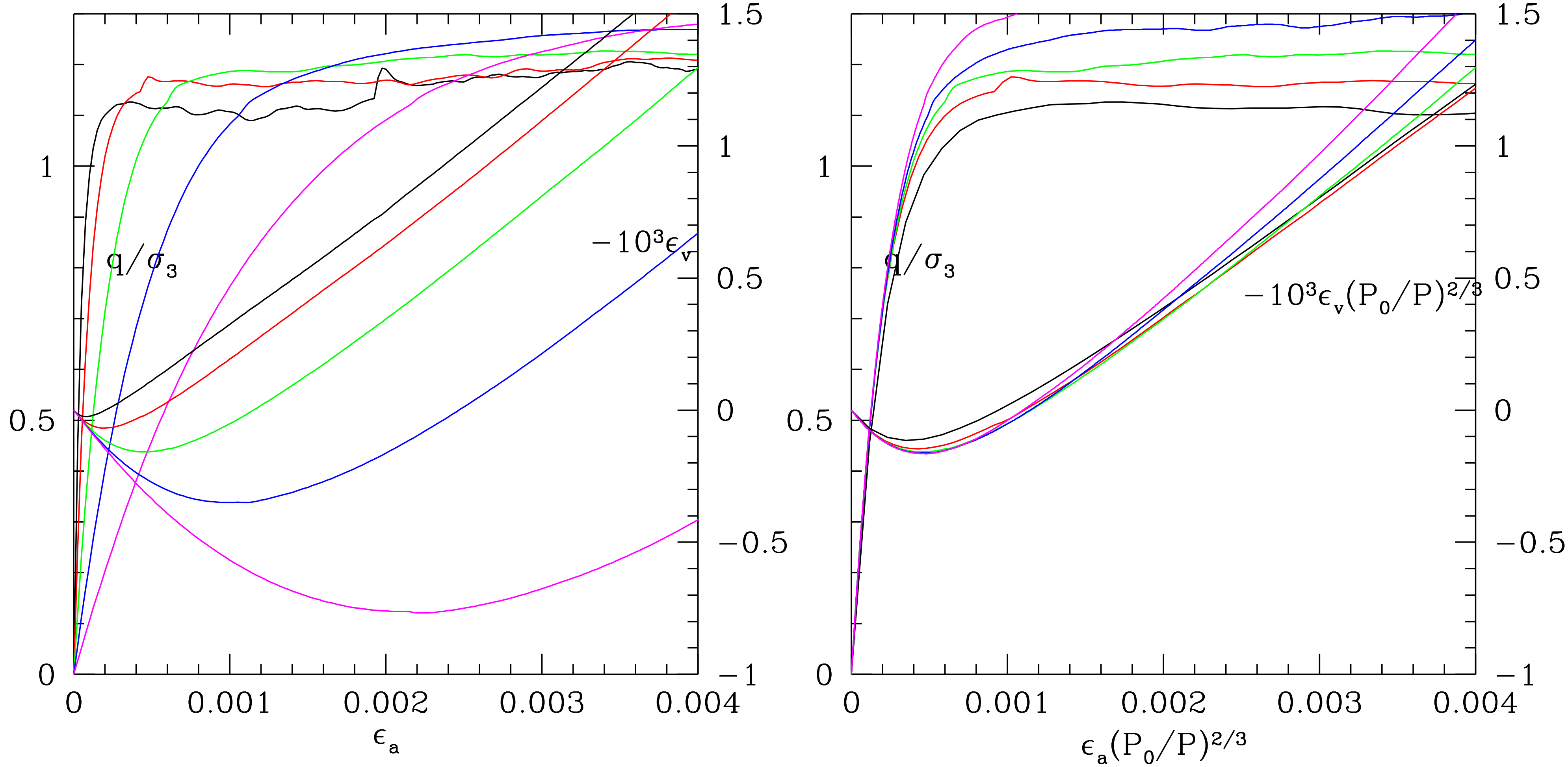




Dense state C ($\Phi \geq 0.635$ for large κ), weak $z^* \simeq 4.6$ if $\kappa \geq 10^4$ (10 kPa). Strain independent of κ except for ϵ_a very weak (slope in insert = elastic modulus)

Type II strains: contact network breaks

Triaxial compression, influence of κ, many contacts in initial state



Dense state A ($\Phi \simeq 0.637$), large $z^* \simeq 6$ if $\kappa \geq 10^4$ (10 kPa). Strain of order κ^{-1}.

Type I strains: initial contact network resists

Properties of régimes I et II

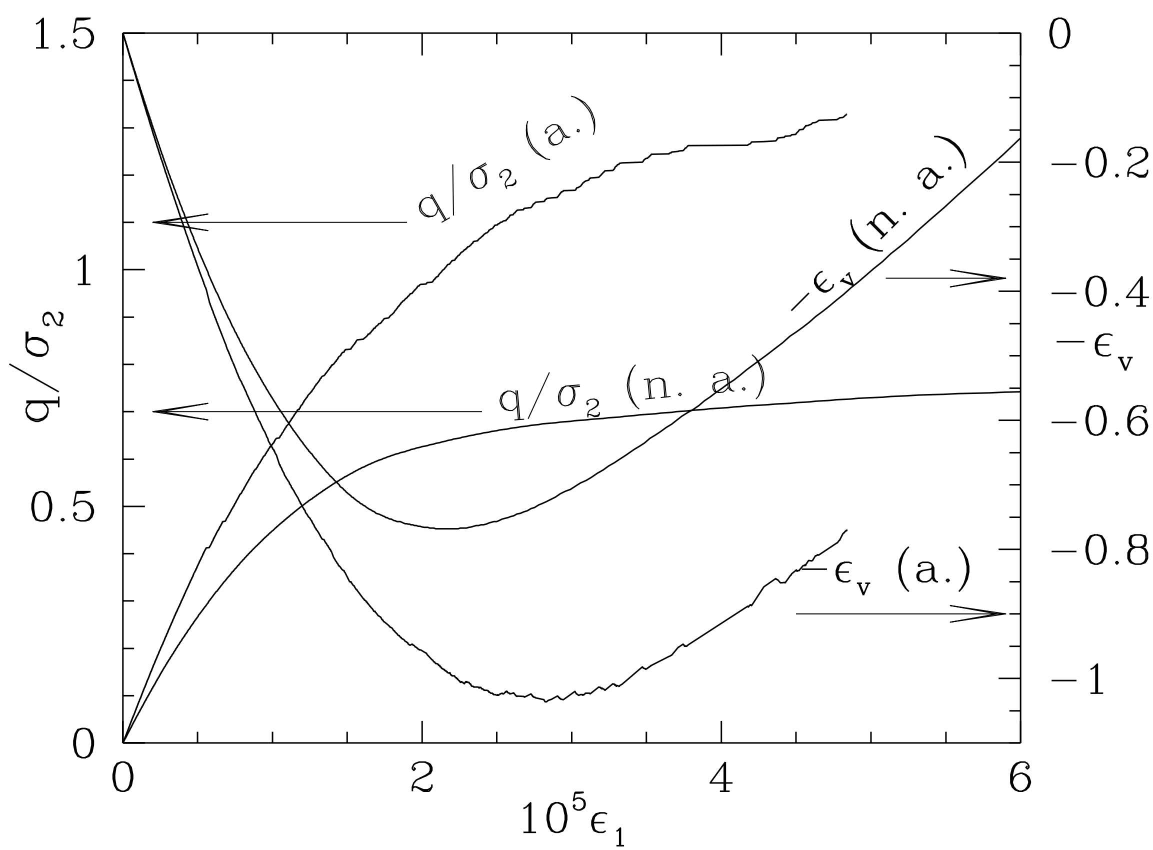
Régime I

- strains inversely proportional to κ (small !), not reversible, contained by contact elasticity
- system evolution = continuous set of (load-dependent) equilibrium configurations
- Contact creation negligible
- little sensitivity to perturbations
- extends to rather large stress interval in well-coordinated systems, or on unloading

Régime II

- larger strains, not sensitive to stiffness level κ, contacts open and close
- larger fluctuations and slower approach to large system limit
- set of equilibrium configurations discontinuous, with “jumps” and bursts of kinetic energy

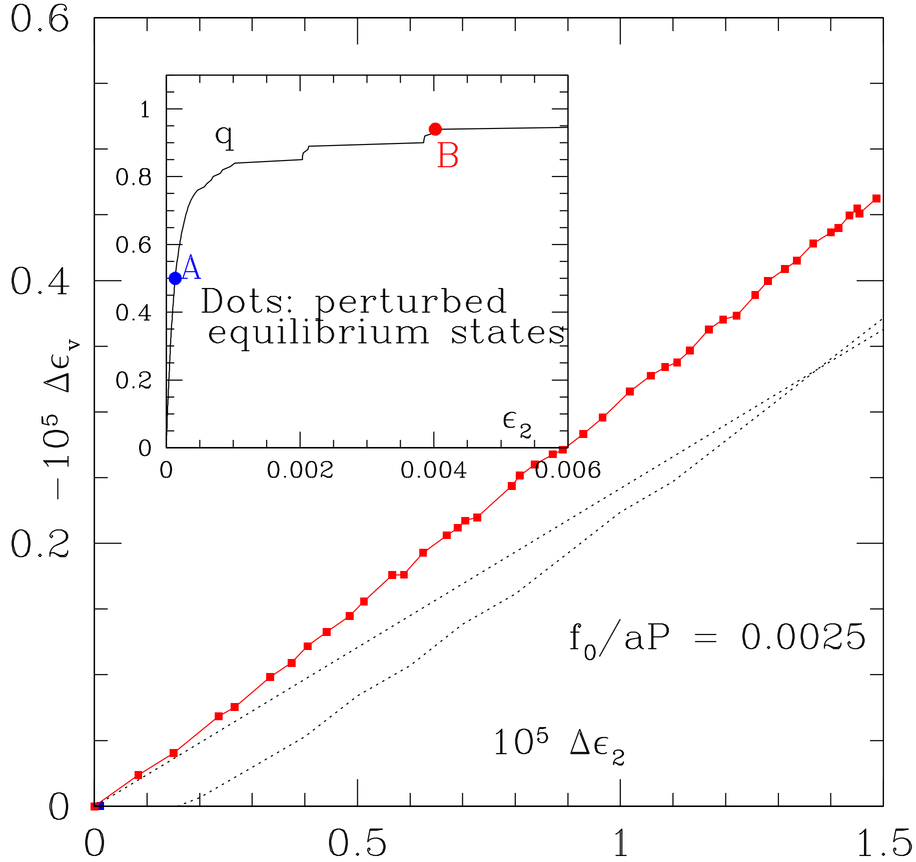
Regime I interval: compared to prediction of limit analysis



Contact network fails before set of admissible contact forces (equilibrium + Coulomb condition) is empty

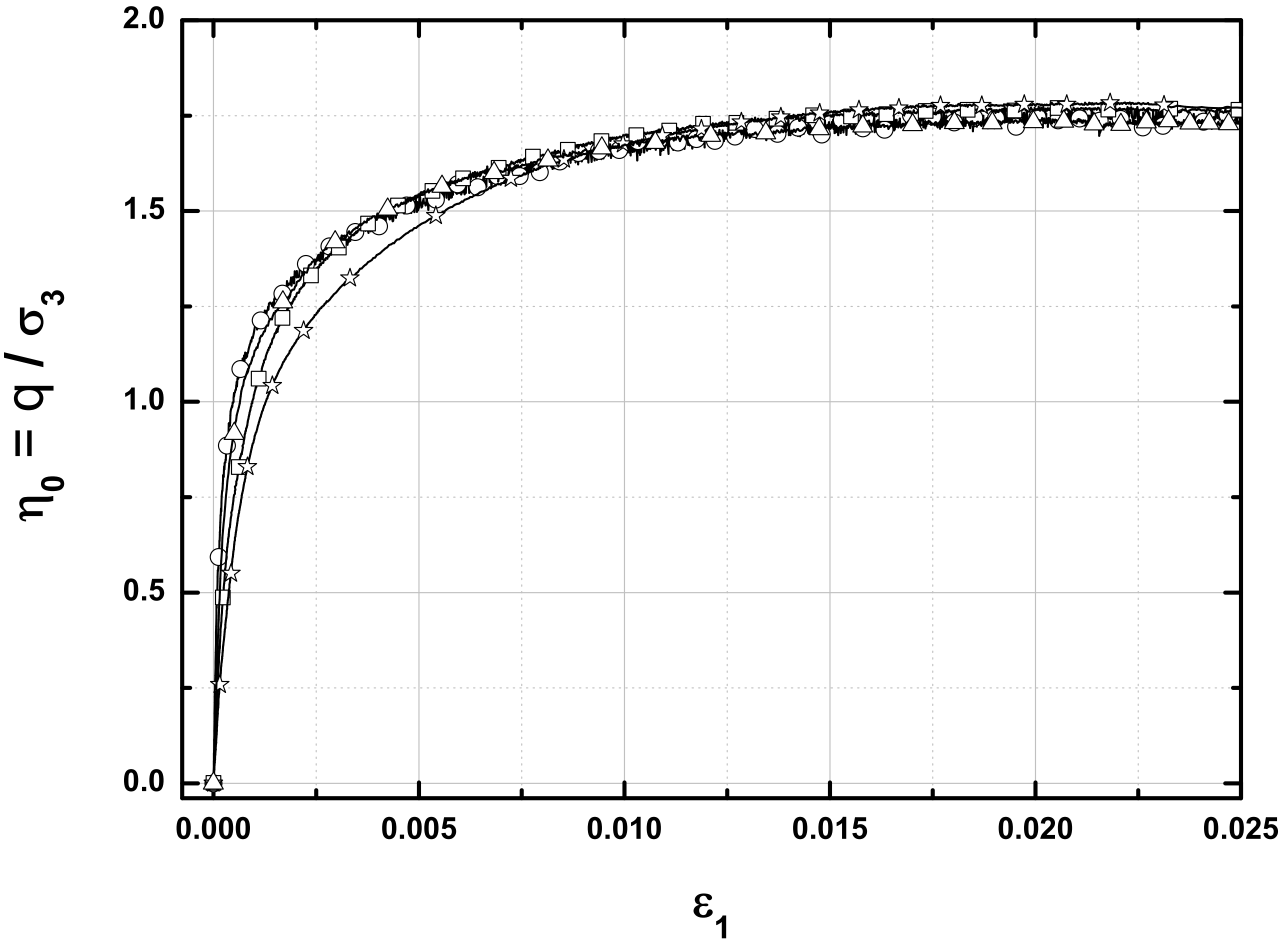
"a"= associated (dilatant friction law in contacts with angle = angle of friction); "n. a." = non-associated (true friction law). Contact network fails while it is still possible to balance external load with contact forces abiding by Coulomb conditions.

Sensitivity to perturbations and creep



Repeated applications of random forces on all grains → creep in regime II, undetectable in regime I

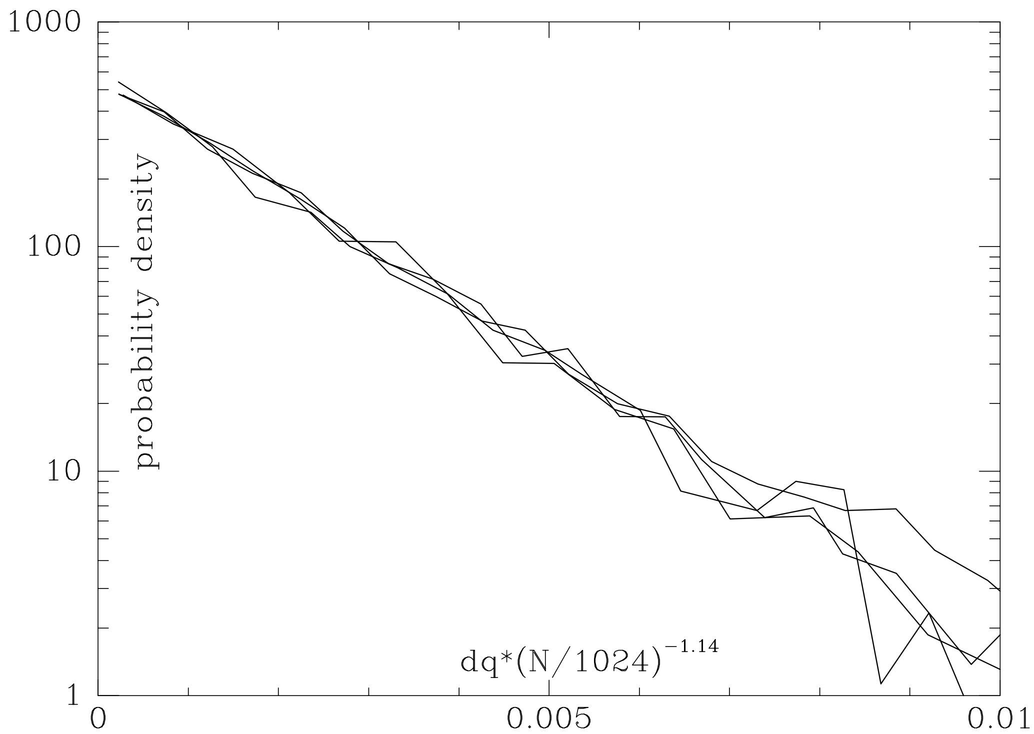
Laboratory triaxial tests: effect of confinement



Glass beads, $50\text{kPa} \leq P \leq 400\text{kPa}$, $\eta_0 = q/\sigma_3$

Note softer behaviour under larger confining pressure, suggesting type I strains

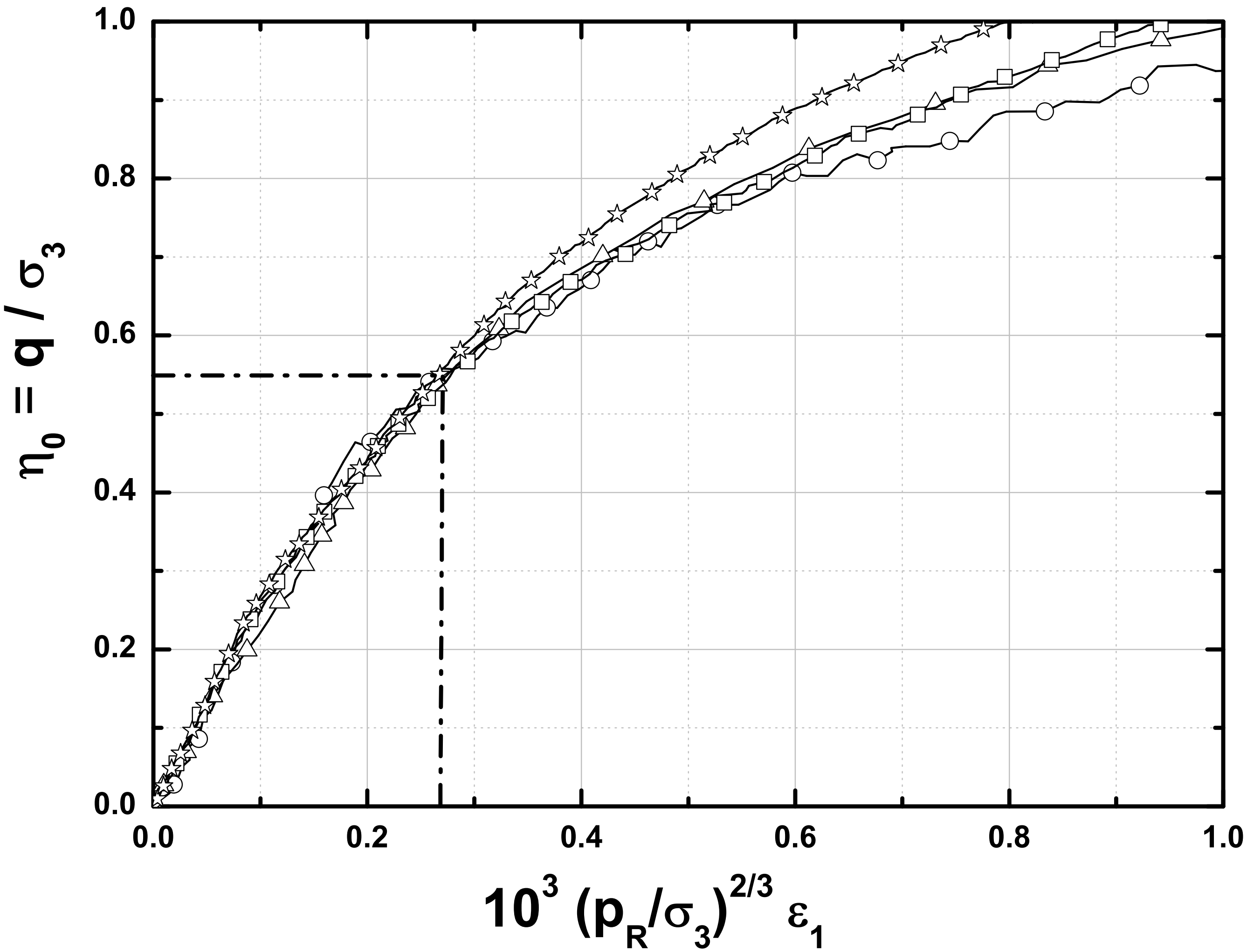
The special case of rigid, frictionless grains (disks)



Stability range (dq=deviator interval /P) of equilibrated configurations, for different numbers N of disks

No régime I, no elastic range !

Laboratory triaxial tests: strain scale



Deviator interval in regime I. Larger φ compared to simulations (particle shape, slightly non-sperical, matters)

Prediction of quasistatic rheology from micromechanics ?

Difficult ! Should involve two stages

1. Stability of contact network, determination of unstable initial motion
   Depends on microstructure and forces: coordination, fabric, mobilization of friction

2. Determination of the net result of rearrangements:
   Dilatancy, fabric evolution as a function of strain, etc.

For item 1, note that instability occurs before prediction of “limit analysis”

Frictionless systems simpler ? Yes, for mechanical properties. No, because of anomalies and difficulties at statistical level...

Conclusions, questions

- Classification of initial states depending on assembling procedure
  In practice many open questions are related to assembling, elaboration methods
- Interesting to use elastic moduli to probe microstructure (not only for initial isotropic states)
- Comparison with experiments yields encouraging results
- contact deformability plays a role in stability interval of given contact networks (see regimes I and II)
- Further studies of network stability properties ? Length scales ? ($\rightarrow$ simulation of large samples)
- Use fabric (contact orientations) as hardening variable. DIfficulty is to relate fabric evolution to strain

Other perspectives

- Cohesive materials: parametric study, behaviour under non-proportional loading path
- Behaviour of loose states
- Strain localization phenomena from a discrete approach
- role of interstitial fluid
- Other particle shapes, size distribution