Partial Differential Equations: Variational Approaches

Frédéric LEGOLL

April 2022

Contents

General introduction

Many phenomena in the engineering sciences (in solid and fluid mechanics, in finance, in traffic flow, . . .) can be modelled using Partial Differential Equations (PDEs).

The aim of this course is to introduce the students to some modern mathematical tools to address these types of equations. The formalization of these tools has started in the middle of the 20th century (in particular by Laurent Schwartz). In these lecture notes, we first focus on the Poisson equation in a bounded domain, which reads

$$\left\{\begin{array}{l} \text{Find a function } u : \Omega \longrightarrow \mathbb{R} \text{ such that} \\ -\Delta u = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{array}\right. \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^d$, $\partial\Omega$ is its boundary and f is a given function, defined on Ω and valued in $\mathbb{R}$. Problem (1) arises for instance in mechanics: the solution u to (1) is the vertical displacement of an elastic membrane submitted to the load f, and clamped at its boundary. It also arises in electrostatics, where u is the electric potential and f is the charge.

Along this course, we will also consider several variants of (1), in different directions:

- non-symmetric problems (such as advection-diffusion problems);
- vector-valued problems (such as linear elasticity problems);
- boundary conditions different from homogeneous Dirichlet boundary conditions.

We will also introduce the students to the energetic viewpoint, where (1) is seen as the Euler-Lagrange equation arising from the minimization of some energy.

This course is devoted to the *mathematical analysis* of Problem (1) and its variants. The questions we address are the following:

- Does the problem (1) have a solution? Is it unique?
- If this is the case, is the map $f \mapsto u$ continuous?

Problem (1) is a *boundary value problem* in the sense that it is composed of a Partial Differential Equation (PDE) and a boundary condition ($u = 0$ on $\partial\Omega$). Its mathematical analysis allows to emphasize a generic trend in modern analysis: the use of *Geometry* tools in infinite dimensional vector spaces allows to obtain results for *Analysis* problems such as (1) in a simple and elegant manner. An important step to study (1) is to introduce the appropriate spaces in which to look for the solution.

We underline that we do not claim to provide an explicit expression, in closed form, of the solution to (1). Except in very specific cases (one-dimensional problems, or constant coefficient problems posed in the full space $\mathbb{R}^d$), such closed form expressions are not available. Once theoretical questions have been understood (this is the aim of this course!), we are left with a series of *numerical analysis* questions: how to compute an approximation u_h of the exact solution u, how to guarantee that u_h is an accurate approximation of u, ... These questions will be addressed in another course.

These lecture notes are organized as follows. The first part is devoted to the construction of appropriate functional spaces in which the solution to (1) should be looked for. We start by an introduction to the distribution theory (in Chapter 1), followed by the construction of the Sobolev spaces (in Chapter 2), which are the relevant spaces of functions when dealing with PDEs.

The mathematical analysis of (1) and its variants is carried out in Chapter 3. To that aim, we introduce the notion of *variational formulation* of a boundary value problem, the analysis of which is performed using the Lax-Milgram theorem. In some cases, an energetic viewpoint can be associated to partial differential equations (this is the case for (1)). This additional perspective is discussed in Chapter 4.

This course is based on notions introduced in two ENPC courses, *Outils Mathématiques pour l'Ingénieur* and *Analyse et Calcul Scientifique*. However, the present lecture notes are, as much as possible, self-contained. Nevertheless, the reader is supposed to be familiar with Banach and Hilbert spaces. Some basic facts concerning the Lebesgue integral and L^p spaces have been collected in Appendix A. For historical reasons, these lectures notes are written in English. A glossary of the main mathematical terms can be found in Appendix B.

I am indebted to Eric Cancès and Alexandre Ern, who have been teaching an *Analyse* course at ENPC for many years. The structure of this document is very much inspired from their lectures notes [2, 4] and from [3]. I am also grateful to the colleagues who gave me opportunities to teach, at ENPC and elsewhere. It has always been a wonderful experience, and I am very pleased to thank them.

As always, there are certainly still several typos in these notes, despite our best efforts. Any suggestions on these notes are therefore warmly welcome.

Frédéric Legoll, Champs sur Marne, April 28, 2022

Chapter 1

Introduction to distribution theory

The distribution theory, introduced by Laurent Schwartz in 1946, is a fundamental piece of Analysis. Distributions generalize the notion of functions, and satisfy the following two striking properties, that make them particularly easy to manipulate:

- any distribution is differentiable and its derivative is again a distribution;
- if a sequence of distributions T_n converges to some distribution T, then the derivatives of T_n converge to the derivatives of T.

In our context, distributions will be useful to build the appropriate functional spaces for the study of Partial Differential Equations (see Chapter 2).

In what follows, Ω is an open subset (which may be either bounded or unbounded) of $\mathbb{R}^d$, with $d \geq 1$. Following the international convention, we denote by (a, b) the one-dimensional open interval $\{x \in \mathbb{R},\ a < x < b\}$.

1.1 Definitions

Roughly speaking, a distribution is a linear and continuous form on the vector space $\mathcal{D}(\Omega)$ of the functions which are in $C^\infty(\Omega)$ and which have a compact support in Ω. We start by recalling the definition of the support of a continuous function.

Definition 1.1. *Let Ω be an open subset of $\mathbb{R}^d$ and $\phi : \Omega \longrightarrow \mathbb{R}$ be a continuous function. The* support *of ϕ is*

$$Supp(\phi) = \overline{\{x \in \Omega,\ \phi(x) \neq 0\}}^{\Omega}.$$

We recall that the notation $\overline{A}^{\Omega}$ means "closure of the set A in Ω". The set $\overline{A}^{\Omega}$ is hence the set of points in Ω that are limits of sequences of points of A. Here are some examples of support computations:

1. $\Omega = \mathbb{R}$ and $\phi(x) = \sin x$: then

$$\{x \in \mathbb{R},\ \phi(x) \neq 0\} = \mathbb{R} \setminus \pi\mathbb{Z}, \qquad \mathrm{Supp}(\phi) = \mathbb{R}.$$

2. $\Omega = (-1, 2)$ and $\phi(x) = \begin{cases} x+1 & \text{if } x \in (-1, 0], \\ 1-x & \text{if } x \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$

   Then $\{x \in \Omega,\ \phi(x) \neq 0\} = (-1, 1)$ and $\mathrm{Supp}(\phi) = (-1, 1]$.

3. $\Omega = (-2,2)$ and ϕ is the same function as above, that is $\phi(x) = \begin{cases} x+1 & \text{if } x \in (-1,0], \\ 1-x & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$

   Then $\{x \in \Omega,\ \phi(x) \neq 0\} = (-1,1)$ and $\text{Supp}(\phi) = [-1,1]$.

The support of ϕ is compact only in the last example (we recall that the compact sets of $\mathbb{R}^d$ are the sets which are bounded and closed).

Exercise 1.2. *Let f and g be continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$.*

- *Show that Supp$(fg) \subset$ Supp$(f) \cap$ Supp(g) and that the inclusion may be strict.*
- *Show that Supp$(f+g) \subset$ Supp$(f) \cup$ Supp(g) and that the inclusion may be strict.*

Definition 1.3. *Let $\mathcal{D}(\Omega)$ be the vector space of functions defined on Ω, which are infinitely differentiable (hence, $\mathcal{D}(\Omega) \subset C^\infty(\Omega)$), and which have a compact support. The functions in $\mathcal{D}(\Omega)$ are called* test functions. *For any compact set $K \subset \Omega$, we denote by $\mathcal{D}_K(\Omega)$ the set of test functions with support contained in K.*

We recall the following notation: for any $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we set $|\alpha| = \sum_{i=1}^{d} \alpha_i$ and

$$\partial^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}} = \frac{\partial^{\alpha_1+\dots+\alpha_d}\phi}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}}. \tag{1.1}$$

Definition 1.4. *A* distribution *T in the open set Ω is a linear form on $\mathcal{D}(\Omega)$ which satisfies the following "continuity property": for any compact set $K \subset \Omega$, there exists an integer p and a constant C such that*

$$\forall \phi \in \mathcal{D}_K(\Omega), \qquad |\langle T, \phi \rangle| \leq C \sup_{x \in K,\ |\alpha| \leq p} |\partial^\alpha \phi(x)|. \tag{1.2}$$

The vector space of distributions in Ω is denoted $\mathcal{D}'(\Omega)$.

Remark 1.5. *On a vector space E endowed with a norm $\|\cdot\|$, the continuity property of a linear form T simply reads*

$$\forall x \in E, \qquad |\langle T, x \rangle| \leq C\|x\|.$$

The more complex form of the "continuity property" (1.2) *stems from the fact that the topology with which the vector space $\mathcal{D}(\Omega)$ should be endowed such that its dual $\mathcal{D}'(\Omega)$ has good properties is* not *a topology arising from a norm.*

Definition 1.6. *When the integer p can be chosen independently of K, one says that the distribution T has a* finite order. *The smallest possible value of p is the* order *of T.*

Remark 1.7. *Heuristically, the larger the order of a distribution T is, the more singular T is.*

1.2 First examples of distributions

We now give some classical examples. First, to any function $f \in L^1_{\mathrm{loc}}(\Omega)$, one can associate the distribution T_f (also denoted f) defined by

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \langle T_f, \phi \rangle = \int_\Omega f\, \phi. \tag{1.3}$$

Next, for any $a \in \Omega \subset \mathbb{R}$, we denote by δ_a the distribution defined by

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \langle \delta_a, \phi \rangle = \phi(a) \tag{1.4}$$

and by δ'_a the distribution defined by

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \langle \delta'_a, \phi \rangle = -\phi'(a). \tag{1.5}$$

Last, consider a locally bounded measure μ, i.e. a measure μ defined on the set $\mathcal{B}(\Omega)$ of the borelian subsets of $\Omega \subset \mathbb{R}^d$ and such that $\mu(K) < \infty$ for any compact subset $K \subset \Omega$. We denote T_μ (or simply μ) the distribution defined by

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \langle T_\mu, \phi \rangle = \int_\Omega \phi\, d\mu. \tag{1.6}$$

Exercise 1.8. *The aim of this exercise is to compute the order of some particular distributions.*

- *Show that the linear forms T_f defined by* (1.3)*, δ_a defined by* (1.4) *and T_μ defined by* (1.6) *are distributions of order 0 on Ω.*
- *Show that the linear form δ'_a defined by* (1.5) *is a distribution of order 1 on Ω (Hint: consider the simple case $\Omega = \mathbb{R}$, and consider the action of δ'_a on the sequence $\phi_j(x) = \phi_0(x)\, atan(jx)$, where $\phi_0 \in \mathcal{D}(\mathbb{R})$ is an even function that is equal to* 1 *in the neighborhood of* 0*; such a test function indeed exists in view of Lemma 1.18 below).*

Remark 1.9. *There does not exist any function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $T_f = \delta_0$. This shows that the space of distributions is (much) larger than the space of functions. The proof of this statement is not easy. However, it is easy to see that there exists no function $f \in C^0(\mathbb{R})$ such that $T_f = \delta_0$.*

1.3 Derivation in the sense of distributions

Let $f \in C^1(\mathbb{R})$. The integration by parts formula reads

$$\forall \phi \in \mathcal{D}(\mathbb{R}), \qquad \int_{\mathbb{R}} \frac{df}{dx}\, \phi = -\int_{\mathbb{R}} f\, \frac{d\phi}{dx}.$$

Likewise, for any $f \in C^1(\mathbb{R}^d)$, we have, for any $1 \le i \le d$,

$$\forall \phi \in \mathcal{D}(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i}\, \phi = -\int_{\mathbb{R}^d} f\, \frac{\partial \phi}{\partial x_i}.$$

The definition of the derivative of a distribution is inspired by this formula. Note that, in contrast to the case of functions, *all distributions are differentiable*!

Definition 1.10. *Let $T \in \mathcal{D}'(\Omega)$. The derivative of T with respect to the variable x_i, which is denoted by $\frac{\partial T}{\partial x_i}$, is defined by*

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \langle \frac{\partial T}{\partial x_i}, \phi \rangle = -\langle T, \frac{\partial \phi}{\partial x_i} \rangle.$$

It is an easy exercise to check that the linear form $\frac{\partial T}{\partial x_i}$ defined above is indeed a distribution (i.e. that it satisfies the continuity property (1.2)).

Exercise 1.11. *Compute the first and second derivative (in the sense of distributions) of the "hat" function (see Figure 1.1) defined on $\mathbb{R}$ by*

$$f(x) = \begin{cases} x+1 & \text{if } -1 < x < 0, \\ 1-x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

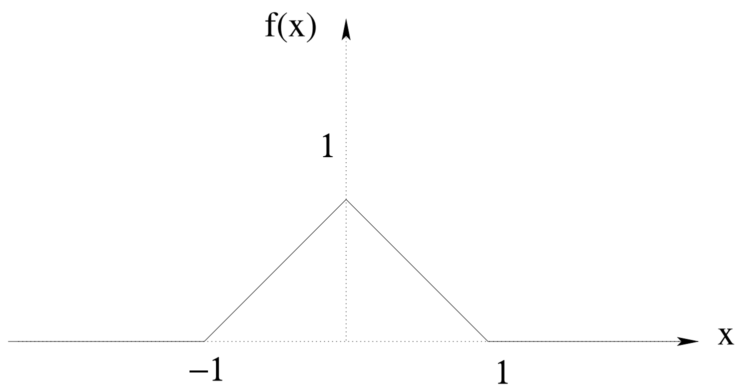

Figure 1.1: Hat function

Exercise 1.12. *Check that the distribution δ'_a defined by* (1.5) *is the derivative of δ_a in the sense of distributions.*

Exercise 1.13. *Consider the Heaviside function H defined on $\mathbb{R}$ by $H(x) = 1$ if $x > 0$, $H(x) = 0$ otherwise. Why can H be considered to be a distribution on $\mathbb{R}$? Compute its derivative in the sense of distributions.*

Remark 1.14. *Any function $f \in L^1_{\text{loc}}(\mathbb{R})$ can be considered to be a distribution. It is thus possible to define its derivative f' in the sense of distributions. In general, f' is* not *a function, but a distribution only.*

1.4 The space of test functions

The space of test functions plays an important role in the construction and the manipulation of distributions. For instance, to show that the distribution δ'_a is of order 1 (see the exercise 1.8), we use a test function $\phi_0 \in \mathcal{D}(\mathbb{R})$ that is even and is equal to 1 in the neighborhood of 0. This section collects some results ensuring the existence of such test functions. It is important to know that there exist test functions satisfying the properties discussed below. In contrast, the proof of these results are often technical. They will all be skipped here.

We start with the following important result:

Theorem 1.15. *The space $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \leq p < +\infty$.*

Remark 1.16. *Note that the case $p = \infty$ is not allowed in the above theorem. It turns out that $\mathcal{D}(\Omega)$ is not dense in $L^\infty(\Omega)$. Consider indeed the following example: on $\Omega = (0,1)$, let $f \in L^\infty(0,1)$ be defined by $f(x) = 1$ almost everywhere. Then, for any $\phi \in \mathcal{D}(0,1)$, we have $\|f - \phi\|_{L^\infty} \geq 1$.*

Corollary 1.17. *For any integer $k \geq 0$ and any $1 \leq p < \infty$, the space $C^k(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$.*

Proof. This is a direct consequence of Theorem 1.15 as $\mathcal{D}(\Omega) \subset C^k(\Omega) \cap L^p(\Omega) \subset L^p(\Omega)$. □

We now list several technical results.

Lemma 1.18. *Let Ω be an open subset of $\mathbb{R}^d$ and let $B_a(r)$ be the ball of center $a \in \Omega$ and of radius $r > 0$. If $B_a(r) \subset \Omega$, there exists $\phi \in \mathcal{D}(\Omega)$, with support in $B_a(r)$, and satisfying $\phi \geq 0$ and $\displaystyle\int_\Omega \phi = 1$.*

Lemma 1.19. *Let $K \subset \Omega$ be compact. There exists a test function $\phi \in \mathcal{D}(\Omega)$ such that $0 \leq \phi \leq 1$ on Ω and $\phi \equiv 1$ on K (see Figure 1.2).*

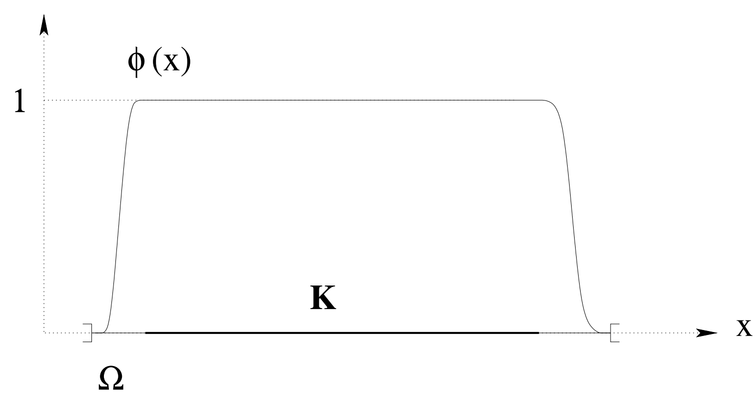

Figure 1.2: A specific test function

Lemma 1.20 (Partition of unity)**.** *Let $\Omega_1, \ldots, \Omega_n$ be open subsets of $\mathbb{R}^d$ and let K be a compact set of $\mathbb{R}^d$ such that*

$$K \subset \bigcup_{k=1}^{n} \Omega_k.$$

There exist test functions $\alpha_1, \ldots, \alpha_n$ such that

- *$Supp(\alpha_k) \subset \Omega_i$,*
- *$0 \leq \alpha_k \leq 1$,*
- *$\displaystyle\sum_{k=1}^{n} \alpha_k = 1$ on K.*

Lemma 1.21 (Borel lemma)**.** *For any sequence* $(a_\alpha)_{\alpha\in\mathbb{N}^d}$*, there exists* $\phi \in \mathcal{D}(\mathbb{R}^d)$ *such that*

$$\forall \alpha \in \mathbb{N}^d, \qquad \partial^\alpha \phi(0) = a_\alpha.$$

Lemma 1.22 (Hadamard lemma)**.** *Let* $\phi \in \mathcal{D}(\mathbb{R})$ *be such that* $\phi^{(k)}(0) = 0$ *for any integer* $0 \le k \le n$*. Then there exists* $\psi \in \mathcal{D}(\mathbb{R})$ *such that, for any* $x \in \mathbb{R}$*,* $\phi(x) = x^n \psi(x)$*.*

Proof. This follows by using the Taylor formula with a remainder in integral form. □

Remark 1.23. *A similar result holds in dimension higher than* $d = 1$*.*

1.5 Distributions and locally integrable functions

We have seen above that we can associate a distribution T_f to any function $f \in L^1_{\rm loc}(\Omega)$ (see formula (1.3)). It turns out that the space of test functions is sufficiently large as to ensure that the map $f \mapsto T_f$ from $L^1_{\rm loc}(\Omega)$ to $\mathcal{D}'(\Omega)$ is actually injective. This result, which is very important, is formalized in the theorem below.

Theorem 1.24. *Let* f *and* g *be functions in* $L^1_{\rm loc}(\Omega)$*. Then*

$$f = g \text{ almost everywhere} \quad \Leftrightarrow \quad T_f = T_g \text{ in } \mathcal{D}'(\Omega).$$

Proof. The implication $\Rightarrow$ is straightforward. We show the converse implication in the simplified case when $\Omega = \mathbb{R}^d$ and when f and g are in $L^1(\mathbb{R}^d)$.

Consider f and g in $L^1(\mathbb{R}^d)$ such that $T_f = T_g$ and set $h = f - g$. We hence have $h \in L^1(\mathbb{R}^d)$ such that

$$\forall \phi \in \mathcal{D}(\mathbb{R}^d), \qquad \int_\Omega h\,\phi = 0. \tag{1.7}$$

Our aim is to show that $h = 0$ almost everywhere. The proof falls in two steps.

Step 1 (approximation of identity): Consider a non-negative function $\chi \in \mathcal{D}(\mathbb{R}^d)$, with support in the unit ball and with integral equal to 1. Define the sequence $(\chi_n)_{n\in\mathbb{N}^\star}$ by

$$\chi_n(x) = n^{-d}\chi(x/n).$$

Such a sequence is an approximation of the identity (see Section 1.7). We recall that, for any n, $\|\chi_n\|_{L^1(\mathbb{R}^d)} = \|\chi\|_{L^1(\mathbb{R}^d)} = 1$. Let $\psi \in \mathcal{D}(\mathbb{R}^d)$. We claim that

$$\|\psi - \psi \star \chi_n\|_{L^1(\mathbb{R}^d)} \underset{n\to+\infty}{\longrightarrow} 0, \tag{1.8}$$

where $\psi \star \chi_n$ is the convolution product of ψ with χ_n (see Definition A.18). To prove (1.8), we first note, using the change of variables $z = y/n$, that

$$\psi \star \chi_n(x) = n^{-d} \int_{\mathbb{R}^d} \psi(x-y)\,\chi(y/n)\,dy = \int_{\mathbb{R}^d} \psi(x - z/n)\,\chi(z)\,dz.$$

Using the dominated convergence theorem (see Theorem A.1), we obtain that, for any $x \in \mathbb{R}^d$, $\psi \star \chi_n(x)$ converges to $\psi(x)$ when $n \to \infty$. We next note that

$$|\psi \star \chi_n(x)| = \left| \int_{\mathbb{R}^d} \psi(x-y)\,\chi_n(y)\,dy \right| \le \|\psi\|_{L^\infty(\mathbb{R}^d)}\,\|\chi_n\|_{L^1(\mathbb{R}^d)} = \|\psi\|_{L^\infty(\mathbb{R}^d)},$$

and that

$$\mathrm{Supp}(\psi \star \chi_n) \subset \mathrm{Supp}(\psi) + \mathrm{Supp}(\chi_n) \subset \mathrm{Supp}(\psi) + \mathrm{Supp}(\chi).$$

This implies that

$$\begin{cases} |\psi - \psi \star \chi_n| \le 2 \, \|\psi\|_{L^\infty(\mathbb{R}^d)} \, 1_{\text{Supp}(\psi)+\text{Supp}(\chi)} & \text{for any } n, \\ \psi(x) - \psi \star \chi_n(x) \underset{n\to+\infty}{\longrightarrow} 0 & \text{for any } x \in \mathbb{R}^d. \end{cases}$$

We can hence use the dominated convergence theorem again, and obtain the convergence (1.8).

Step 2: Let $\epsilon > 0$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\|h - \psi\|_{L^1(\mathbb{R}^d)} \le \epsilon/3. \tag{1.9}$$

The existence of such a function ψ stems from the density of $\mathcal{D}(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$ (see Theorem 1.15). Let $(h_n)_{n\in\mathbb{N}^\star}$ be defined by

$$h_n(x) = (h \star \chi_n)(x) = \int_{\mathbb{R}^d} h(y)\, \chi_n(x-y)\, dy,$$

where χ_n was defined in Step 1. Since, for any $n \in \mathbb{N}^\star$, $\chi_n \in \mathcal{D}(\mathbb{R}^d)$, we observe that h_n vanishes on $\mathbb{R}^d$ for any n, in view of (1.7). We hence write

$$\|h\|_{L^1(\mathbb{R}^d)} = \|h - h \star \chi_n\|_{L^1(\mathbb{R}^d)} \le \|h - \psi\|_{L^1(\mathbb{R}^d)} + \|\psi - \psi \star \chi_n\|_{L^1(\mathbb{R}^d)} + \|(h-\psi) \star \chi_n\|_{L^1(\mathbb{R}^d)}. \tag{1.10}$$

By definition of the convolution product (see the Definition A.18), we have

$$\|(h-\psi) \star \chi_n\|_{L^1(\mathbb{R}^d)} \le \|h - \psi\|_{L^1(\mathbb{R}^d)} \, \|\chi_n\|_{L^1(\mathbb{R}^d)} = \|h - \psi\|_{L^1(\mathbb{R}^d)} \le \epsilon/3. \tag{1.11}$$

Using (1.8), we know that we can choose N such that, for any $n \ge N$, we have

$$\|\psi - \psi \star \chi_n\|_{L^1(\mathbb{R}^d)} \le \epsilon/3. \tag{1.12}$$

Collecting (1.10), (1.9), (1.12) and (1.11), we get that

$$\|h\|_{L^1(\mathbb{R}^d)} \le \epsilon.$$

This upper bound holds for any $\epsilon > 0$. This implies that $h = 0$ in $L^1(\mathbb{R}^d)$. □

It is thus possible to identify any function of $L^1_{\text{loc}}(\Omega)$ with its associated distribution, which can be denoted f instead of T_f. The previous result can be recast as follows.

Theorem 1.25. *Let f and g in $L^1_{\text{loc}}(\Omega)$. Then*

$$f = g \text{ a.e.} \quad \Leftrightarrow \quad f = g \text{ in } \mathcal{D}'(\Omega).$$

Remark 1.26. *We therefore write $L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$ just like we write $\mathbb{N} \subset \mathbb{R} \subset \mathbb{C}$. The notation $A \subset B$ here means that there exists a canonical injection from A to B.*

Remark 1.27. *The notion of distributions generalizes the notion of functions, but it does not mean that any function is a distribution. Only functions in L^1_{loc} are distributions. More singular functions are not necessarily distributions. For instance, the function $f(x) = 1/|x|$ is a distribution in $\mathbb{R}^d$ for any $d \ge 2$ but not in $\mathbb{R}$. See Section 1.10 for more details in that direction.*

1.6 Multiplication by C^∞ functions

Definition 1.28. *Let $T \in \mathcal{D}'(\Omega)$ and $g \in C^\infty(\Omega)$. The distribution $g\,T$ is defined by*

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \langle g\,T, \phi \rangle = \langle T, g\,\phi \rangle.$$

It is an easy exercise to check that the linear form $g\,T$ is indeed a distribution.

Remark 1.29. *The multiplication of two distributions is* not defined*. Consider for instance the function $f(x) = \dfrac{1}{\sqrt{|x|}}$, which belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ and hence defines a distribution. But the function f^2 does not define a distribution (recall that $f^2(x) = 1/|x|$ does not belong to $L^1_{\mathrm{loc}}(\mathbb{R})$). More generally, the product $T_1\,T_2$ with $T_1 \in \mathcal{D}'(\Omega)$ and $T_2 \in \mathcal{D}'(\Omega)$ is not defined (see for instance the exercise 1.56).*

Exercise 1.30. *Show that $x\,\delta_0 = 0$.*

1.7 Convergence of distributions

Definition 1.31. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of distributions in Ω. This sequence is said to* converge to some *$T \in \mathcal{D}'(\Omega)$ when $n \to \infty$ if*

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \langle T_n, \phi \rangle \underset{n\to+\infty}{\longrightarrow} \langle T, \phi \rangle.$$

The example of the exercise below is often useful.

Exercise 1.32. *Let $\chi \in \mathcal{D}(\mathbb{R}^d)$ be a non-negative function, with support in the unit ball and of integral equal to 1. Set*

$$\chi_n(x) = n^d \chi(n\,x).$$

Show that

$$\chi_n \underset{n\to\infty}{\longrightarrow} \delta_0 \text{ in } \mathcal{D}'(\mathbb{R}^d).$$

Definition 1.33. *A sequence $(\chi_n)_{n\in\mathbb{N}}$ of test functions in $\mathcal{D}(\mathbb{R}^d)$ satisfying*

$$\chi_n \underset{n\to+\infty}{\longrightarrow} \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^d)$$

is called an approximation of the identity.

Remark 1.34. *This name comes from the fact that the distribution δ_0 is the identity for the convolution operation (this operation is defined in Section A.5 for functions in $L^1(\mathbb{R}^d)$ and it can be extended to some distributions).*

Proposition 1.35. *Let $1 \le p \le +\infty$. Convergence in $L^p_{\mathrm{loc}}(\Omega)$ implies convergence in $\mathcal{D}'(\Omega)$.*

Proof. Consider $(f_n)_{n\in\mathbb{N}}$ that converges to some f in $L^p_{\text{loc}}(\Omega)$ when $n \to \infty$. Let $\phi \in \mathcal{D}(\Omega)$. Using the Hölder inequality, we have, for any $1 < p < +\infty$,

$$\begin{aligned}
|\langle f_n, \phi\rangle - \langle f, \phi\rangle| &= \left|\int_\Omega (f_n - f)\phi\right| \\
&\leq \int_\Omega |f_n - f|\,|\phi| \\
&\leq \|\phi\|_{L^\infty} \int_{\text{Supp}(\phi)} |f_n - f| \\
&\leq \|\phi\|_{L^\infty} \left(\int_{\text{Supp}(\phi)} |f_n - f|^p\right)^{1/p} \left(\int_{\text{Supp}(\phi)} 1^{p'}\right)^{1/p'} \\
&\leq \|\phi\|_{L^\infty}\, \|f_n - f\|_{L^p(\text{Supp}(\phi))}\, |\text{Supp}(\phi)|^{1/p'}.
\end{aligned}$$

We hence get that $\lim\limits_{n\to\infty} |\langle f_n, \phi\rangle - \langle f, \phi\rangle| = 0$. If $p = 1$ or $p = +\infty$, then the proof, which is even simpler, is left to the reader. □

Remark 1.36. *The fact that a sequence of functions f_n converges almost everywhere to some function f (i.e. $\lim\limits_{n\to\infty} f_n(x) = f(x)$ a.e. on Ω) does not imply convergence in $\mathcal{D}'$. The converse is also false. The three situations below are possible:*

1. *a sequence converges in $\mathcal{D}'$ but not almost everywhere;*
2. *a sequence converges almost everywhere but not in $\mathcal{D}'$;*
3. *a sequence converges almost everywhere and in $\mathcal{D}'$ but the limits are different.*

We refer to the exercise 1.73 for some examples.

With the above definitions, differentiation is a continuous operation in $\mathcal{D}'(\Omega)$, in the following sense.

Proposition 1.37. *Consider a sequence (T_n) that converges to T in $\mathcal{D}'(\Omega)$ when $n \to \infty$. Then, for any $\alpha \in \mathbb{N}^d$, $\partial^\alpha T_n$ converges to $\partial^\alpha T$ in $\mathcal{D}'(\Omega)$ when $n \to \infty$.*

Proof. Let $\phi \in \mathcal{D}(\Omega)$. We have

$$\langle \partial^\alpha T_n, \phi\rangle = (-1)^{|\alpha|}\langle T_n, \partial^\alpha \phi\rangle \longrightarrow (-1)^{|\alpha|}\langle T, \partial^\alpha\phi\rangle = \langle \partial^\alpha T, \phi\rangle.$$

We hence get $\partial^\alpha T_n \longrightarrow_{n\to\infty} \partial^\alpha T$. □

The following proposition is a direct corollary of the above result.

Proposition 1.38. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}'(\Omega)$. Assume that the series $\sum\limits_{n\in\mathbb{N}} T_n$ converges in $\mathcal{D}'(\Omega)$ to a distribution T. Then, for any $\alpha \in \mathbb{N}^d$, the series $\sum\limits_{n\in\mathbb{N}} \partial^\alpha T_n$ converges in $\mathcal{D}'(\Omega)$ and we have*

$$\partial^\alpha T = \sum_{n\in\mathbb{N}} \partial^\alpha T_n.$$

In $\mathcal{D}'$, it is thus possible to differentiate a series as soon as

$$T = \sum_{n\in\mathbb{N}} T_n,$$

this equality being an equality between distributions. This result is of course in sharp contrast with the results when manipulating functions.

The following result is very useful, and make the manipulation of distributions easy. We omit its proof.

Theorem 1.39. *Consider a sequence $(T_n)_{n\in\mathbb{N}}$ of distributions in Ω. Assume that, for any $\phi \in \mathcal{D}(\Omega)$, the sequence $(\langle T_n, \phi\rangle)_{n\in\mathbb{N}}$ converges to some limit ℓ_ϕ. Then, the map T defined by*

$$\langle T, \phi\rangle = \ell_\phi$$

is a distribution on Ω.

Note that T is obviously a linear form on $\mathcal{D}(\Omega)$. To show that it is a distribution, we are left with showing the continuity property (1.2), which is actually not easy. The above theorem states that this property indeed holds.

Exercise 1.40. *Let $a \in \mathbb{R}^d$ and $e \in \mathbb{R}^d$ such that $|e| = 1$. Identify the limit in $\mathcal{D}'(\mathbb{R}^d)$, as ϵ tends to 0, of the family of distributions*

$$T_\epsilon = \frac{1}{\epsilon}\delta_{a+\epsilon e} - \frac{1}{\epsilon}\delta_a.$$

Exercise 1.41. *Let $h \in L^1_{\rm loc}(\mathbb{R}^2)$ and $x_0 \in \mathbb{R}$. We define the* single layer *distribution $SL(h, x_0) \in \mathcal{D}'(\mathbb{R}^3)$ by*

$$\forall \phi \in \mathcal{D}(\mathbb{R}^3), \qquad \langle SL(h, x_0), \phi\rangle = \int_{\mathbb{R}^2} h(y, z)\, \phi(x_0, y, z)\, dy\, dz.$$

Show that $SL(h, x_0)$ is indeed a distribution on $\mathbb{R}^3$.

Let $g \in L^1_{\rm loc}(\mathbb{R}^2)$. Identify the limit in $\mathcal{D}'(\mathbb{R}^3)$, as ϵ tends to 0, of the family of distributions

$$T_\epsilon = SL(g/\epsilon, \epsilon) - SL(g/\epsilon, 0).$$

Exercise 1.42. *Show that $\displaystyle\sum_{n\in\mathbb{Z}} \delta_n$ converges in $\mathcal{D}'(\mathbb{R})$.*

Exercise 1.43. *Does the series*

$$\sum_{n=1}^{+\infty} \delta_{1/n}$$

converge in $\mathcal{D}'(\mathbb{R})$? What about in $\mathcal{D}'(0, +\infty)$?

1.8 More on differentiation: the one-dimensional case

1.8.1 The case of C^1 functions

Consider $f \in C^1(a,b)$. The functions f and $\dfrac{df}{dx}$ are in $L^1_{\text{loc}}(a,b)$, and hence in $\mathcal{D}'(a,b)$. We have

$$\left\langle \frac{d}{dx} T_f, \phi \right\rangle = -\int_a^b f \, \frac{d\phi}{dx} = \int_a^b \frac{df}{dx} \, \phi = \left\langle T_{\frac{df}{dx}}, \phi \right\rangle.$$

This means that

$$\frac{d}{dx} T_f = T_{\frac{df}{dx}},$$

namely that the derivation in the sense of distributions coincides with the standard derivation for functions of class C^1.

1.8.2 The case of functions that are piecewise C^1

Definition 1.44. *A function f is* piecewise C^k *in (a,b) if, for any compact interval $[\alpha,\beta]$ contained in (a,b), there exists a finite number of points $\alpha = a_0 < a_1 < \ldots < a_{N+1} = \beta$ such that*

- *in each interval (a_i, a_{i+1}), f is C^k;*
- *f and its derivatives up to the order k can be extended by continuity on each side of the points $a_1, \ldots, a_N$ (note that the left and right values of f – or its derivatives – at points a_i are not necessarily equal).*

Note that a piecewise C^1 function has at most a countable number of discontinuity points and that the possible accumulation points of the set of discontinuity points are only a and b.

Here are some examples and counter-examples:

- the Heaviside function is piecewise C^1;
- the function $x \mapsto |x|$ is piecewise C^1;
- the function $x \mapsto \tan x$ is not piecewise C^1;
- the function $x \mapsto \sqrt{|x|}$ is not piecewise C^1.

Theorem 1.45 (Jump formula)**.** *Let f be a piecewise C^1 function on (a,b). With the above notation, we have*

$$f' = f'_{\text{reg}} + \sum_{i \in \mathcal{I}} [f(c_i + 0) - f(c_i - 0)] \, \delta_{c_i},$$

where f' is the derivative of f in the sense of distributions, $\{c_i\}_{i \in \mathcal{I}}$ are the points where f is discontinuous (recall that $\mathcal{I}$ is countable and its only possible accumulation points are a and b), f'_{reg} is the piecewise continuous function defined (except at the points c_i) as the standard derivative of f, $f(c_i + 0)$ and $f(c_i - 0)$ are the (right and left) limits of f in c_i, and δ_{c_i} is the Dirac mass in c_i.

Proof. The proof is based on the integration by part formula. □

Exercise 1.46. *Let f be a piecewise C^2 function on (a,b). Check that*

$$f'' = f''_{\text{reg}} + \sum_{i \in \mathcal{I}} [f'(c_i + 0) - f'(c_i - 0)] \, \delta_{c_i} + \sum_{i \in \mathcal{I}} [f(c_i + 0) - f(c_i - 0)] \, \delta'_{c_i}$$

for some countable set $\mathcal{I}$ and some points $\{c_i\}_{i \in \mathcal{I}}$.

1.8.3 Linear differential equations in $\mathcal{D}'(a,b)$

Theorem 1.47. *Consider the open interval (a,b).*

1. *The distributions T on (a,b) satisfying $T'=0$ in $\mathcal{D}'(a,b)$ are the constant functions.*
2. *For any $S \in \mathcal{D}'(a,b)$, there exists $T \in \mathcal{D}'(a,b)$ such that $T'=S$.*

The first statement means that, if $T'=0$ in $\mathcal{D}'(a,b)$, then there exists a constant C such that $T=C$, which means that, for any $\phi \in \mathcal{D}(a,b)$, we have $\langle T, \phi\rangle = C\int_a^b \phi$.

Proof. We first note that a test function $\phi \in \mathcal{D}(a,b)$ admits a primitive in $\mathcal{D}(a,b)$ if and only if $\int_a^b \phi = 0$ (in this case, the primitive is $\int_a^x \phi$). Let $\rho \in \mathcal{D}(a,b)$ such that $\int_a^b \rho = 1$. Since the integral of $\phi - \left(\int_a^b \phi\right)\rho$ vanishes, there exists $\psi \in \mathcal{D}(a,b)$ such that

$$\psi'(x) = \phi(x) - \left(\int_a^b \phi\right)\rho(x). \tag{1.13}$$

Consider now $T \in \mathcal{D}'(a,b)$ such that $T'=0$. We write

$$\langle T, \phi\rangle = \left(\int_a^b \phi\right)\langle T, \rho\rangle + \langle T, \psi'\rangle = \left(\int_a^b \phi\right)\langle T, \rho\rangle - \langle T', \psi\rangle = \left(\int_a^b \phi\right)\langle T, \rho\rangle.$$

Denoting C the constant $\langle T, \rho\rangle$, we have hence shown that $T=C$.

Consider now $S \in \mathcal{D}'(a,b)$. For any $\phi \in \mathcal{D}(a,b)$, we set

$$\langle T, \phi\rangle = -\langle S, \psi\rangle,$$

where $\psi \in \mathcal{D}(a,b)$ is uniquely defined by (1.13). It is then easy to show that T is a distribution and that $T'=S$. ☐

Theorem 1.47 can be used to show existence and uniqueness results for differential equations in $\mathcal{D}'$. The case of the equation $xT'+T=0$ is considered in the exercise 1.76.

1.8.4 Link between the standard derivative and the derivative in the sense of distributions

For a function $f \in L^1_{\text{loc}}$, the relation between the standard derivative and the derivative in the sense of distributions is not straightforward:

1. for a C^1 function, the two concepts lead to the same object;
2. for a piecewise C^1 function, the standard derivative is defined almost everywhere, but does not fully represent the variations of f since it ignores jumps; the derivative in the sense of distributions is the correct concept;
3. when f is everywhere differentiable without being piecewise C^1, the situation is more complex. When f' is locally integrable, the two concepts coincide (see Exercise 1.77). Otherwise, the derivative in the sense of distributions is somehow the finite part of the standard derivative (see the exercises 1.54 and 1.58).

1.9 More on differentiation: the multi-dimensional case

1.9.1 Schwarz Theorem

Theorem 1.48 (Schwarz Theorem). *Let $T \in \mathcal{D}'(\Omega)$. We have*

$$\partial^\alpha \partial^\beta T = \partial^\beta \partial^\alpha T = \partial^{\alpha+\beta} T.$$

Proof. The proof is straightforward. □

1.9.2 The case of C^1 functions

As in the one-dimensional case, we have the following result. Consider Ω an open subset of $\mathbb{R}^d$ and $f \in C^1(\Omega)$. Then, for any $1 \leq i \leq d$,

$$\frac{\partial}{\partial x_i} T_f = T_{\frac{\partial f}{\partial x_i}}.$$

1.9.3 Stokes formula and applications

Definition 1.49. *A bounded open set $\Omega \subset \mathbb{R}^d$ is* smooth *(of class C^1) if, at any $x \in \partial\Omega$, it is possible to define an exterior normal vector, denoted $n(x)$, and if the function $x \mapsto n(x)$ from $\partial\Omega$ to $\mathbb{R}^d$ is continuous.*

Examples: the circle (in dimension $d = 2$) or the sphere (when $d = 3$) are smooth. The square (when $d = 2$) or the cube (when $d = 3$) are not smooth.

Theorem 1.50 (Stokes formula). *Let Ω be a bounded smooth open subset of $\mathbb{R}^d$ and let X be a field defined in $\overline{\Omega}$, which is vector valued, and such that all components belong to $C^1(\overline{\Omega})$. Then*

$$\int_\Omega \text{div}\,(X)\, dx = \int_{\partial\Omega} X \cdot n\, d\sigma.$$

Corollary 1.51 (Integration by part formula). *Let Ω be a bounded smooth open subset of $\mathbb{R}^d$ and let f and g in $C^1(\overline{\Omega})$. Then*

$$\int_\Omega \frac{\partial f}{\partial x_i}\, g\, dx = \int_{\partial\Omega} f\, g\, (n \cdot e_i)\, d\sigma - \int_\Omega f\, \frac{\partial g}{\partial x_i}\, dx.$$

Proof. Use the Stokes formula for the field $X(x) = f(x)\, g(x)\, e_i$. □

Corollary 1.52 (Green formula). *Let Ω be a bounded smooth open subset of $\mathbb{R}^d$ and let f and g in $C^2(\overline{\Omega})$. Then*

$$\int_\Omega f\, \Delta g = -\int_\Omega \nabla f \cdot \nabla g + \int_{\partial\Omega} f \frac{\partial g}{\partial n}.$$

Proof. Use the Stokes formula for the field $X(x) = f(x)\, \nabla g(x)$. □

1.10 Principal values and finite parts

Principal values and finite parts of functions are distributions that naturally appear when one wants to associate a distribution with some singular functions (i.e. functions not in L^1_{loc}). Here, we only consider two specific examples.

1.10.1 Principal value of $1/x$

We define the principal value of the function $x \in \mathbb{R} \mapsto 1/x$ by

$$\forall \phi \in \mathcal{D}(\mathbb{R}), \qquad \left\langle \mathrm{PV}\left(\frac{1}{x}\right), \phi \right\rangle = \lim_{\epsilon\to 0,\ \epsilon>0} \int_{|x|>\epsilon} \frac{\phi(x)}{x}\, dx. \tag{1.14}$$

Exercise 1.53. *Show that $PV\left(\frac{1}{x}\right)$ is indeed a distribution, of order 1. Hint: show that the distribution is of order at most 1, and then that its order is exactly 1. To do this, compute its action on the sequence $\phi_j(x) = \phi_0(x)\, atan(jx)$, where $\phi_0 \in \mathcal{D}(\mathbb{R})$ is even and equal to 1 in the neighborhood of 0.*

Exercise 1.54. *Show that the function $x \in \mathbb{R} \mapsto \ln|x|$ defines a distribution in $\mathcal{D}'(\mathbb{R})$ and compute its derivative.*

Exercise 1.55. *Show that $x\, PV\left(\frac{1}{x}\right) = 1$.*

Exercise 1.56. *Explain why the quantities*

$$\left(x\, PV\left(\frac{1}{x}\right)\right)\delta_0 \qquad \textit{and} \qquad (x\,\delta_0)\, PV\left(\frac{1}{x}\right).$$

are well defined in $\mathcal{D}'(\mathbb{R})$ and compute them. In view of this computation, does it seem possible to define the product of any two distributions?

1.10.2 Finite part of $H(x)x^\alpha$

Let $\phi \in \mathcal{D}(\mathbb{R})$ and H be the Heaviside function. The integral

$$\int_{\mathbb{R}} H(x)x^\alpha \phi(x)\, dx = \int_0^{+\infty} x^\alpha \phi(x)\, dx$$

is well-defined for $\alpha > -1$, but not for any $\alpha \leq -1$ if $\phi(0) \neq 0$. In the second case, $H(x)x^\alpha$ is therefore not a distribution. It is nevertheless still possible to associate a distribution to the function $H(x)x^\alpha$, as we explain now.

Using the Hadamard lemma 1.22, one can show that, for any $\epsilon > 0$,

$$\int_\epsilon^{+\infty} \phi(x)x^\alpha\, dx = P_\phi(\epsilon) + R_\phi(\epsilon),$$

where P_ϕ is a linear combination of negative powers of ϵ (and of $\ln\epsilon$ when α is a negative integer) and where $R_\phi(\epsilon)$ converges to a finite limit as ϵ tends to 0. One then defines the finite part of the function $H(x)x^\alpha$, which is denoted $\mathrm{FP}(H(x)x^\alpha)$, by the formula

$$\forall \phi \in \mathcal{D}(\mathbb{R}), \qquad \langle \mathrm{FP}(H(x)x^\alpha), \phi\rangle = \lim_{\epsilon\to 0,\ \epsilon>0} R_\phi(\epsilon).$$

Exercise 1.57. *Show that the finite part of the function $H(x)x^\alpha$ is a distribution of order the integer part of α.*

Exercise 1.58. *Let $-1 < \alpha < 0$. Show that the derivative (in the sense of distributions) of the function $H(x)x^\alpha$ is $\alpha FP\Big(H(x)x^{\alpha-1}\Big)$.*

1.11 Distributions with compact support

1.11.1 Definitions and first properties

Definition 1.59. *Let $T \in \mathcal{D}'(\Omega)$.*

1. *Let ω be an open subset of Ω. The distribution T is said to vanish on ω if, for any $\phi \in \mathcal{D}(\Omega)$ such that Supp$(\phi) \subset \omega$, we have $\langle T, \phi\rangle = 0$.*
2. *The* support *of T is the complement in Ω of the union of the open subsets of Ω on which T vanishes.*

Note that, by construction, T vanishes on $\Omega \setminus \mathrm{Supp}(T)$. The notion of support of a distribution generalizes the notion of support of a function, as shown by Exercise 1.60 below.

Exercise 1.60. *Consider $f \in C^0(\mathbb{R})$. Since $C^0(\mathbb{R}) \subset L^1_{\mathrm{loc}}(\mathbb{R})$, a distribution T_f can naturally be associated to the function f (see (1.3)). Show that Supp$(T_f) =$ Supp(f).*

Exercise 1.61. *Show that the support of $\delta_a \in \mathcal{D}'(\mathbb{R})$ is $\{a\}$.*

Definition 1.62. *The vector space of distributions on Ω with compact support is denoted $\mathcal{E}'(\Omega)$.*

Theorem 1.63. *If a distribution $T \in \mathcal{D}'(\Omega)$ has a compact support, then it is of finite order.*

Proof. Let K be the support of T and $\alpha = d(K, \mathbb{R}^d \setminus \Omega)$ be the distance between K and $\mathbb{R}^d \setminus \Omega$ (if $\Omega = \mathbb{R}^d$, then we take $\alpha = +\infty$). We set $\beta = \inf(1, \alpha)$ and consider the sets

$$K' = \left\{ x \in \mathbb{R}^d, \quad d(x, K) \le \frac{\beta}{3} \right\}, \qquad \Omega' = \left\{ x \in \mathbb{R}^d, \quad d(x, K) < \frac{2\beta}{3} \right\}.$$

The set K' is compact, the set Ω' is open and with compact closure and we have

$$K \subset K' \subset \Omega' \subset \overline{\Omega'} \subset \Omega.$$

Let $p \in \mathbb{N}$ and $C \in \mathbb{R}$ such that

$$\forall \phi \in \mathcal{D}_{\overline{\Omega'}}(\Omega), \qquad |\langle T, \phi\rangle| \le C \sup_{x \in \overline{\Omega'},\, |\alpha| \le p} |\partial^\alpha \phi(x)|.$$

Consider now $\rho \in \mathcal{D}(\Omega)$ which is equal to 1 on K' and with support in Ω'. For any $\phi \in \mathcal{D}(\Omega)$, we have

$$\langle T, \phi\rangle = \langle T, \rho\phi\rangle + \langle T, (1-\rho)\phi\rangle$$

and $\langle T, (1-\rho)\phi\rangle = 0$ since the supports of T and of $(1-\rho)\phi$ are disjoint. Moreover, since $\mathrm{Supp}(\rho\phi) \subset \overline{\Omega'}$, we have

$$\forall \phi \in \mathcal{D}(\Omega), \qquad |\langle T, \phi\rangle| \le C \sup_{x \in \Omega,\, |\alpha| \le p} |\partial^\alpha (\rho\phi)(x)|.$$

Using Leibniz formula, we get

$$\forall \phi \in \mathcal{D}(\Omega), \qquad |\langle T, \phi\rangle| \le C' \sup_{x \in \Omega,\, |\alpha| \le p} |\partial^\alpha \phi(x)|$$

with $C' = C \displaystyle\sup_{x \in \Omega,\, |\alpha| \le p,\, \beta \le \alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} |\partial^\beta \rho(x)|$. We thus obtain that T has a finite order, lower than or equal to p. ∎

Exercise 1.64. *Let $T \in \mathcal{D}'(\Omega)$ with $\Omega \subset \mathbb{R}^d$. Then, for any $1 \leq i \leq d$, we have Supp$\left(\frac{\partial T}{\partial x_i}\right) \subset$ Supp(T).*

Remark 1.65. *Let $T \in \mathcal{D}'(\Omega)$ and consider $\phi \in \mathcal{D}(\Omega)$ such that ϕ vanishes on Supp(T). This does not imply that $\langle T, \phi \rangle = 0$.*

Consider indeed the case of $T = \delta_0'$, the support of which is $\{0\}$ (in view of Exercises 1.61 and 1.64, we have Supp$(\delta_0') \subset \{0\}$, and the support cannot be empty as δ_0' is not equal to 0). Consider $\phi(x) = x\rho(x)$, with $\rho \in \mathcal{D}(\mathbb{R})$ equal to 1 on a neighborhood of 0. We observe that ϕ vanishes on Supp(δ_0') while $\langle \delta_0', \phi \rangle \neq 0$.

However, if ϕ vanishes on an open neighborhood of Supp(T), then $\langle T, \phi \rangle = 0$ (see the proof of Proposition 1.66 below where this argument is used).

1.11.2 Distributions with support restricted to a point

Proposition 1.66. *Let $T \in \mathcal{D}'(\Omega)$ of order 0 such that Supp$(T) = \{a\}$. Then there exists a constant c such that*

$$T = c\,\delta_a.$$

Proof. Let $\epsilon > 0$ such that $B_a(\epsilon) \subset \Omega$. Let $K = \overline{B_a(\epsilon/2)}$ and C be a constant such that

$$\forall \phi \in \mathcal{D}_K(\Omega), \qquad |\langle T, \phi \rangle| \leq C \sup_{x \in K} |\phi(x)|.$$

Let $0 < r < \epsilon$ and $\rho \in \mathcal{D}(\Omega)$, with compact support in $B_a(r)$ and such that $0 \leq \rho \leq 1$ and $\rho = 1$ in K. For any $\phi \in \mathcal{D}(\Omega)$, we have

$$\langle T, \phi \rangle = \langle T, \rho\phi \rangle + \langle T, (1-\rho)\phi \rangle.$$

The function $(1-\rho)\phi$ vanishes in a neighborhood of $\{a\} = \text{Supp}(T)$, hence $\langle T, (1-\rho)\phi \rangle = 0$. We also observe that $\text{Supp}(\rho\phi) \subset K$, hence

$$|\langle T, \phi \rangle| = |\langle T, \rho\phi \rangle| \leq C \sup_{\Omega} |\rho\phi| \leq C \sup_{B_a(r)} |\phi|.$$

The constant C does not depend on r. Taking the limit $r \to 0$, we thus deduce that

$$|\langle T, \phi \rangle| \leq C|\phi(a)|. \tag{1.15}$$

Now consider an arbitrary function $\psi \in \mathcal{D}(\Omega)$. We write

$$\psi = \psi(a)\rho + (\psi - \psi(a)\rho)$$

and see that the function $\phi = \psi - \psi(a)\rho$ is such that $\phi(a) = 0$. Using (1.15), we obtain that $\langle T, \psi - \psi(a)\rho \rangle = 0$. Thus

$$\langle T, \psi \rangle = \langle T, \psi(a)\rho \rangle = \langle T, \rho \rangle\, \psi(a).$$

We hence get that $T = c\delta_a$, where $c = \langle T, \rho \rangle$. ☐

The next proposition (the proof of which is omitted) generalizes Proposition 1.66 by characterizing the distributions (of any order) that have a support equal to the point $\{a\}$.

Proposition 1.67. *Let $T \in \mathcal{D}'(\Omega)$ such that Supp$(T) = \{a\}$. Then T is of the form*

$$T = \sum_{|\alpha| \leq p} c_\alpha \partial^\alpha \delta_a,$$

where p is the order of T and where c_α, $|\alpha| \leq p$, are constant.

Exercise 1.68. *Show that δ_a and $PV\left(\frac{1}{x}\right)$ have compact support and compute their support.*

1.12 Exercises

We collect here some exercises that are, in general, less straightforward that the previous ones.

Exercise 1.69. *For the following maps $T : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$, identify which ones are distributions:*

- $\langle T, \phi \rangle = \displaystyle\int_{\mathbb{R}} |\phi(t)|\, dt.$
- $\langle T, \phi \rangle = \displaystyle\sum_{n=0}^{\infty} \phi^{(n)}(n).$

Exercise 1.70. *Show that the map $T : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ defined by $\langle T, \phi \rangle = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \left(\phi\left(\frac{1}{n}\right) - \phi(0) \right)$ is a distribution of order lower or equal to 1.*

Exercise 1.71. *Consider $T : \mathcal{D}(\mathbb{R}^2) \to \mathbb{R}$ defined by*

$$\langle T, \phi \rangle = \int_0^{\infty} \phi(z, 2z)\, dz.$$

Show that T is a distribution and that $\dfrac{\partial T}{\partial x} + 2\dfrac{\partial T}{\partial y} = \delta_{(0,0)}$ in $\mathcal{D}'(\mathbb{R}^2)$.

Exercise 1.72. *Consider $v_n(x) = \dfrac{n}{1 + n^2 x^2}$ and $w_n(x) = atan(nx)$.*

1. *Show that $v_n(x)$ converges to some $v(x)$ almost everywhere on $\mathbb{R}$. Identify v.*
2. *Show that, for any $\alpha > 0$, v_n converges in $L^1((-\infty, -\alpha) \cup (\alpha, \infty))$.*
3. *Show that w_n converges in $\mathcal{D}'(\mathbb{R})$ to $\dfrac{\pi}{2} sgn(x)$.*
4. *Compute the limit of v_n in $\mathcal{D}'(\mathbb{R})$ by two different methods.*

Exercise 1.73. *The aim of this exercise is to study the links between convergence almost everywhere and convergence in $\mathcal{D}'$.*

1. *Consider $f \in \mathcal{D}(\mathbb{R})$ and the sequence $f_n(x) = f(x - n)$. Study the almost everywhere convergence of f_n, the convergence in $\mathcal{D}'(\mathbb{R})$ and the convergence in $L^1(\mathbb{R})$.*
2. *Consider $f_n(x) = e^{inx}$. Show that f_n does not converge almost everywhere but that f_n converges to 0 in $\mathcal{D}'(\mathbb{R})$.*
3. *Consider $f_n(x) = \displaystyle\sum_{k=1}^{n} \chi_{1/n^2}\left(x - \frac{1}{k}\right)$, where χ_ϵ is an approximation of identity, that is $\chi_\epsilon(x) = \epsilon\chi(\epsilon x)$ where $\chi \in \mathcal{D}(\mathbb{R})$, with support in the unit ball and with integral equal to 1. Show that $f_n \to 0$ almost everywhere but that f_n does not converge in $\mathcal{D}'(\mathbb{R})$.*
4. *Consider $f_n(x) = \dfrac{1}{\sigma_n \sqrt{2\pi}}\, e^{-x^2/2\sigma_n^2}$ with $\sigma_n \to 0$. Show that $f_n \to 0$ almost everywhere and that $f_n \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$.*

Exercise 1.74. *Identify all the distributions* $T \in \mathcal{D}'(\mathbb{R})$ *such that* $x\,T = 0$.

Exercise 1.75. *For any* $\phi \in \mathcal{D}(\mathbb{R})$*, we define*

$$\left\langle \frac{1}{x+i0}, \phi \right\rangle = \lim_{\epsilon \to 0,\ \epsilon > 0} \int_{\mathbb{R}} \frac{\phi(x)}{x+i\epsilon}\, dx \quad \text{and} \quad \left\langle \frac{1}{x-i0}, \phi \right\rangle = \lim_{\epsilon \to 0,\ \epsilon > 0} \int_{\mathbb{R}} \frac{\phi(x)}{x-i\epsilon}\, dx.$$

Show that $\dfrac{1}{x+i0}$ *and* $\dfrac{1}{x-i0}$ *define distributions of order 1. Show that*

$$\frac{1}{x+i0} - \frac{1}{x-i0} = -2i\pi\delta_0 \quad \text{and} \quad \frac{1}{x+i0} + \frac{1}{x-i0} = 2\,PV\left(\frac{1}{x}\right).$$

Exercise 1.76. *Identify all distributions* $T \in \mathcal{D}'(\mathbb{R})$ *such that* $x\,T' + T = 0$. *(Hint: consider* $S = x\,T$*).*

Exercise 1.77. *Let* $f \in L^1_{\rm loc}(\mathbb{R})$ *and* $F(x) = \displaystyle\int_0^x f$. *Show that the derivative (in the sense of distributions) of* F *is equal to* f.

Chapter 2

Sobolev spaces

Using the distribution theory exposed in Chapter 1, we are now in position to build the appropriate spaces of functions for the study of Partial Differential Equations.

2.1 The spaces $H^k(\Omega)$

We start with the simplest Sobolev space, namely the $H^1(\Omega)$ space. We recall that we can associate to any $f \in L^2(\Omega)$ a distribution, which is again denoted f. Since f is a distribution, it can be differentiated (in the sense of distribution), and each component $\dfrac{\partial f}{\partial x_i}$ of its gradient ($1 \leq i \leq d$) is hence a distribution. When this distribution can be represented by a function in $L^2(\Omega)$, that is when there exists some function $F_i \in L^2(\Omega)$ such that

$$\forall \phi \in \mathcal{D}(\Omega), \quad \left\langle \frac{\partial f}{\partial x_i}, \phi \right\rangle = \int_\Omega F_i \, \phi,$$

then f is said to belong to the space $H^1(\Omega)$. In other words, we have the following definition:

Definition 2.1. *Let Ω be an open subset of $\mathbb{R}^d$. The set of functions of $L^2(\Omega)$, the first derivatives (in the sense of distributions) of which are in $L^2(\Omega)$, is denoted $H^1(\Omega)$:*

$$H^1(\Omega) = \left\{ f \in L^2(\Omega), \quad \forall 1 \leq i \leq d, \quad \frac{\partial f}{\partial x_i} \in L^2(\Omega) \right\}.$$

This definition can be generalized for higher-order derivatives.

Definition 2.2. *Let Ω be an open subset of $\mathbb{R}^d$ and let $k \in \mathbb{N}$. The set of functions of $L^2(\Omega)$, the derivative (in the sense of distributions) of which, up to order k, are in $L^2(\Omega)$, is denoted $H^k(\Omega)$:*

$$H^k(\Omega) = \left\{ f \in L^2(\Omega), \quad \forall \alpha \in \mathbb{N}^d, \ |\alpha| \leq k, \quad \partial^\alpha f \in L^2(\Omega) \right\}.$$

We recall that, for any distribution f and for any $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we denote $\partial^\alpha f = \dfrac{\partial^{\alpha_1 + \cdots + \alpha_d} f}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_d} x_d}$ (see (1.1)) the distribution obtained by differentiating f. In the above definition of $H^k(\Omega)$, we wrote that $\partial^\alpha f \in L^2(\Omega)$: this means that there exists a function $F_\alpha \in L^2(\Omega)$ such that

$$\forall \phi \in \mathcal{D}(\Omega), \quad \langle \partial^\alpha f, \phi \rangle = \int_\Omega F_\alpha \, \phi.$$

Theorem 2.3. *$H^k(\Omega)$ is a vector space. Endowed with the scalar product*

$$(f,g)_{H^k} = \sum_{|\alpha|\le k} \int_\Omega \partial^\alpha f(x)\, \partial^\alpha g(x)\, dx,$$

it is a Hilbert space. Its norm is denoted $\|\cdot\|_{H^k}$.

Proof. Obviously, $H^k(\Omega)$ is a vector space and $(\cdot,\cdot)_{H_k}$ is a scalar product on $H^k(\Omega)$. We are left with showing that $H^k(\Omega)$ is a complete space for the norm $\|\cdot\|_{H^k}$. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of $H^k(\Omega)$. We first observe that

$$\|f_p - f_q\|_{L^2} \le \|f_p - f_q\|_{H^k},$$

hence (f_n) is a Cauchy sequence in L^2, hence it converges in L^2 to some $f \in L^2$. Likewise, for any $|\alpha| \le k$,

$$\|\partial^\alpha f_p - \partial^\alpha f_q\|_{L^2} \le \|f_p - f_q\|_{H^k},$$

hence $\partial^\alpha f_n$ converges in L^2 to some function g_α. In addition, convergence in L^2 implies convergence in $\mathcal{D}'$. Hence

$$f_n \longrightarrow f \qquad \text{in } \mathcal{D}'$$

and

$$\partial^\alpha f_n \longrightarrow g_\alpha \qquad \text{in } \mathcal{D}'.$$

From the first assertion, we deduce that

$$\partial^\alpha f_n \longrightarrow \partial^\alpha f \qquad \text{in } \mathcal{D}'.$$

By uniqueness of the limit in $\mathcal{D}'(\Omega)$, we find that $\partial^\alpha f = g_\alpha \in L^2$. Hence $f \in H^k$. Since $\partial^\alpha f_n \longrightarrow \partial^\alpha f$ in L^2, we get that $f_n \longrightarrow f$ in H^k. □

A function u in $H^1(\Omega)$ is more regular than a function $u \in L^2(\Omega)$, since its gradient is also a square integrable function. One may wonders if this additional regularity also shows up on u itself. This is indeed the case:

- in dimension $d = 1$, functions in H^1 have a continuous representation (see Exercise 2.17);
- in dimension $d \ge 2$, functions in H^1 are not necessarily continuous (Exercise 2.18 provides a counter-example); however, it turns out that, in dimension d and for any open set $\Omega \subset \mathbb{R}^d$, any $u \in H^1(\Omega)$ belongs to $L^p(\Omega)$ for some $p > 2$. If u has some singularities, there are not only square-integrable but their power p is also integrable.

2.2 The space $H^1_0(\Omega)$

2.2.1 Definition

We start by an important density result.

Proposition 2.4.

1. *If $\Omega = \mathbb{R}^d$, then $\mathcal{D}(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$.*
2. *If $\Omega \subset \mathbb{R}^d$, $\Omega \ne \mathbb{R}^d$, then $\mathcal{D}(\Omega)$ is not dense in $H^1(\Omega)$.*

Proof. The first assertion can be shown following a technical proof using truncation and regularization. We omit it here and focus on the second assertion, in dimension 1, when $\Omega = (0,1)$. Let $\phi \in \mathcal{D}(0,1)$. We have

$$\forall x \in (0,1), \qquad \phi(x) = \int_0^x \phi'(t)\, dt,$$

hence

$$\forall x \in (0,1),\ |\phi(x)| \leq \int_0^x |\phi'(t)|\, dt \leq \left(\int_0^x |\phi'(t)|^2\, dt\right)^{1/2} \left(\int_0^x 1^2\, dt\right)^{1/2} \leq \|\phi'\|_{L^2(0,1)}.$$

We therefore obtain

$$\|\phi\|_{L^2(0,1)} = \left(\int_0^1 |\phi(x)|^2\, dx\right)^{1/2} \leq \|\phi'\|_{L^2(0,1)},$$

an estimate which is valid for any $\phi \in \mathcal{D}(0,1)$.

If $\mathcal{D}(0,1)$ were dense in $H^1(0,1)$, then this estimate would remain true for any $\phi \in H^1(0,1)$. However, consider the function $f = 1$ on $(0,1)$. We see that $f \in H^1(0,1)$, $\|f\|_{L^2(0,1)} = 1$ and $\|f'\|_{L^2(0,1)} = 0$, hence f does not satisfy the above estimate. □

Definition 2.5. *The closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ is denoted $H^1_0(\Omega)$.*

Remark 2.6. *In view of Proposition 2.4, we have $H^1_0(\mathbb{R}^d) = H^1(\mathbb{R}^d)$. In contrast, $H^1_0(\Omega)$ is a strict subspace of $H^1(\Omega)$ when $\Omega \subset \mathbb{R}^d$ with $\Omega \neq \mathbb{R}^d$.*

Proposition 2.7. *The space $H^1_0(\Omega)$ is a vector space. It is a Hilbert space for the scalar product $(\cdot,\cdot)_{H^1}$.*

Proof. It is obvious that $H^1_0(\Omega)$ is a vector space (it is the closure of a vector space for a given norm). The scalar product of $H^1(\Omega)$ restricted to $H^1_0(\Omega)$ of course defines a scalar product on $H^1_0(\Omega)$. Since $H^1_0(\Omega)$ is closed in a complete space, it is itself complete. □

2.2.2 Poincaré inequality

We now state a very important result, that we will often need in these lecture notes (see e.g. Section 3.3).

Theorem 2.8 (Poincaré inequality)**.** *Let Ω be a bounded open subset of $\mathbb{R}^d$. There exists a constant C_Ω such that*

$$\forall u \in H^1_0(\Omega), \qquad \|u\|_{L^2(\Omega)} \leq C_\Omega\, \|\nabla u\|_{L^2(\Omega)}. \tag{2.1}$$

In the language of functional analysis, it is said that the L^2 norm of u is *controlled* by the L^2 norm of its gradient.

The two assumptions that Ω is bounded and that u vanishes on the boundary of Ω are critical:

- for any bounded open set Ω, consider the function $u(x) = 1$ on Ω. We have $u \in H^1(\Omega)$, but u does not vanish on $\partial\Omega$ and (2.1) of course does not hold;

- in the one-dimensional case, consider the non-bounded choice $\Omega = \mathbb{R}$ and a non-negative, even function u_N which is equal to 1 on $(-N, N)$, which vanishes outside $(-N-1, N+1)$, and which goes smoothly from 1 to 0 on $(N, N+1)$ in a manner independent of N (and likewise on $(-N-1, -N)$): for any $x \in (N, N+1)$, $u_N(x) = v(x-N)$ for some fixed, smooth, non-negative v with $v(0) = 1$ and $v(1) = 0$. We observe that $u_N \in H^1(\mathbb{R}) = H^1_0(\mathbb{R})$. We then compute

$$\|u_N\|^2_{L^2(\mathbb{R})} = 2\left(\int_0^N u_N^2 + \int_N^{N+1} u_N^2\right) = 2\,N + 2\int_0^1 v^2$$

and

$$\|u'_N\|^2_{L^2(\mathbb{R})} = 2\int_N^{N+1} (u'_N)^2 = 2\int_0^1 (v')^2.$$

The inequality (2.1) cannot be satisfied with a fixed constant $C_\mathbb{R}$ and for any function u_N.

Proof of Theorem 2.8. Since the open set Ω is bounded, there exists $L > 0$ such that $\Omega \subset [-L, L]^d$. Let $\phi \in \mathcal{D}(\Omega)$ and ψ the extension of ϕ by 0 to the whole space $\mathbb{R}^d$. We have that $\psi \in C^\infty(\mathbb{R}^d)$ and that

$$\forall x \in [-L, L]^d, \qquad \psi(x) = \int_{-L}^{x_1} \frac{\partial \psi}{\partial x_1}(t, x_2, \cdots, x_d)\, dt.$$

We hence have

$$\begin{aligned} |\psi(x)|^2 &= \left(\int_{-L}^{x_1} \frac{\partial \psi}{\partial x_1}(t, x_2, \cdots, x_d)\, dt\right)^2 \\ &\leq \left(\int_{-L}^{x_1} \left|\frac{\partial \psi}{\partial x_1}(t, x_2, \cdots, x_d)\right|^2 dt\right)\left(\int_{-L}^{x_1} 1^2\, dt\right) \\ &\leq 2L \int_{-L}^{L} \left|\frac{\partial \psi}{\partial x_1}(t, x_2, \cdots, x_d)\right|^2 dt. \end{aligned}$$

Integrating the above inequality on $[-L, L]^d$, we deduce that

$$\int_{[-L,L]^d} |\psi|^2 \leq 4L^2 \int_{[-L,L]^d} \left|\frac{\partial \psi}{\partial x_1}\right|^2 \leq 4L^2 \int_{[-L,L]^d} |\nabla \psi|^2.$$

Since Supp (ψ) = Supp $(\phi) \subset \Omega$, Supp $(\nabla\psi)$ = Supp $(\nabla\phi) \subset \Omega$, and since $\phi = \psi$ and $\nabla\phi = \nabla\psi$ on Ω, we deduce that

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \|\phi\|_{L^2(\Omega)} \leq 2L\, \|\nabla\phi\|_{L^2(\Omega)}\,. \tag{2.2}$$

In addition, the mappings

$$\begin{array}{ccc} H^1(\Omega) & \longrightarrow & \mathbb{R} \\ u & \mapsto & \|u\|_{L^2(\Omega)} \end{array} \quad \text{and} \quad \begin{array}{ccc} H^1(\Omega) & \longrightarrow & \mathbb{R} \\ u & \mapsto & \|\nabla u\|_{L^2(\Omega)} \end{array}$$

are continuous. Since $\mathcal{D}(\Omega)$ is dense in $H^1_0(\Omega)$ under the H^1 norm, we get that the estimate (2.2) remains valid for any function in $H^1_0(\Omega)$. □

2.3 On the notion of traces

We now introduce the notion of trace, which generalizes the notion of the restriction on the boundary $\partial\Omega$ of a function defined on Ω. For any function $u \in C^0(\overline{\Omega})$, the trace of u on $\partial\Omega$ is defined as

$$\begin{array}{rccl} \gamma(u) : \partial\Omega & \longrightarrow & \mathbb{R} \\ x & \mapsto & u(x). \end{array}$$

Put differently, we have $\gamma(u) = u|_{\partial\Omega}$. For a function u in $C^0(\overline{\Omega})$, the trace of u on $\partial\Omega$ is simply the restriction of u on $\partial\Omega$.

Note that the trace mapping

$$\begin{array}{rccl} \gamma : C^0(\overline{\Omega}) & \longrightarrow & C^0(\partial\Omega) \\ u & \mapsto & \gamma(u) \end{array}$$

is linear and continuous:

- for any u and v in $C^0(\overline{\Omega})$ and any $\lambda \in \mathbb{R}$, we have

$$\gamma(u + \lambda v) = (u + \lambda v)|_{\partial\Omega} = u|_{\partial\Omega} + \lambda\, v|_{\partial\Omega} = \gamma(u) + \lambda\gamma(v),$$

  thus the linearity of γ;

- for any u in $C^0(\overline{\Omega})$, we have

$$\|\gamma(u)\|_{C^0(\partial\Omega)} = \sup_{x\in\partial\Omega} |\gamma(u)(x)| = \sup_{x\in\partial\Omega} |u(x)| \leq \sup_{x\in\overline{\Omega}} |u(x)| = \|u\|_{C^0(\overline{\Omega})},$$

  thus the continuity of γ.

The question is now to extend the notion of trace to functions that are less regular than $C^0(\overline{\Omega})$. This is not always possible. Here are some answers:

1. It is not possible to define the trace of a function $u \in L^2(\Omega)$.

   Consider indeed the function $u : x \mapsto \sin(1/x)$ on $\Omega = (0, 1)$, which belongs to $L^\infty(\Omega) \subset L^2(\Omega)$. The boundary $\partial\Omega$ is the union of two points, 0 and 1. The function u is continuous at 1 and we can hence define its trace in 1 (it is the real number $\sin 1$). In contrast, the set of limit points of u at 0 is the whole interval $[-1, 1]$. Hence, there is no natural way to define the trace of u (the value of u) at 0.

2. As pointed out above, in dimension $d = 1$, a function in $H^1(a, b)$ admits a continuous representation (the proof of this statement is carried out in Exercise 2.17). Hence, one can define the trace in a and in b of a function belonging to $H^1(a, b)$.

3. In dimension $d \geq 2$, a function in $H^1(\Omega)$ is not necessarily continuous (see Exercise 2.18). However, one can still define its trace on $\partial\Omega$, according to the following result.

Theorem 2.9. *There exists a linear and continuous mapping*

$$\begin{array}{rccl} \gamma : H^1(\Omega) & \longrightarrow & L^2(\partial\Omega) \\ u & \mapsto & \gamma(u) \end{array}$$

such that, for any $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$, we have $\gamma(u) = u|_{\partial\Omega}$.

As above, the mapping γ is linear in the sense that $\gamma(u+\lambda v) = \gamma(u)+\lambda\gamma(v)$, and it is continuous in the sense that there exists some constant (depending on Ω) such that

$$\|\gamma(u)\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}.$$

For functions $u \in H^1(\Omega)$ which happen to also be continuous on $\overline{\Omega}$, $\gamma(u)$ is the usual restriction of u. In that sense, the trace generalizes the usual notion of restriction.

Remark 2.10. *Actually, the function $\gamma(u)$ is more regular than simply $L^2(\partial\Omega)$, it belongs to the space $H^{1/2}(\partial\Omega)$ (we refer to [1] for more details). In other words, the trace mapping is not surjective from $H^1(\Omega)$ to $L^2(\partial\Omega)$, its range belongs to a strictly smaller set than $L^2(\partial\Omega)$.*

We close this section with the following characterization of $H_0^1(\Omega)$ *via* the notion of traces. It is often said that $H_0^1(\Omega)$ is the space of functions in $H^1(\Omega)$ that "vanish on the boundary". The formalization of this statement is the following proposition, which provides a definition of $H_0^1(\Omega)$ which is equivalent to that given in Definition 2.5.

Proposition 2.11. *For any open set $\Omega \subset \mathbb{R}^d$, we have*

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega), \quad \gamma(u) = 0\right\}.$$

2.4 The space $H^{-1}(\Omega)$

Definition 2.12. *Let Ω be an open subset of $\mathbb{R}^d$. We denote by $H^{-1}(\Omega)$ the vector space of distributions $T \in \mathcal{D}'(\Omega)$ for which there exists a constant C such that*

$$\forall \phi \in \mathcal{D}(\Omega), \qquad |\langle T, \phi\rangle| \leq C\|\phi\|_{H^1(\Omega)}.$$

Remark 2.13. *It is clear that $L^2(\Omega) \subset H^{-1}(\Omega)$. Indeed, if $f \in L^2(\Omega)$, then*

$$\forall \phi \in \mathcal{D}(\Omega), \qquad |\langle f, \phi\rangle| = \left|\int_\Omega f\phi\right| \leq \|f\|_{L^2(\Omega)}\ \|\phi\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\ \|\phi\|_{H^1(\Omega)}.$$

Theorem 2.14. *It is possible to identify $H^{-1}(\Omega)$ with the topological dual of $H_0^1(\Omega)$ (i.e. with the vector space of linear and continuous forms on $H_0^1(\Omega)$).*

The proof of Theorem 2.14 is given below. We note here that, as a consequence of the above results, we have the following inclusions:

$$\mathcal{D}(\Omega) \subset H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) \subset \mathcal{D}'(\Omega)$$

and, for any $T \in L^2(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$, we have

$$\langle T, \phi\rangle = \langle T, \phi\rangle_{H^{-1},H_0^1} = \langle T, \phi\rangle_{L^2} = \int_\Omega T\,\phi.$$

The space L^2 is the so-called "pivot" space for all of these dualities: if $T \in L^2(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$, then $\langle T, \phi\rangle$ can be understood in $(\mathcal{D}', \mathcal{D})$, (H^{-1}, H_0^1) or (L^2, L^2).

Proof of Theorem 2.14. Let $T \in H^{-1}(\Omega)$. The linear map

$$\phi \mapsto \langle T, \phi \rangle$$

is continuous on the space $\mathcal{D}(\Omega)$ *endowed with the* H^1 *norm*. Since $\mathcal{D}(\Omega)$ is dense in $H^1_0(\Omega)$ for that norm, we take as given the fact that the above mapping can be extended (in a unique way) to some linear and continuous mapping on $H^1_0(\Omega)$, that we denote

$$\phi \mapsto \langle T, \phi \rangle_{H^{-1}, H^1_0},$$

and which satisfies

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \langle T, \phi \rangle_{H^{-1}, H^1_0} = \langle T, \phi \rangle.$$

Hence, to any $T \in H^{-1}(\Omega)$, we can associate an element of the topological dual of $H^1_0(\Omega)$ (i.e. the vector space of linear and continuous forms on $H^1_0(\Omega)$). Hence, we define

$$\begin{array}{rcl} \alpha \,:\, H^{-1}(\Omega) & \longrightarrow & (H^1_0(\Omega))' \\ T & \mapsto & \langle T, \cdot \rangle_{H^{-1}, H^1_0}. \end{array}$$

Conversely, let $L \in (H^1_0(\Omega))'$. There exists a constant C such that

$$\forall \phi \in H^1_0(\Omega), \qquad |L(\phi)| \leq C \|\phi\|_{H^1}.$$

We restrict L to $\mathcal{D}(\Omega) \subset H^1_0(\Omega)$, and we hence get a linear form on $\mathcal{D}(\Omega)$ that satisfies

$$\forall \phi \in \mathcal{D}(\Omega), \qquad |L(\phi)| \leq C \|\phi\|_{H^1}.$$

We are left to show that L is a distribution, i.e. that L satisfies the "continuity property" (1.2). Let K be a compact set contained in Ω and let $\phi \in \mathcal{D}_K(\Omega)$. We see that

$$|L(\phi)| \leq C \|\phi\|_{H^1} \leq C \left(\|\phi\|^2_{L^2} + \|\nabla \phi\|^2_{L^2} \right)^{1/2}.$$

Using that $\|\phi\|_{L^2} \leq \sqrt{|K|} \sup |\phi|$ and $\|\nabla \phi\|_{L^2} \leq \sqrt{|K|} \sup |\nabla \phi|$, we obtain that

$$|L(\phi)| \leq C' \sup_{|\alpha| \leq 1,\ x \in K} |\partial^\alpha \phi|.$$

Hence L defines a distribution (of order lower or equal to 1). We hence define

$$\begin{array}{rcl} \beta \,:\, (H^1_0(\Omega))' & \longrightarrow & H^{-1}(\Omega) \\ L & \mapsto & L_{\mathcal{D}(\Omega)}. \end{array}$$

It is easy to check that $\alpha \circ \beta = I_{(H^1_0(\Omega))'}$ and that $\beta \circ \alpha = I_{H^{-1}(\Omega)}$. □

Proposition 2.15 (Characterization of the distributions in H^{-1})**.** *Let Ω be an open subset of $\mathbb{R}^d$. A distribution T belongs to $H^{-1}(\Omega)$ if and only if there exist, for any $|\alpha| \leq 1$, a function $g_\alpha \in L^2(\Omega)$ such that*

$$T = \sum_{|\alpha| \leq 1} \partial^\alpha g_\alpha.$$

Proof. It is obvious that, if T is of the form

$$T = \sum_{|\alpha|\le 1} \partial^\alpha g_\alpha$$

with $g_\alpha \in L^2(\Omega)$, we have

$$\begin{aligned}
\forall \phi \in \mathcal{D}(\Omega), \qquad |\langle T, \phi\rangle| &= \left|\langle \sum_{|\alpha|\le 1} \partial^\alpha g_\alpha, \phi\rangle\right| \\
&\le \left|\sum_{|\alpha|\le 1} (-1)^{|\alpha|} \langle g_\alpha, \partial^\alpha \phi\rangle\right| \\
&\le \sum_{|\alpha|\le 1} \|g_\alpha\|_{L^2} \|\partial^\alpha \phi\|_{L^2} \\
&\le \left(\sum_{|\alpha|\le 1} \|g_\alpha\|_{L^2}\right) \|\phi\|_{H^1}.
\end{aligned}$$

Hence $T \in H^{-1}(\Omega)$. The converse statement is more challenging to prove. We omit its proof. □

2.5 Exercises

Exercise 2.16.

1. *For which values of k (we recall that $k \in \mathbb{N}$) do the following functions belong to $H^k(-1,1)$: $x \mapsto \sin x$, $x \mapsto |x|$ and $x \mapsto sgn(x)$?*
2. *For which values of the integer k and of the real number α does the function $x \mapsto \dfrac{1}{x^\alpha}$ belong to $H^k(0,1)$?*
3. *For which values of the integer k, the real number α and the integer d does the function $x \mapsto \dfrac{1}{|x|^\alpha}$ belong to $H^k\left(\mathbb{R}^d \setminus \overline{B_0(1)}\right)$?*

Exercise 2.17. *Consider two real numbers a and b with $a < b$. We want to show that any function u in $H^1(a,b)$ has a continuous representation. To this end, we proceed in three steps:*

1. *Let $f \in L^1_{\text{loc}}(\mathbb{R})$ and let $F(x) = \displaystyle\int_0^x f$. Show that the derivative of F, in the sense of distribution, is equal to f (hint: show that $F \in L^1_{\text{loc}}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ and next compute F' using the Fubini theorem).*
2. *Introduce $U(x) = \displaystyle\int_0^x u'$. For any $a \le x_1 \le x_2 \le b$, show that $|U(x_2) - U(x_1)| \le \sqrt{|x_2 - x_1|}\, \|u'\|_{L^2(a,b)}$. Deduce that U is continuous on $[a,b]$.*
3. *Using the first question, compute $(u - U)'$. Deduce that there exists a constant C such that $u(x) = U(x) + C$ and conclude.*

Exercise 2.18. *Let Ω be the disk centered at 0 of radius $1/e$ in $\mathbb{R}^2$. Show that the function*

$$u(x,y) = \ln\left[-\ln\left(\sqrt{x^2+y^2}\right)\right]$$

belongs to $H^1_0(\Omega)$ but that it is not continuous at 0.

Exercise 2.19. *Consider the vector space*

$$V = \left\{v \in L^2(\Omega),\ \Delta v \in L^2(\Omega)\right\}$$

endowed with the scalar product $(v,w)_V = (v,w)_{L^2} + (\Delta v, \Delta w)_{L^2}$. Show that V is a Hilbert space.

Exercise 2.20. *Let p be a real number with $1 \le p \le +\infty$. Let $k \in \mathbb{N}$. We set*

$$W^{k,p}(\Omega) = \left\{u \in L^p(\Omega),\ \partial^\alpha u \in L^p(\Omega) \text{ for any } \alpha \in \mathbb{N}^d,\ |\alpha| \le k\right\}.$$

This vector space is endowed with the norm

$$\|u\|_{W^{k,p}} = \sum_{\alpha\in\mathbb{N}^d,\ |\alpha|\le k} \|\partial^\alpha u\|_{L^p}.$$

Explain why we can consider the derivative (in the sense of distributions) of u, and show that the space $W^{k,p}(\Omega)$ is a Banach space.

Exercise 2.21. *Let $u \in H^1(\mathbb{R})$. Show that*

$$\|u\|^2_{L^\infty(\mathbb{R})} \le \|u\|_{L^2(\mathbb{R})}\ \|u'\|_{L^2(\mathbb{R})}.$$

Hint: use the density of $\mathcal{D}(\mathbb{R})$ in $H^1(\mathbb{R})$.

Exercise 2.22. *Consider two real numbers a and b with $a < b$. Let $f \in H^1(a,b)$. Show that $|f| \in H^1(a,b)$. It will be useful to show that*

$$\frac{d|f|}{dx}(x) = \begin{cases} \dfrac{f(x)}{|f(x)|}\ \dfrac{df}{dx}(x) & \text{if } f(x) \neq 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

Deduce that, if f and g belong to $H^1(a,b)$, then $\max(f,g)$ and $\min(f,g)$ belong to $H^1(a,b)$.

Exercise 2.23. *Let $u \in H^1(\mathbb{R}^d)$. We denote $(e_1,\dots,e_d)$ the canonical basis of $\mathbb{R}^d$. Show that, for any $1 \le i \le d$,*

$$\int_{\mathbb{R}^d}\left|\frac{\partial u}{\partial x_i}\right|^2 = \lim_{t\to 0}\ \frac{1}{t^2}\ \int_{\mathbb{R}^d} |u(x+t\,e_i) - u(x)|^2\ dx.$$

Exercise 2.24. *Let Ω be a bounded open subset of $\mathbb{R}^d$ and $f \in L^2(\Omega)$. On $H^1_0(\Omega)$, consider the functional J defined by*

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f\,u.$$

The quantity $J(u)$ is the energy that naturally arises in thermal or elasticity problems. Show that J is infinite at infinity, namely that $J(u) \to \infty$ when $\|u\|_{H^1} \to \infty$.

Exercise 2.25. *The aim of this exercise is to show the Hardy inequality in $\mathcal{D}(\mathbb{R}^3)$.*

1. *Let $f \in C^\infty([0,\infty))$ and such that there exists $A > 0$ such that $f(r) = 0$ for any $r \geq A$. Show that*
$$\int_0^\infty f^2(r)\,dr = -2\int_0^\infty r\,f'(r)\,f(r)\,dr$$
and deduce that
$$\int_0^\infty f^2(r)\,dr \leq 4\int_0^\infty r^2\,(f'(r))^2\,dr.$$

2. *Deduce that*
$$\forall \psi \in \mathcal{D}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} \frac{\psi^2(x)}{|x|^2}\,dx \leq 4\int_{\mathbb{R}^3} |\nabla\psi(x)|^2\,dx. \tag{2.3}$$
Hint: note that, in spherical coordinates,
$$\int_{\mathbb{R}^3} \frac{\psi^2(x)}{|x|^2}\,dx = \int_0^\pi \int_0^{2\pi} \left(\int_0^\infty \psi^2(r,\theta,\varphi)\,dr\right)\sin(\theta)\,d\theta d\varphi$$
and that $\nabla\psi = \dfrac{\partial\psi}{\partial r}\,e_r + \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}\,e_\theta + \dfrac{1}{r\,\sin(\theta)}\dfrac{\partial\psi}{\partial\varphi}\,e_\varphi$, where the spherical basis $(e_r, e_\theta, e_\varphi)$ is orthonormal.

Exercise 2.26. *Consider the space*
$$\mathcal{A} = \left\{u \in H^1(\mathbb{R}^3),\ \|u\|_{L^2(\mathbb{R}^3)} = 1\right\}$$
and the function
$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|}\,dx.$$
The quantity $\mathcal{E}$ represents the electronic energy of the Hydrogen atom in quantum physics.

1. *Show that $\mathcal{E}(\psi)$ is well-defined whenever $\psi \in \mathcal{D}(\mathbb{R}^3)$.*

2. *Using the inequality (2.3), show that, for any $\psi \in \mathcal{D}(\mathbb{R}^3)$, we have*
$$\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|}\,dx \leq 2\|\psi\|_{L^2(\mathbb{R}^3)}\,\|\nabla\psi\|_{L^2(\mathbb{R}^3)}. \tag{2.4}$$

3. *Let $u \in H^1(\mathbb{R}^3)$. The aim of this question is to show that*
$$\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|}\,dx \leq 2\|u\|_{L^2(\mathbb{R}^3)}\,\|\nabla u\|_{L^2(\mathbb{R}^3)}. \tag{2.5}$$
   (a) *Recall the reason for which there exists a sequence of functions $\psi_n \in \mathcal{D}(\mathbb{R}^3)$ that converges to u in $H^1(\mathbb{R}^3)$.*

   (b) *Let $\phi \in \mathcal{D}(\mathbb{R}^3)$. Show that $\dfrac{\phi(x)}{|x|} \in L^2(\mathbb{R}^3)$ and that*
$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{\psi_n(x)\phi(x)}{|x|}\,dx = \int_{\mathbb{R}^3} \frac{u(x)\phi(x)}{|x|}\,dx.$$

(c) *Using the inequality* (2.3)*, show that the sequence of functions* $\dfrac{\psi_n(x)}{|x|}$ *is a Cauchy sequence in* $L^2(\mathbb{R}^3)$.

(d) *Deduce that there exists* $v \in L^2(\mathbb{R}^3)$ *such that* $\dfrac{\psi_n(x)}{|x|}$ *converges to* v *in* $L^2(\mathbb{R}^3)$.

(e) *Using the questions* (3a)–(3d)*, show that* $\dfrac{u(x)}{|x|} = v(x)$ *in* $L^2(\mathbb{R}^3)$.

(f) *Show that*

$$\left| \int_{\mathbb{R}^3} \frac{\psi_n(x)^2}{|x|}\, dx - \int_{\mathbb{R}^3} \frac{u(x)^2}{|x|}\, dx \right| \le \left\| \frac{\psi_n - u}{|x|} \right\|_{L^2(\mathbb{R}^3)} \|\psi_n + u\|_{L^2(\mathbb{R}^3)}$$

and deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{\psi_n(x)^2}{|x|}\, dx = \int_{\mathbb{R}^3} \frac{u(x)^2}{|x|}\, dx.$$

(g) *Using inequality* (2.4) *on* ψ_n *and taking the limit* $n \to \infty$*, show that* (2.5) *holds.*

4. *Consider the optimization problem*

$$I = \inf \{\mathcal{E}(u);\ u \in \mathcal{A}\}.$$

Using (2.5)*, show that* $I > -\infty$.

Exercise 2.27 (Hardy inequality)**.** *Consider the vector space*

$$W^1(\mathbb{R}^3) = \left\{ u \in L^2_{\rm loc}(\mathbb{R}^3),\ \frac{u(x)}{|x|} \in L^2(\mathbb{R}^3),\ \nabla u \in \left(L^2(\mathbb{R}^3)\right)^3 \right\}.$$

We recall that $L^2_{\rm loc}(\mathbb{R}^3)$ *is the vector space of functions* $u : \mathbb{R}^3 \mapsto \mathbb{R}$ *that are measurable and such that* $\int_K |u|^2 < \infty$ *for any compact set* $K \subset \mathbb{R}^3$*. Since* $L^2_{\rm loc}(\mathbb{R}^3) \subset L^1_{\rm loc}(\mathbb{R}^3)$*, we have that* ∇u *is well-defined in the sense of distributions.*

For any u *and* v *in* $W^1(\mathbb{R}^3)$*, we set*

$$(u, v)_{W^1} = \int_{\mathbb{R}^3} \frac{u(x)\, v(x)}{|x|^2}\, dx + \int_{\mathbb{R}^3} \nabla u \cdot \nabla v. \tag{2.6}$$

1. *Show that, for any* u *and* v *in* $W^1(\mathbb{R}^3)$*, the right-hand side of* (2.6) *is well-defined, and that* $(\cdot,\cdot)_{W^1}$ *defines a scalar product on* $W^1(\mathbb{R}^3)$.

2. *Show that* $\mathcal{D}(\mathbb{R}^3) \subset W^1(\mathbb{R}^3)$.

3. *Using the question* (3e) *of Exercise 2.26, show that* $H^1(\mathbb{R}^3) \subset W^1(\mathbb{R}^3)$.

4. *We admit that* $W^1(\mathbb{R}^3)$*, endowed with the above scalar product, is a Hilbert space and that* $\mathcal{D}(\mathbb{R}^3)$ *is dense in* $W^1(\mathbb{R}^3)$ *(when the space* $W^1(\mathbb{R}^3)$ *is endowed with the norm* $\|\cdot\|_{W^1}$ *associated to the scalar product* $(\cdot,\cdot)_{W^1}$*).*

   We recall (see Exercise 2.25) that, for any $\psi \in \mathcal{D}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \frac{\psi(x)^2}{|x|^2}\, dx \le 4 \int_{\mathbb{R}^3} |\nabla \psi|^2.$$

   Show that, for any $u \in W^1(\mathbb{R}^3)$,

$$\left\| \frac{u}{|x|} \right\|_{L^2} \le 2\, \|\nabla u\|_{L^2}.$$

Chapter 3

Linear elliptic boundary value problems

This chapter is devoted to the mathematical analysis of linear elliptic boundary value problems. A boundary value problem is usually composed of

- one (or several) Partial Differential Equation (PDE),
- boundary conditions (or asymptotic conditions if the domain on which the problem is posed is the whole space $\mathbb{R}^d$).

For time-dependent problems (which will not be considered here), one has to additionally consider initial conditions.

We restrict ourselves to linear problems, where the solution depends linearly (or in an affine manner) on the right-hand side of the PDE and on the imposed boundary conditions.

The term *elliptic* designates a class of partial differential equations, typical examples in that class being the Poisson equation $-\Delta u = f$ studied in Section 3.3 and the advection-diffusion equation studied in Section 3.4. Other classes of PDEs, that will not be considered here, are *parabolic* problems, such as the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f \qquad \text{(complemented by appropriate boundary and initial conditions)},$$

or *hyperbolic* problems (such as the wave equation)

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \qquad \text{(complemented by appropriate boundary and initial conditions)}.$$

In this chapter, we show that the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

is well-posed in $H^1(\Omega)$, in the sense that it has a unique solution, and that this solution depends in a continuous manner on the right-hand side f. Problem (3.1) appears e.g. in electrostatic and is often written in the form $-\Delta V = \rho$, where V is the electric potential and ρ is the distribution of electric charges. Problem (3.1) is also encountered in mechanics (it describes the vertical displacement u of a membrane submitted to some forces f and clamped at its boundary). Besides its own interest, Problem (3.1) is also interesting because many boundary value problems in the engineering sciences are (more or less elaborated) variants of (3.1).

To show the well-posedness of (3.1), we proceed as follows:

1. First, we show that Problem (3.1) is equivalent to another problem, written in the form of a variational formulation of the type

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ \forall v \in V, \quad a(u,v) = b(v) \end{cases} \tag{3.2}$$

   for some appropriate choices of a Hilbert space V, a bilinear form a and a linear form b (see Section 3.3 for details in the particular case of Problem (3.1)). Problems (3.1) and (3.2) are equivalent in the sense that any solution in $H^1(\Omega)$ of Problem (3.1) is a solution to Problem (3.2), and the converse is also true.

2. Second, using the *Lax-Milgram theorem*, we show that Problem (3.2) is well-posed (i.e. has a unique solution).

Besides the specific problem (3.1), other problems are also studied in this chapter. Several exercises guide the reader through more and more complex examples.

As pointed out above, this chapter is devoted to the *mathematical analysis* of some boundary value problems, establishing that these problems are well-posed. Questions related to the approximation of the exact solutions, to the estimation of the error, and more generally to the *numerical analysis* of these problems, will be addressed in another course.

3.1 Lax-Milgram theorem

Let H be a vector space endowed with a norm $\|\cdot\|$. We recall that a linear form b on H is continuous if and only if there exists c such that

$$\forall v \in H, \quad |b(v)| \le c\|v\|.$$

Likewise, a bilinear form a on $H \times H$ is continuous if and only if there exists M such that

$$\forall (u,v) \in H \times H, \quad |a(u,v)| \le M\|u\|\,\|v\|.$$

Definition 3.1. *Let H be a Hilbert space and a be a bilinear form on H. We say that a is* coercive *on H if there exists a real number $\alpha > 0$ such that*

$$\forall u \in H, \qquad a(u,u) \ge \alpha\,\|u\|^2.$$

Theorem 3.2 (Lax-Milgram theorem)**.** *Let H be a Hilbert space, a be a bilinear form on H, continuous and coercive. Let b be a continuous linear form on H. Then the problem*

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ \forall v \in H, \qquad a(u,v) = b(v) \end{cases} \tag{3.3}$$

has a unique solution.

In the case when a is symmetric, that is $a(u,v) = a(v,u)$ for any u and v in H, the proof of Theorem 3.2 is simple and this is the first proof we provide below. In the general case when a is not necessarily symmetric, then the proof is more involved (except in the finite-dimensional case). We consider this case in a second stage.

Proof of Theorem 3.2 in the symmetric case. Introduce

$$(u, v)_a = a(u, v).$$

Since a is symmetric and coercive, it is clear that $(\cdot, \cdot)_a$ is a scalar product on H. In addition, the norms $\|\cdot\|$ and $\|\cdot\|_a$ are equivalent. Indeed, using the coercivity and the continuity of the bilinear form a, we obtain that there exists $\alpha > 0$ and M such that

$$\forall v \in H, \qquad \alpha\|v\|^2 \leq \|v\|_a^2 = a(v, v) \leq M\|v\|^2.$$

The space H, which is complete for the norm $\|\cdot\|$, is thus also complete for the norm $\|\cdot\|_a$. The space H, endowed with the scalar product $(\cdot, \cdot)_a$, is hence a Hilbert space. The bilinear form b is continuous for the norm $\|\cdot\|$, thus also for the equivalent norm $\|\cdot\|_a$. Using the Riesz theorem, we thus obtain that there exists a unique $u \in H$ such that

$$\forall v \in H, \qquad a(u, v) = (u, v)_a = b(v),$$

which concludes the proof in the symmetric case. □

Proof of Theorem 3.2 in the non-symmetric, finite-dimensional case. The proof of Theorem 3.2 in the finite dimensional case is easy. Consider the Hilbert space $H = \mathbb{R}^n$, the bilinear form $a(u, v) = v^T A u$ for any u and v in H, where A is a $n \times n$ matrix, and the linear form $b(v) = v^T B$ for any v in H, where B is a vector of size n.

We first show that Ker $A = \{0\}$. Let $u \in H$ such that $Au = 0$. Then $a(u, u) = u^T A u = 0$. Thanks to the coercivity assumption on a, this implies that $u = 0$.

Using the fact that dim Ker $A +$ dim Ran $A =$ dim H, we get that dim Ran $A =$ dim H, and hence the matrix A is invertible.

The problem "find $u \in H = \mathbb{R}^n$ such that $a(u, v) = b(v)$ for any $v \in H = \mathbb{R}^n$" reads: "find $u \in \mathbb{R}^n$ such that $v^T A u = v^T B$ for any $v \in \mathbb{R}^n$", which is equivalent to "find $u \in \mathbb{R}^n$ such that $Au = B$". This problem has a unique solution since A is invertible. □

Proof of Theorem 3.2 in the general case. Consider the mapping

$$\begin{array}{rcl} \Phi : H & \longrightarrow & H \\ u & \mapsto & b \text{ such that } a(u, \cdot) = \langle b, \cdot \rangle_H. \end{array}$$

Note that, for any $u \in H$, we have used the Riesz theorem to represent the continuous and linear form $v \in H \mapsto a(u, v) \in \mathbb{R}$ by an element of H, denoted $b = \Phi(u)$. The mapping Φ is of course linear, and it is also continuous:

$$\|b\|^2 = \langle b, b \rangle_H = a(u, b) \leq M\|u\| \, \|b\|$$

hence $\|\Phi(u)\| = \|b\| \leq M\|u\|$. We wish to show that Φ is bijective. The proof falls in three steps.

Step 1: Φ is injective: Let $u \in H$ such that $\Phi(u) = 0$. For any $v \in H$, we have $a(u, v) = 0$, thus $a(u, u) = 0$. Using the coercivity of a, we deduce that

$$0 = a(u, u) \geq \alpha\|u\|^2.$$

Hence $u = 0$.

Step 2: Φ is surjective: Let $V = \text{Ran}(\Phi) \subset H$. We show that $V = H$.

(a) Let us first show that $X = \mathrm{Ran}(\Phi)$ is closed. Let $u \in H$, $u \neq 0$. We have

$$\sup_{v \in H,\, v \neq 0} \frac{\langle \Phi(u), v \rangle_H}{\|v\|} = \sup_{v \in H,\, v \neq 0} \frac{a(u,v)}{\|v\|} \geq \frac{a(u,u)}{\|u\|} \geq \alpha \|u\|.$$

In addition, for any $v \neq 0$, we have

$$\frac{\langle \Phi(u), v \rangle_H}{\|v\|} \leq \|\Phi(u)\|.$$

We thus obtain that

$$\|\Phi(u)\| \geq \alpha \|u\|$$

and this estimate also holds if $u = 0$. Consider now a sequence $(b_n)_{n \in \mathbb{N}}$ in X that converges in H to some $b \in H$. We want to show that $b \in X$. Let $u_n \in H$ such that $\Phi(u_n) = b_n$. Since $(b_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, we see that $(u_n)_{n \in \mathbb{N}}$ is also a Cauchy sequence. Indeed,

$$\|b_p - b_q\| = \|\Phi(u_p) - \Phi(u_q)\| = \|\Phi(u_p - u_q)\| \geq \alpha \|u_p - u_q\|.$$

Therefore, $(u_n)_{n \in \mathbb{N}}$ converges in H to some $u \in H$. In addition, Φ is continuous, so

$$b_n = \Phi(u_n) \longrightarrow \Phi(u) \quad \text{in } H.$$

Hence $b = \Phi(u) \in X$, by uniqueness of the limit in H. The vector space X is hence closed.

(b) Let $b_0 \in H$. Since X is closed in H, we can consider the orthogonal projection of b_0 on X, that we denote $b_1 \in X$. For any $v \in X$, we have $\langle v, b_0 - b_1 \rangle_H = 0$. Hence, for any $w \in H$, we have

$$0 = \langle \Phi(w), b_0 - b_1 \rangle_H = a(w, b_0 - b_1).$$

Taking $w = b_0 - b_1$ and using the coercivity of a, we deduce that $b_0 = b_1 \in X$, which implies that $H = X$.

Step 3: Conclusion: Since b is a continuous linear form on H, it can be represented by some $f \in H$. The problem (3.3) can thus be recast as finding $u \in H$ such that $a(u, \cdot) = \langle f, \cdot \rangle_H$, that is finding $u \in H$ such that $\Phi(u) = f$. In view of the above two steps, the mapping Φ is bijective, so this problem has a unique solution. □

3.2 A first linear elliptic boundary value problem

Let Ω be an open subset of $\mathbb{R}^d$, λ be a positive real number ($\lambda > 0$) and $f \in L^2(\Omega)$. We look for $u \in H^1(\Omega)$ solution to

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Remark 3.3. *All what follows can be extended to the case $f \in H^{-1}(\Omega)$. We have chosen $f \in L^2(\Omega)$ only for the sake of simplicity.*

3.2.1 Variational formulation of (3.4)

We set

- $H = H^1_0(\Omega)$,

- $a(u,v) = \int_\Omega \nabla u \cdot \nabla v + \lambda \int_\Omega u\, v,$
- $b(v) = \int_\Omega f\, v,$

and consider the problem

$$\left\{ \begin{array}{l} \text{Find } u \in H \text{ such that} \\ \forall v \in H, \qquad a(u,v) = b(v). \end{array} \right. \tag{3.5}$$

Proposition 3.4. *Problems* (3.4) *and* (3.5) *are equivalent.*

Proof. We first show that a is continuous on $H \times H$ and that b is continuous on H:

$$\forall v \in H = H_0^1(\Omega), \qquad |b(v)| = \left| \int_\Omega f\, v \right| \leq \|f\|_{L^2}\, \|v\|_{H^1},$$

which means that b is continuous on H. Furthermore, we have

$$\begin{array}{rcl} \forall (u,v) \in H \times H, \qquad |a(u,v)| & = & \left| \int_\Omega \nabla u \cdot \nabla v + \lambda \int_\Omega u\, v \right| \\ & \leq & \left| \int_\Omega \nabla u \cdot \nabla v \right| + \lambda \left| \int_\Omega u\, v \right| \\ & \leq & \left(\int_\Omega |\nabla u|^2 \right)^{1/2} \left(\int_\Omega |\nabla v|^2 \right)^{1/2} + \lambda \left(\int_\Omega u^2 \right)^{1/2} \left(\int_\Omega v^2 \right)^{1/2} \\ & \leq & (1+\lambda) \|u\|_{H^1}\, \|v\|_{H^1}, \end{array}$$

which means that a is continuous on $H \times H$.

We now show that (3.5) implies (3.4). Let u be a solution to (3.5). We already have that $u \in H_0^1(\Omega)$, hence $u \in H^1(\Omega)$ and $u = 0$ on $\partial\Omega$. Let now $\phi \in \mathcal{D}(\Omega)$. We have $\phi \in H_0^1(\Omega)$, thus

$$\begin{array}{rcl} \langle f, \phi \rangle & = & \int_\Omega f\, \phi \\ & = & b(\phi) \\ & = & a(u,\phi) \\ & = & \int_\Omega \nabla u \cdot \nabla \phi + \lambda \int_\Omega u\, \phi \\ & = & \langle \nabla u, \nabla \phi \rangle + \lambda \langle u, \phi \rangle \\ & = & \langle -\Delta u + \lambda u, \phi \rangle. \end{array}$$

Therefore $-\Delta u + \lambda u = f$ in $\mathcal{D}'(\Omega)$. Thus u is a solution to (3.4).

Conversely, let u be a solution to (3.4). We already have that $u \in H^1(\Omega)$ and $u = 0$ on $\partial\Omega$, thus $u \in H_0^1(\Omega)$. Let $\phi \in \mathcal{D}(\Omega)$. We have

$$\begin{array}{rcl} b(\phi) & = & \int_\Omega f\, \phi \\ & = & \langle f, \phi \rangle \\ & = & \langle -\Delta u + \lambda u, \phi \rangle \\ & = & \langle \nabla u, \nabla \phi \rangle + \lambda \langle u, \phi \rangle \\ & = & a(u,\phi). \end{array}$$

Hence,

$$\forall \phi \in \mathcal{D}(\Omega), \qquad a(u,\phi) = b(\phi).$$

Since $a(u,\cdot)$ and b are continuous on $H = H^1_0(\Omega)$ and since $\mathcal{D}(\Omega)$ is dense in $H^1_0(\Omega)$, we conclude that

$$\forall v \in H, \qquad a(u,v) = b(v).$$

Therefore, u is a solution to (3.5). □

3.2.2 Checking the assumptions of the Lax-Milgram theorem

We now check that Problem (3.5) falls within the assumptions of the Lax-Milgram theorem:

- $H = H^1_0(\Omega)$ is a Hilbert space;
- b is linear and continuous on H;
- a is bilinear and continuous on $H \times H$. In addition, a is coercive: for any $v \in H$,

$$\begin{aligned} a(v,v) &= \int_\Omega |\nabla v|^2 + \lambda \int_\Omega v^2 \\ &\geq \min(1,\lambda) \left(\int_\Omega |\nabla v|^2 + \int_\Omega v^2 \right) \\ &= \min(1,\lambda) \|v\|^2_{H^1}. \end{aligned}$$

We are thus in position to state that Problem (3.5) has a unique solution, in view of the Lax-Milgram theorem 3.2. The problems (3.4) and (3.5) are equivalent. The problem (3.4) thus has a unique solution.

Remark 3.5 (Link with other courses)**.** *Note that what we call here variational formulation, namely the formulation* (3.5)*, is often called "virtual works principle" (principe des travaux virtuels) in mechanics. The virtual works of mechanics courses identically correspond to our test functions v, which span the whole space $H^1_0(\Omega)$.*

3.2.3 On the choice of the variational formulation

Being given a PDE, there is not necessarily a unique appropriate variational formulation (see Remark 3.9). We show here why, for the choice of a and b made above, the choice of working in the space $H = H^1_0(\Omega)$ is natural.

First, it is not possible to work in a space larger than $H^1(\Omega)$ (for instance $L^2(\Omega)$), otherwise the bilinear form a is not defined. In addition, the boundary condition $u = 0$ makes no sense for $u \in L^2(\Omega)$, since the trace is not defined on $L^2(\Omega)$. The largest space on which one can work is therefore $H^1_0(\Omega)$.

Working in a space strictly included in $H^1_0(\Omega)$ is not possible either, for the following reason. Consider for instance the choice $H = H^2(\Omega) \cap H^1_0(\Omega)$. Then the bilinear form a and the linear form b are well-defined on H, and both are continuous on H. However, the bilinear form a is not coercive on H, i.e. there exists no α such that

$$\forall u \in H^2(\Omega) \cap H^1_0(\Omega), \qquad a(u,u) \geq \alpha \|u\|^2_{H^2(\Omega)}.$$

We argue by contradiction, assume that such a constant α exists and consider (in the one-dimensional case $\Omega = (0,2\pi)$) the function $u_n(x) = \sin(nx)$. We observe that $u_n \in C^\infty(\Omega)$ (it thus belongs to $H^2(\Omega)$) and that u_n vanishes on the boundary of Ω. We hence have checked that $u_n \in H^2(\Omega) \cap H^1_0(\Omega)$. We next compute

$$a(u_n,u_n) = n^2 \int_\Omega \cos^2(nx) + \lambda \int_\Omega \sin^2(nx) \sim_{n\to\infty} \frac{n^2}{2}$$

while

$$\|u_n\|^2_{H^2(\Omega)} \sim_{n\to\infty} \frac{n^4}{2}.$$

The constant α should thus satisfy, for sufficiently large n, that $\frac{n^2}{2} \geq \alpha \frac{n^4}{2}$, which is not possible.

3.3 Poisson equation on a bounded open set

Let Ω be an open, bounded subset of $\mathbb{R}^d$ and $f \in L^2(\Omega)$. We look for $u \in H^1(\Omega)$ solution to

$$\begin{cases} -\Delta u = f & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Remark 3.6. *Again, as in Section 3.2, it is possible to take $f \in H^{-1}(\Omega)$ rather than $f \in L^2(\Omega)$.*

3.3.1 Variational formulation of (3.6)

We set

- $H = H^1_0(\Omega)$,
- $a(u,v) = \displaystyle\int_\Omega \nabla u \cdot \nabla v$,
- $b(v) = \displaystyle\int_\Omega f\, v$,

and consider the problem

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ \forall v \in H, \qquad a(u,v) = b(v). \end{cases} \tag{3.7}$$

Proposition 3.7. *Problems (3.6) and (3.7) are equivalent.*

Proof. The proof follows the same steps as the proof of Proposition 3.4. □

3.3.2 Checking the assumptions of the Lax-Milgram theorem

We now check that Problem (3.7) falls within the assumptions of the Lax-Milgram theorem:

- $H = H^1_0(\Omega)$ is a Hilbert space;
- b is linear and continuous on H;
- a is bilinear and continuous on $H \times H$.

We are left with showing that a is coercive. Using the Poincaré inequality (see Theorem 2.8), we see that, for any $v \in H^1_0(\Omega)$,

$$\|v\|^2_{H^1} = \|v\|^2_{L^2} + \|\nabla v\|^2_{L^2} \leq (1 + C^2_\Omega)\ \|\nabla v\|^2_{L^2} = (1 + C^2_\Omega)\ a(v,v).$$

Hence

$$a(v,v) \geq \frac{1}{1 + C^2_\Omega}\ \|v\|^2_{H^1}.$$

In view of Lax-Milgram theorem 3.2, we deduce that Problem (3.7) has a unique solution. The problems (3.6) and (3.7) are equivalent. The problem (3.6) thus has a unique solution.

3.4 A non-symmetric boundary value problem

Let Ω be a open, bounded subset of $\mathbb{R}^d$ and $f \in L^2(\Omega)$. Let $c : \Omega \to \mathbb{R}^d$ be a vector field of class $C^1(\overline{\Omega})$ which is divergence-free:

$$\text{div } c = \sum_{i=1}^{d} \frac{\partial c_i}{\partial x_i} = 0 \quad \text{in } \Omega.$$

We look for $u \in H^1(\Omega)$ solution to

$$\begin{cases} -\Delta u + c \cdot \nabla u = f & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Note that the product $c \cdot \nabla u$ is well-defined in $\mathcal{D}'(\Omega)$. This comes from the fact that (i) ∇u is a function in $L^2(\Omega)$, hence in $L^1(\Omega)$ since Ω is bounded, and (ii) c is a bounded function (it is a continuous function on a compact set). The product $c \cdot \nabla u$ is thus in $L^1(\Omega)$, and therefore in $\mathcal{D}'(\Omega)$.

3.4.1 Variational formulation of (3.8)

We set

- $H = H_0^1(\Omega)$,
- $a(u,v) = \displaystyle\int_\Omega \nabla u \cdot \nabla v + \int_\Omega (c \cdot \nabla u)\, v$,
- $b(v) = \displaystyle\int_\Omega f\, v$,

and consider the problem

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ \forall v \in H, \qquad a(u,v) = b(v). \end{cases} \tag{3.9}$$

Proposition 3.8. *Problems* (3.8) *and* (3.9) *are equivalent.*

Proof. In the proof of Proposition 3.4, we have already shown the continuity of b. We now show the continuity of a. For any $(u,v) \in H \times H$, we compute

$$\begin{aligned} |a(u,v)| &= \left| \int_\Omega \nabla u \cdot \nabla v + \int_\Omega (c \cdot \nabla u)\, v \right| \\ &\leq \left| \int_\Omega \nabla u \cdot \nabla v \right| + \left| \int_\Omega (c \cdot \nabla u)\, v \right| \\ &\leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \sup_{\overline{\Omega}} |c| \; \|\nabla u\|_{L^2} \|v\|_{L^2} \\ &\leq (1 + \sup_{\overline{\Omega}} |c|) \; \|u\|_{H^1} \|v\|_{H^1}. \end{aligned}$$

Thus a is continuous in $H \times H$.

The sequel of the proof follows the arguments of the proof of Proposition 3.4. □

Remark 3.9. *For advection-diffusion problems, it is sometimes useful to work with a different bilinear form. For any u and v in $H^1_0(\Omega)$ (and without making any assumption on* div c*), we note that*

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega (c\cdot\nabla u)\,v + \frac{1}{2}\int_\Omega (v\,c)\cdot\nabla u \\ &= \int_\Omega \nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega (c\cdot\nabla u)\,v - \frac{1}{2}\int_\Omega u\,\mathrm{div}\,(v\,c) \\ &= \int_\Omega \nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega (c\cdot\nabla u)\,v - \frac{1}{2}\int_\Omega (c\cdot\nabla v)\,u - \frac{1}{2}\int_\Omega u\,v\,\mathrm{div}\,c. \end{aligned}$$

This thus leads to introduce (again without making any assumption on div c*) the bilinear form*

$$\overline{a}(u,v) = \int_\Omega \nabla u \cdot \nabla v - \frac{1}{2}\int_\Omega u\,v\,\mathrm{div}\,c + \frac{1}{2}\int_\Omega (c\cdot\nabla u)\,v - \frac{1}{2}\int_\Omega (c\cdot\nabla v)\,u,$$

which has the advantage that the first two terms are symmetric, while the last two terms are skew-symmetric (establishing the coercivity of $\overline{a}$ may thus be easier than for a, since the last two terms vanish in $\overline{a}(u,u)$). Whenever working in the $H^1_0(\Omega)$ space, solving $a(u,v) = b(v)$ for any v or solving $\overline{a}(u,v) = b(v)$ for any v is equivalent. However, for discretization purposes, it is sometimes useful to introduce finite-dimensional spaces which are not *subset of $H^1_0(\Omega)$. Working on the basis of a or of $\overline{a}$ may then leads to a different discretization.*

3.4.2 Checking the assumptions of the Lax-Milgram theorem

We now check that Problem (3.9) falls within the assumptions of the Lax-Milgram theorem:

- $H = H^1_0(\Omega)$ is a Hilbert space;
- b is linear and continuous on H;
- a is bilinear and continuous.

We are left with showing that a is coercive. We note that, for any $\phi \in \mathcal{D}(\Omega)$,

$$\int_\Omega (c\cdot\nabla\phi)\,\phi = \int_\Omega c\cdot\nabla\left(\frac{\phi^2}{2}\right) = \int_{\partial\Omega} (c\cdot n)\left(\frac{\phi^2}{2}\right) - \int_\Omega \left(\frac{\phi^2}{2}\right)(\mathrm{div}\ c)$$

and the first term in the right-hand side vanishes as ϕ vanishes in the neighborhood of $\partial\Omega$. We thus have, using that c is divergence-free and the Poincaré inequality,

$$\begin{aligned} a(\phi,\phi) &= \int_\Omega |\nabla\phi|^2 + \int_\Omega (c\cdot\nabla\phi)\phi \\ &= \int_\Omega |\nabla\phi|^2 - \int_\Omega \left(\frac{\phi^2}{2}\right)(\mathrm{div}\ c) \\ &= \int_\Omega |\nabla\phi|^2 \\ &\geq \frac{1}{1+C_\Omega^2}\|\phi\|_{H^1}. \end{aligned}$$

By continuity of a and density of $\mathcal{D}(\Omega)$ in $H = H^1_0(\Omega)$, we deduce that a is coercive on H.

In view of Lax-Milgram theorem 3.2, we deduce that Problem (3.9) has a unique solution. The problems (3.8) and (3.9) are equivalent. The problem (3.8) thus has a unique solution.

3.5 Exercises

Exercise 3.10. *Let Ω be a bounded open subset of $\mathbb{R}^d$ and let $f \in L^2(\Omega)$. We consider a matrix field $A : \Omega \longrightarrow \mathbb{R}^{d\times d}$ that belongs to $(L^\infty(\Omega))^{d\times d}$ and such that there exists $\alpha > 0$ such that*

$$\forall \xi \in \mathbb{R}^d, \qquad \xi^T A(x)\xi \geq \alpha \xi^T \xi \quad \text{almost everywhere on } \Omega.$$

We consider the following boundary value problem: Find $u \in H^1_0(\Omega)$ such that

$$-\text{div } (A\nabla u) = f \qquad \text{in } \mathcal{D}'(\Omega),$$

where we recall that, for any vector-valued function σ, we have $\text{div } \sigma = \displaystyle\sum_{i=1}^d \frac{\partial \sigma_i}{\partial x_i}$.

1. *Show that the above boundary value problem is equivalent to the following variational formulation: Find $u \in H^1_0(\Omega)$ such that*

$$\forall v \in H^1_0(\Omega), \qquad a(u,v) = b(v)$$

with

$$a(u,v) = \int_\Omega (\nabla v)^T A \nabla u = \int_\Omega (\nabla v)\cdot(A\nabla u), \qquad b(v) = \int_\Omega f\, v.$$

2. *Show that the bilinear form a is coercive on $H^1_0(\Omega)$.*

3. *Show that the above variational problem is well-posed.*

Exercise 3.11. *Let Ω be a bounded open subset of $\mathbb{R}^d$ and let $f \in L^2(\Omega)$. We consider*

- *a matrix field $A : \Omega \longrightarrow \mathbb{R}^{d\times d}$ that belongs to $(L^\infty(\Omega))^{d\times d}$,*
- *two vector fields b and c in $(L^\infty(\Omega))^d$,*
- *a scalar-valued field $d \in L^\infty(\Omega)$.*

Write the variational formulation of the following boundary value problem: Find $u \in H^1_0(\Omega)$ such that

$$-\text{div } (A\nabla u) + \text{div } (bu) + c\cdot\nabla u + du = f \qquad \text{in } \mathcal{D}'(\Omega).$$

We now assume that the matrix field is uniformly coercive on Ω, that is that there exists $\alpha > 0$ such that

$$\forall \xi \in \mathbb{R}^d, \qquad \xi^T A(x)\xi \geq \alpha \xi^T \xi \quad \text{almost everywhere on } \Omega.$$

Identify conditions on A, b, c and d under which the above boundary value problem has a unique solution in $H^1_0(\Omega)$.

Exercise 3.12 (Advection-diffusion problem)**.** *Let Ω be a open, bounded subset of $\mathbb{R}^d$ and $f \in L^2(\Omega)$. Let $c : \Omega \to \mathbb{R}^d$ be a vector field of class $C^1(\overline{\Omega})$. We look for $u \in H^1(\Omega)$ solution to* (3.8)*, namely*

$$\begin{cases} -\Delta u + c\cdot\nabla u = f & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We have seen in Section 3.4 that, if $\text{div } c = 0$ *on Ω, then the above problem is well-posed. We now consider several alternative assumptions on the advection field c.*

1. *Check that, if* $\text{div } c \leq 0$ *on Ω, then the above problem is again well-posed.*

2. *Check that, if* $|\text{div } c| \leq m$ *on* Ω*, then the bilinear form* a *defined in Section 3.4 satisfies*

$$\forall \phi \in \mathcal{D}(\Omega), \qquad a(\phi,\phi) \geq \int_\Omega |\nabla \phi|^2 - \frac{m}{2}\int_\Omega \phi^2.$$

Using the Poincaré inequality, show that, if m *is sufficiently small, then* a *is coercive on* $H^1_0(\Omega)$*.*

3. *Check that, if* $|c| \leq m$ *on* Ω*, then the bilinear form* a *defined in Section 3.4 satisfies*

$$\forall u \in H^1_0(\Omega), \qquad a(u,u) \geq \|\nabla u\|^2_{L^2(\Omega)} - m\,\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}$$

Using again the Poincaré inequality, show that, if m *is sufficiently small, then* a *is coercive on* $H^1_0(\Omega)$*.*

Exercise 3.13 (Regularization)**.** *Let* Ω *be an open subset of* $\mathbb{R}^d$ *and* $f \in L^2(\Omega)$*.*

1. *Let* $\varepsilon > 0$*. We look for* $u_\varepsilon \in H^1(\Omega)$ *solution to*

$$\begin{cases} -\varepsilon\,\Delta u_\varepsilon + u_\varepsilon = f & \text{in } \mathcal{D}'(\Omega), \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Show that this problem is well-posed and that

$$\forall \varepsilon, \quad \|u_\varepsilon\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$

2. *We denote* u_n *the solution in* $H^1(\Omega)$ *to*

$$\begin{cases} -\dfrac{1}{n}\,\Delta u_n + u_n = f & \text{in } \mathcal{D}'(\Omega), \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

(a) *Let* $\phi \in \mathcal{D}(\Omega)$*,* $q \in \mathbb{N}^\star$ *and* $b_n = \langle\, u_n, \phi\,\rangle$*. Show that*

$$|b_{n+q} - b_n| \;\leq\; \frac{1}{n+q}\,\|u_{n+q}\|_{L^2(\Omega)}\|\Delta\phi\|_{L^2(\Omega)} + \frac{1}{n}\,\|u_n\|_{L^2(\Omega)}\|\Delta\phi\|_{L^2(\Omega)}$$

and that

$$|b_{n+q} - b_n| \;\leq\; \frac{2}{n}\,\|f\|_{L^2(\Omega)}\|\Delta\phi\|_{L^2(\Omega)}.$$

(b) *Deduce that* u_n *converges in* $\mathcal{D}'(\Omega)$ *to some distribution that we denote* T*.*

(c) *Using* (3.10)*, show that* $T = f$*.*

3. *We now assume that* $f \in \mathcal{D}(\Omega)$*. Let* $v_n = u_n - f$*.*

(a) *Write the problem that* v_n *satisfies.*

(b) *Using its variational formulation, show that*

$$\|\nabla v_n\|_{L^2(\Omega)} \;\leq\; \|\nabla f\|_{L^2(\Omega)}.$$

(c) *Deduce that*

$$\lim_{n\to\infty} \|v_n\|_{L^2(\Omega)} = 0$$

(d) *What does this imply on the convergence of* u_n*?*

Exercise 3.14. *Let $u_0 \in L^2(\mathbb{R}^d)$ and $\tau > 0$. Consider the sequence $u_n \in H^1(\mathbb{R}^d)$ recursively defined by: for all $n \geq 0$,*

$$\frac{u_{n+1} - u_n}{\tau} = \Delta u_{n+1}. \tag{3.11}$$

1. *Show that the sequence $\{u_n\}_{n\geq 1}$ is well-defined.*
2. *In the case $d = 1$, consider u defined by*

$$u(x) = \begin{cases} \cos(\omega x) & \text{if } x \leq -2\pi/\omega, \\ 1 & \text{if } -2\pi/\omega \leq x \leq 2\pi/\omega, \\ \cos(\omega x) & \text{if } x \geq 2\pi/\omega, \end{cases}$$

   where $\omega = 1/\sqrt{\tau}$. Compute the second derivative (in the sense of distributions) of u.
3. *Using the previous question, build a function $u_0 \in L^2(\mathbb{R})$ such that the problem: Find $u_1 \in H^1(\mathbb{R})$ such that*

$$\frac{u_1 - u_0}{\tau} = u_0''$$

   does not have any solution.
4. *In which context does (3.11) appear? What conclusion can you draw from the above question?*

Exercise 3.15. *Let $f \in L^2(\mathbb{R}^2)$ and $\lambda \in \mathbb{R}$. Consider the problem:*

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ \forall v \in V, \qquad a_\lambda(u, v) = b(v) \end{cases} \tag{3.12}$$

where $V = H^2(\mathbb{R}^2)$, $b(v) = \int_{\mathbb{R}^2} f\, v$ and

$$a_\lambda(u, v) = \int_{\mathbb{R}^2} \Delta u\, \Delta v + \lambda \int_{\mathbb{R}^2} \nabla u \cdot \nabla v + \int_{\mathbb{R}^2} u\, v.$$

1. *Show that, for any u and v in $H^2(\mathbb{R}^2)$, we have*

$$\int_{\mathbb{R}^2} \Delta u\, \Delta v = \sum_{i,j=1}^{2} \int_{\mathbb{R}^2} \frac{\partial^2 u}{\partial x_i \partial x_j}\, \frac{\partial^2 v}{\partial x_i \partial x_j}.$$

2. *Show that, if $\lambda > 0$, then Problem (3.12) is well-posed.*
3. *Show that, for any $\phi \in \mathcal{D}(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} |\nabla \phi|^2 = -\int_{\mathbb{R}^2} \phi\, \Delta \phi,$$

   and deduce from this that, for any u in $H^2(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} |\nabla u|^2 \leq \frac{1}{2} \left(\int_{\mathbb{R}^2} u^2 + \int_{\mathbb{R}^2} |\Delta u|^2 \right).$$

   Deduce that, if $\lambda > -2$, then Problem (3.12) is well-posed.

Exercise 3.16 (On the requirement of coercivity)**.** *We consider the finite dimensional Hilbert space* $H = \mathbb{R}^n$*, endowed with the Euclidean scalar product* $\langle u, v \rangle = \sum_{i=1}^{n} u_i\, v_i$*. Let* $A \in \mathbb{R}^{n \times n}$ *be a symmetric matrix, and let* $B \in \mathbb{R}^n$*. We consider the problem: Find* $u \in \mathbb{R}^n$ *such that* $Au = B$*.*

1. *Using standard arguments, show that the above problem is well-posed if and only if all the eigenvalues of* A *do not vanish.*
2. *We now follow a variational approach.*
   (a) *Show that the above problem is equivalent to:*
   $$\text{Find } u \in \mathbb{R}^n \text{ such that } a(u,v) = b(v) \text{ for any } v \in \mathbb{R}^n, \tag{3.13}$$
   with $a(u, v) = v^T A u$ *and* $b(v) = v^T B$*.*
   (b) *Show that, if all the eigenvalues of* A *are positive, then* a *is coercive. Provide an expression for the coercivity constant of* a *in terms of the eigenvalues of* A*.*
   (c) *Conversely, show that, if* a *is coercive, then all the eigenvalues of* A *are positive, and provide a lower bound on these eigenvalues in terms of the coercivity constant of the bilinear form* a*.*
   (d) *Conclude on whether or not the bilinear form* a *has to be coercive for the variational problem* (3.13) *to be well-posed.*

This exercise shows that, for general variational problems of the form (3.2)*, the coercivity of the bilinear form is a sufficient, but not necessary, condition for* (3.2) *to be well-posed.*

Exercise 3.17 (Non homogeneous Dirichlet boundary conditions)**.** *Let* Ω *be a bounded open subset of* $\mathbb{R}^d$*,* $f \in L^2(\Omega)$*,* $u_0 \in H^1(\Omega)$ *(this assumption will be made stronger below). We consider the following problem:*
$$\left\{ \begin{array}{l} \text{Find } u \in H^1(\Omega) \text{ such that} \\ -\Delta u = f \text{ in } \mathcal{D}'(\Omega), \\ u = u_0 \text{ on } \partial\Omega. \end{array} \right. \tag{3.14}$$
The functions u *and* u_0 *are in* $H^1(\Omega)$*, thus they both have a trace on* $\partial\Omega$*. The last equation above simply means that the two traces are equal.*

Note also that the function u_0 *is defined on purpose on* Ω*, although only its trace on the boundary* $\partial\Omega$ *appears in* (3.14)*.*

1. *Introduce* $w = u - u_0$*, and show that* u *is a solution to* (3.14) *if and only if* w *is a solution to*
   $$\left\{ \begin{array}{l} \text{Find } w \in H^1(\Omega) \text{ such that} \\ -\Delta w = g \text{ in } \mathcal{D}'(\Omega), \\ w = 0 \text{ in } \partial\Omega, \end{array} \right. \tag{3.15}$$
   for a distribution g *that will be made precise. The interest of* (3.15) *is that it is a problem with* homogeneous *boundary conditions.*
2. *We assume for now that* $u_0 \in H^2(\Omega)$*. Deduce that* $g \in L^2(\Omega)$*. Write a variational formulation (with unknown* w*) equivalent to* (3.15)*, and show the existence and uniqueness of a solution to* (3.14)*.*

3. *Using the Green formula, show that the solution u to (3.14) satisfies*

$$\begin{cases} u \in H^1(\Omega),\ u = u_0 \text{ on } \partial\Omega, \text{ and} \\ \forall \varphi \in H^1_0(\Omega), \quad \displaystyle\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi. \end{cases}$$

Note that the solution u and the test function φ do not belong to the same space.

The following exercise aims at proving the Theorem 3.18 below, which is a functional inequality that belongs to the same family as the Poincaré inequality of Theorem 2.8:

Theorem 3.18. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, open and connected subset of $\mathbb{R}^d$, and let β be a real number $\beta > 0$. There exists $C > 0$ such that*

$$\forall v \in H^1(\Omega), \quad \|\nabla v\|^2_{L^2(\Omega)} + \beta \|v\|^2_{L^2(\partial\Omega)} \geq C\|v\|^2_{H^1(\Omega)}. \tag{3.16}$$

Exercise 3.19. *Our aim is to prove Theorem 3.18. We proceed by contradiction and assume that there exists no constant C such that (3.16) holds.*

1. *Show that there exists a sequence $\{u_n\}_{n\in\mathbb{N}^\star}$ such that $\|u_n\|_{H^1(\Omega)} = 1$ and*

$$\|\nabla u_n\|^2_{L^2(\Omega)} + \beta\|u_n\|^2_{L^2(\partial\Omega)} \leq \frac{1}{n}. \tag{3.17}$$

   What can be said of the limits (when $n \to \infty$) of $\|\nabla u_n\|_{L^2(\Omega)}$ and of $\|u_n\|_{L^2(\partial\Omega)}$?

2. *We admit that the bound $\|u_n\|_{H^1(\Omega)} = 1$ implies that there exists $u_\star \in H^1(\Omega)$ and a subsequence $u_{n'}$ such that $u_{n'}$ converges to $u_\star$ in $L^2(\Omega)$.*

   Using (3.17), show that $\nabla u_\star = 0$. We then deduce from the fact that Ω is connected that $u_\star$ is a constant.

3. *Show that $u_{n'}$ converges to $u_\star$ in $H^1(\Omega)$, and next that the constant $u_\star$ is different from 0.*

4. *Using that the trace mapping is continuous from $H^1(\Omega)$ to $L^2(\partial\Omega)$, deduce that $u_{n'}$ converges to $u_\star$ in $L^2(\partial\Omega)$.*

5. *Reach a contradiction.*

Exercise 3.20 (Robin boundary conditions)**.** *Let Ω be a bounded open subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$, $g \in C^\infty(\partial\Omega)$ and a real number $\beta > 0$. We look for $u \in H^1(\Omega)$ solution to*

$$\begin{cases} -\Delta u = f & \text{in } \mathcal{D}'(\Omega) \\ \dfrac{\partial u}{\partial n} + \beta u = g & \text{on } \partial\Omega, \end{cases} \tag{3.18}$$

where $\dfrac{\partial u}{\partial n} = \nabla u \cdot n$ (n is the outward normal vector to $\partial\Omega$). The precise interpretation of the boundary condition will be done below.

1. *As a first step, we wish to write a variational formulation associated to (3.18). We thus assume that all functions are smooth, that all integrations by parts are allowed, . . .*

   Show that, if u is a sufficiently regular solution to (3.18), then, for any smooth function $v : \Omega \to \mathbb{R}$, we have

$$\int_\Omega \nabla u \cdot \nabla v + \beta \int_{\partial\Omega} u\, v = \int_\Omega f\, v + \int_{\partial\Omega} g\, v.$$

2. *We now establish and study a variational formulation of the problem. We recall that, for any function $u \in H^1(\Omega)$, the trace of u on the boundary $\partial\Omega$ is well-defined, that this is a function of class $L^2(\partial\Omega)$, and that there exists C (that only depends on Ω) such that*

$$\forall u \in H^1(\Omega), \quad \|u\|_{L^2(\partial\Omega)} \le C\|u\|_{H^1(\Omega)}.$$

The above question leads to consider the bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + \beta \int_{\partial\Omega} u\, v$$

and the linear form

$$b(v) = \int_\Omega f\, v + \int_{\partial\Omega} g\, v.$$

   (a) *Show that a and b are well-defined on $H^1(\Omega)$ and that they are continuous.*
   (b) *Show that a is coercive on $H^1(\Omega)$ (use Theorem 3.18). Check also that, if $\beta \le 0$, then a is not coercive on $H^1(\Omega)$.*
   (c) *Deduce from the above questions that the problem*

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ \forall v \in H^1(\Omega), \quad a(u,v) = b(v) \end{cases} \tag{3.19}$$

   is well-posed.

3. *We now go back to the boundary value problem, and establish in which sense the solution u to (3.19) satisfies the boundary value problem (3.18).*
   (a) *Show that the solution u to (3.19) satisfies*

$$-\Delta u = f \text{ in } \mathcal{D}'(\Omega). \tag{3.20}$$

   (b) *We admit that, if g is sufficiently smooth, then the solution u to (3.19) is in $H^2(\Omega)$. We keep this assumption for the sequel of the exercise. Show that $\dfrac{\partial u}{\partial n} \in L^2(\partial\Omega)$.*
   (c) *Multiplying (3.20) by $v \in H^1(\Omega)$ and using a Green formula, show that u is such that, for any $v \in H^1(\Omega)$, we have*

$$\int_{\partial\Omega} v \left(\frac{\partial u}{\partial n} + \beta u \right) = \int_{\partial\Omega} g\, v.$$

   Deduce from this equality that u satisfies the boundary condition $\dfrac{\partial u}{\partial n} + \beta u = g$ (to that aim, admit that the image of the trace mapping $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ is dense in $L^2(\partial\Omega)$).

Exercise 3.21 (Poincaré-Wirtinger inequality). *We show here a functional inequality that belongs to the same family as the Poincaré inequality of Theorem 2.8.*

1. *Show that there exists $C > 0$ such that, for any $\varphi \in C^\infty([0,1])$, we have*

$$\|\varphi - \langle\varphi\rangle\|_{L^2(0,1)} \le C\|\varphi'\|_{L^2(0,1)},$$

   where $\langle\varphi\rangle = \displaystyle\int_0^1 \varphi$ is the mean of φ. Hint: start by showing that there exists $x_0 \in [0,1]$ such that $\langle\varphi\rangle = \varphi(x_0)$.

2. *We admit that $C^\infty([0,1])$ is dense in $H^1(0,1)$. Show that there exists $C>0$ such that*

$$\forall v \in H^1(0,1), \quad \|v - \langle v \rangle\|_{L^2(0,1)} \leq C \|v'\|_{L^2(0,1)}. \tag{3.21}$$

The result of Exercise 3.21, shown in dimension $d=1$, can actually be generalized to the following statement.

Theorem 3.22. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, open and connected subset of $\mathbb{R}^d$. There exists $C>0$ such that*

$$\forall v \in H^1(\Omega), \quad \|v - \langle v \rangle\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}, \tag{3.22}$$

with $\langle v \rangle = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega v.$

Exercise 3.23 (Neumann boundary conditions)**.** *Let Ω be a regular, bounded, open and connected subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$ and $g \in C^\infty(\partial\Omega)$. We consider a matrix field $A : \Omega \longrightarrow \mathbb{R}^{d\times d}$ that belongs to $(L^\infty(\Omega))^{d\times d}$ and such that there exists $\alpha > 0$ such that*

$$\forall \xi \in \mathbb{R}^d, \qquad \xi^T A(x) \xi \geq \alpha \xi^T \xi \quad \text{almost everywhere on } \Omega.$$

We look for $u \in H^1(\Omega)$ solution to

$$\begin{cases} -\text{div } (A\nabla u) = f & \text{in } \mathcal{D}'(\Omega) \\ (A\nabla u)\cdot n = g & \text{on } \partial\Omega, \end{cases} \tag{3.23}$$

where n is the outward normal vector to $\partial\Omega$. The precise interpretation of the boundary condition will be done below.

Verify that Problem (3.23) is ill-posed, in the sense that, if u is solution to (3.23), then $u+c$ is also solution, for any constant c.

We thus introduce the set

$$H^1_m = \left\{ u \in H^1(\Omega), \quad \frac{1}{|\Omega|}\int_\Omega u = 0 \right\}$$

of the H^1 functions with vanishing mean, and look for $u \in H^1_m(\Omega)$ that satisfies (3.23).

1. *As a first step, we wish to write a variational formulation associated to (3.23). We thus assume that all functions are smooth, that all integrations by parts are allowed, ...*

   Show that, if u is a sufficiently regular solution to (3.23), then, for any smooth function $v : \Omega \to \mathbb{R}$, we have

   $$\int_\Omega \nabla v \cdot (A\nabla u) = \int_\Omega fv + \int_{\partial\Omega} gv.$$

   Deduce from this computation that, if (3.23) admits a smooth solution, then

   $$\int_\Omega f + \int_{\partial\Omega} g = 0. \tag{3.24}$$

   We will keep this assumption for the sequel of the exercise.

2. *We now establish and study a variational formulation of the problem. We recall that, for any function* $u \in H^1(\Omega)$*, the trace of* u *on the boundary* $\partial\Omega$ *is well-defined, that this is a function of class* $L^2(\partial\Omega)$*, and that there exists* C *(that only depends on* Ω*) such that*

$$\forall u \in H^1(\Omega), \quad \|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}.$$

The above question leads to consider the bilinear form

$$a(u,v) = \int_\Omega (\nabla v) \cdot (A\nabla u)$$

and the linear form

$$b(v) = \int_\Omega fv + \int_{\partial\Omega} gv.$$

   (a) *Show that* $H^1_m(\Omega)$ *is a Hilbert space for the* H^1 *scalar product.*
   (b) *Show that* a *and* b *are well-defined on* $H^1_m(\Omega)$ *and that they are continuous.*
   (c) *Using Theorem 3.22, show that* a *is coercive on* $H^1_m(\Omega)$*.*
   (d) *Deduce from the above questions that the problem*

$$\begin{cases} \text{Find } u \in H^1_m(\Omega) \text{ such that} \\ \forall v \in H^1_m(\Omega), \quad a(u,v) = b(v) \end{cases} \tag{3.25}$$

   is well-posed.

3. *We now go back to the boundary value problem, and establish in which sense the solution* u *to* (3.25) *satisfies the boundary value problem* (3.23)*. We recall that we assume that* f *and* g *satisfy* (3.24)*.*

   (a) *Show that the solution* u *to* (3.25) *satisfies*

$$\forall v \in H^1(\Omega), \quad a(u,v) = b(v),$$

   where the functions v *can now be chosen in* $H^1(\Omega)$ *and not only in* $H^1_m(\Omega)$*.*
   (b) *Show that the solution* u *to* (3.25) *satisfies*

$$-\text{div } (A\nabla u) = f \text{ in } \mathcal{D}'(\Omega). \tag{3.26}$$

   (c) *We admit that, if* g *is sufficiently smooth, then the solution* u *to* (3.25) *is in* $H^2(\Omega)$*. We also suppose that* A *is continuous on* $\overline{\Omega}$ *(and not only in* $L^\infty(\Omega)$*). We keep these assumptions for the sequel of the exercise. Show that* $n \cdot (A\nabla u) \in L^2(\partial\Omega)$*.*
   (d) *Multiplying* (3.26) *by* $v \in H^1(\Omega)$ *and using a Green formula, show that* u *is such that, for any* $v \in H^1(\Omega)$*, we have*

$$\int_{\partial\Omega} v\,[n \cdot (A\nabla u)] = \int_{\partial\Omega} g\,v.$$

   Deduce from this equality that u *satisfies the boundary condition* $n \cdot (A\nabla u) = g$ *(to that aim, admit that the image of the trace mapping* $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ *is dense in* $L^2(\partial\Omega)$*).*

This exercise shows that, if we want to prescribe a boundary condition on the gradient of u*, the correct way to write it is to impose an equality on the normal flux* $n \cdot (A\nabla u)$ *(i.e. on the normal derivative* $n \cdot \nabla u$ *in the case when* A *is the identity matrix). Imposing a boundary condition on the complete gradient* ∇u *would be too demanding, and leads to ill-posed problems.*

Exercise 3.24 (The Stokes equation). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $f \in (L^2(\Omega))^d$. We introduce the space*

$$L^2_0(\Omega) = \left\{ q \in L^2(\Omega), \quad \int_\Omega q = 0 \right\}$$

of square-integrable functions with vanishing mean and we consider the following problem:

$$\begin{cases} \text{Find } (u,p) \in (H^1_0(\Omega))^d \times L^2_0(\Omega) \text{ such that} \\ -\Delta u + \nabla p = f \quad \text{in } [\mathcal{D}'(\Omega)]^d, \\ \operatorname{div} u = 0 \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{3.27}$$

When $u \in (H^1_0(\Omega))^d$, we recall that Δu is a vector in $\mathbb{R}^d$, the i-th component of which is equal to Δu_i, where u_i is the i-th component of u. The first equation in (3.27) is thus a vectorial equation, and its i-th line reads $-\Delta u_i + \partial_i p = f_i$ in $\mathcal{D}'(\Omega)$.

The problem (3.27) is called the Stokes equation. It is of paramount importance in fluid mechanics and corresponds to the linear version of the Navier-Stokes equation for incompressible fluids.

To deal with the second line of (3.27) (the incompressibility constraint), a possibility is to include this equation in the space in which we work. We thus introduce the space

$$V = \left\{ u \in (H^1_0(\Omega))^d, \quad \operatorname{div} u = 0 \right\}$$

and the variational formulation:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ \forall w \in V, \qquad a(u,w) = g(w) \end{cases} \tag{3.28}$$

where

$$a(u,w) = \int_\Omega \nabla u \cdot \nabla w, \qquad g(w) = \int_\Omega f \cdot w.$$

1. *Explain why we consider the problem (3.27) in $(H^1_0(\Omega))^d \times L^2_0(\Omega)$ and not in $(H^1_0(\Omega))^d \times L^2(\Omega)$.*
2. *We first establish the well-posedness of (3.28).*
   (a) *Recall why the space V is a Hilbert space when endowed with the H^1 scalar product.*
   (b) *Recall why the bilinear form a is continuous on $V \times V$ and why the linear form g is continuous on V.*
   (c) *Show that (3.28) is well-posed.*
3. *Show that, if (u,p) is a solution to (3.27), then u is a solution to (3.28). Hint: consider the scalar product of the first line of (3.27) against some $w \in V$.*

We remark that the formulation (3.28) directly provides the velocity u, but that the pressure p has disappeared from this formulation. It should actually be computed in a second stage, in a non-trivial manner which goes outside the scope of this exercise (this would lead to show that, after having found the velocity u solution to (3.28), it is possible to find a pressure p such that the so-obtained (u,p) is a solution to (3.27)). In practice, it is often the case that the quantity of interest in the Stokes problem is the pressure. This is why alternatives to (3.28) have been considered (the mathematical analysis of these alternatives needs more advanced tools than the Lax-Milgram lemma and is left outside of these lecture notes).

Chapter 4

The energetic viewpoint

4.1 Differential of a function

Let E and F be two real vector spaces, endowed with the norm $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively. We emphasize that E and F may be of infinite dimension. We recall that $\mathcal{L}(E,F)$ is the set of linear and *continuous* maps from E to F, and that we have the following result.

Theorem 4.1. *Let A be a linear map from E to F. The following three assertions are equivalent:*

- *A is continuous on E;*
- *A is continuous at 0*
- *there exists some constant $c \geq 0$ such that*

$$\forall u \in E, \quad \|Au\|_F \leq c\|u\|_E.$$

We consider a function $f : E \to F$ and we wish to define its differential.

Definition 4.2. *We say that f is (Fréchet-)differentiable at $x \in E$ if and only if there exists a linear and continuous map from E to F, which is in general denoted $d_x f$ (or $df(x)$ or $D_x f$ or $Df(x)$ or $f'(x)$, depending on the authors) such that, in a neighborhood of x, we can write*

$$f(x+h) = f(x) + d_x f(h) + o(h). \tag{4.1}$$

In this case, the linear and continuous map $d_x f \in \mathcal{L}(E,F)$ is unique, and it is called the differential of f at $x \in E$.

We recall that the notation $f(x+h) = f(x) + d_x f(h) + o(h)$ means that

$$\forall \varepsilon > 0, \exists \eta > 0 \text{ s.t., for any } h \in E \text{ so that } \|h\|_E \leq \eta, \quad \|f(x+h) - f(x) - d_x f(h)\|_F \leq \varepsilon \, \|h\|_E.$$

Proof of the uniqueness of $d_x f$. Consider two linear and continuous maps L_1 and L_2 such that $f(x+h) = f(x) + L_1(h) + o(h)$ and $f(x+h) = f(x) + L_2(h) + o(h)$, in the sense recalled above. This means that, for any $\varepsilon > 0$, there exists some η such that, for any $h \in E$ so that $\|h\|_E \leq \eta$ and for any $j = 1, 2$, we have

$$\|f(x+h) - f(x) - L_j(h)\|_F \leq \varepsilon \, \|h\|_E.$$

Let $h \in E$ and $0 \leq t \leq \eta/\|h\|_E$. Since the norm of th is smaller than η, we can write

$$\begin{aligned} t\,\|(L_1 - L_2)(h)\|_F &= \|(L_1 - L_2)(th)\|_F \\ &= \Big\|\Big(f(x+th) - f(x) - L_2(th)\Big) - \Big(f(x+th) - f(x) - L_1(th)\Big)\Big\|_F \\ &\leq \|f(x+th) - f(x) - L_2(th)\|_F + \|f(x+th) - f(x) - L_1(th)\|_F \\ &\leq 2\,\varepsilon\,\|th\|_E = 2\,\varepsilon\,t\,\|h\|_E. \end{aligned}$$

We thus deduce that, for any $h \in E$, we have $\|(L_1 - L_2)(h)\|_F \leq 2\,\varepsilon\,\|h\|_E$. Since ε can be arbitrary small, this implies that $\|(L_1 - L_2)(h)\|_F = 0$. Since this is valid for any $h \in E$, this implies that $L_1 = L_2$, thus the uniqueness of the differential. □

Remark 4.3. *The definition of the differential implies the following facts:*

- *a function f which is differentiable in x is continuous at x; indeed, since $d_x f$ is continuous on E and in view of the definition of the notation $o(h)$, we have that $\lim_{h\to 0} f(x+h) = f(x)$;*
- *the differential operation is a linear operation: if f and g are two functions from E to F which are differentiable at $x \in E$, then any linear combination of the form $\lambda\, f + \mu\, g$ (for any real numbers λ and μ) is differentiable at x and $d_x(\lambda\, f + \mu\, g) = \lambda\, d_x f + \mu\, d_x g$;*
- *in the particular case when E is $\mathbb{R}$, f is differentiable in the above sense if and only if f admits a derivative in the usual sense. We then have*

$$\begin{aligned} d_x f : \mathbb{R} &\to F \\ h &\mapsto h\, f'(x) \end{aligned}$$

where $f'(x) = \lim_{t\to 0} \dfrac{f(x+t) - f(x)}{t}$ is the standard, usual derivative of f at x.

Definition 4.4. *Consider some function $f : E \to F$. We say that f is differentiable on E if f is differentiable at any $x \in E$.*

We say that f is of class C^1 on E (or continuously differentiable on E) if the map

$$\begin{aligned} df : E &\to \mathcal{L}(E, F) \\ x &\mapsto d_x f \end{aligned}$$

is continuous on E.

The chain rule again holds:

Theorem 4.5. *Let E, F and G be three normed vector spaces. Let $f : E \to F$ be a function differentiable in x and let $g : F \to G$ be a function differentiable in $y = f(x)$. Then the function $h : E \to G$ defined by $h = g \circ f$ is differentiable in x and $d_x h = d_y g \circ d_x f$.*

Exercise 4.6. *Let Ω be open subset of $\mathbb{R}^d$, $E = H^1(\Omega)$ and $F = \mathbb{R}$. Consider some $f \in L^2(\Omega)$ and some $g \in L^2(\partial\Omega)$. We consider the map J defined on E by $J(u) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 - \int_\Omega f\, u - \int_{\partial\Omega} g\, u$. Compute the differential of J at any $u \in E$.*

Exercise 4.7. *Let H be a Hilbert space, a be a symmetric bilinear continuous form on H and b be a linear continuous form on H. We consider the map J defined on H by $J(u) = \dfrac{1}{2}a(u,u) - b(u)$. Compute the differential of J at any $u \in H$.*

Exercise 4.8. *Let H be a Hilbert space, a be a bilinear continuous form on H and b be a linear continuous form on H. In contrast to Exercise 4.7, we now* do not *assume a to be symmetric. We consider the map J defined on H by $J(u) = \frac{1}{2}a(u,u) - b(u)$. Compute the differential of J at any $u \in H$.*

Exercise 4.9. *Let Ω be open subset of $\mathbb{R}^d$ and consider some real number $p \geq 1$. Let $F = \mathbb{R}$ and let $E = L^p(\Omega)$. Consider some f in $L^q(\Omega)$ with $1/q + 1/p = 1$ and consider the map J defined on E by $J(u) = \int_\Omega f\,u$. Show that J is well-defined and compute the differential of J at any $u \in E$. Hint: use the Hölder inequality.*

4.2 More on the Lax-Milgram theorem in the symmetric case

Let H be a Hilbert space, a be a bilinear form on H, continuous and coercive, and let b be a continuous linear form on H. We have seen that the problem

$$\left\{ \begin{array}{l} \text{Find } u \in H \text{ such that} \\ \forall v \in H, \qquad a(u,v) = b(v) \end{array} \right. \tag{4.2}$$

has a unique solution (see Theorem 3.2).

Under the additional assumption that a is symmetric, that is

$$\forall u, v \in H, \qquad a(u,v) = a(v,u),$$

we are going to give an additional characterization of the solution to (4.2).

Theorem 4.10 (Lax-Milgram theorem, the energetic viewpoint)**.** *Let H be a Hilbert space, a be a bilinear form on H, continuous and coercive. Let b be a continuous linear form on H. Assume furthermore that a is symmetric. Then the unique solution to* (4.2) *is also the unique solution to the minimization problem*

$$\left\{ \begin{array}{l} \text{Find } u \in H \text{ such that} \\ J(u) = \inf\limits_{v \in H} J(v), \end{array} \right. \tag{4.3}$$

where the functional $J(v)$ (the so-called energy functional) is defined by

$$J(v) = \frac{1}{2}a(v,v) - b(v).$$

Proof. Consider the solution u to (4.2) and let $v \in H$. We set $h = v - u$ and compute, using the symmetry of a, that

$$\begin{aligned} J(v) &= J(u+h) \\ &= \frac{1}{2}a(u+h, u+h) - b(u+h) \\ &= J(u) + a(u,h) - b(h) + \frac{1}{2}a(h,h) \qquad \text{[here, we have used the symmetry of } a\text{]} \\ &= J(u) + \frac{1}{2}a(h,h) \qquad [u \text{ is sol. to } (4.2)] \\ &\geq J(u) \qquad [a \text{ is coercive}]. \end{aligned}$$

Hence u is a solution to (4.3).

Conversely, consider a solution u to (4.3). Then, for any $v \in H$ and any $\lambda \in \mathbb{R}$, we have

$$J(u) \leq J(u + \lambda v) = J(u) + \lambda \big(a(u,v) - b(v)\big) + \frac{1}{2}\lambda^2 \, a(v,v).$$

Hence, for any $\lambda \in \mathbb{R}$,

$$\lambda\big(a(u,v) - b(v)\big) + \frac{1}{2}\lambda^2 a(v,v) \geq 0. \tag{4.4}$$

Considering (4.4) for any $\lambda > 0$ and dividing by $\lambda > 0$, we get

$$\big(a(u,v) - b(v)\big) + \frac{1}{2}\lambda \, a(v,v) \geq 0.$$

Letting $\lambda > 0$ go to 0, we deduce that

$$\big(a(u,v) - b(v)\big) \geq 0. \tag{4.5}$$

Considering now (4.4) for $\lambda < 0$ and dividing by λ, we get (note the change of sign in the inequality!)

$$\big(a(u,v) - b(v)\big) + \frac{1}{2}\lambda \, a(v,v) \leq 0.$$

Letting $\lambda < 0$ go to 0, we now deduce that

$$\big(a(u,v) - b(v)\big) \leq 0. \tag{4.6}$$

Collecting (4.5) and (4.6), we deduce that $a(u,v) = b(v)$, an equality that holds for any $v \in H$. Hence u is a solution to (4.2). □

Heuristically, minimizing J (namely, solving (4.3)) amounts to looking for some $u \in H$ such that the derivative of J at u vanishes, that is looking for some $u \in H$ such that $d_u J = 0$, that is looking for some $u \in H$ such that $d_u J(v) = 0$ for any $v \in H$. We have seen in Exercise 4.7 that the differential of J is $d_u J(v) = a(u,v) - b(v)$. The conclusion of this formal argument is that minimizing J amounts to looking for some $u \in H$ such that $a(u,v) - b(v)$ for any $v \in H$, and thus solving (4.2). We recover by a formal argument the statement of Theorem 4.10.

Exercise 4.11. *Let H be a Hilbert space, a be a bilinear form on H, continuous and coercive. Let b be a continuous linear form on H. We do not assume a to be symmetric. We consider the minimization problem*

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ J(u) = \inf_{v \in H} J(v), \end{cases} \tag{4.7}$$

where the functional $J(v)$ is defined by

$$J(v) = \frac{1}{2}a(v,v) - b(v).$$

Write the variational problem which is equivalent to (4.7).

Appendix A

Basic facts on the Lebesgue integral and L^p spaces

We briefly collect here some basic facts concerning the Lebesgue integral and L^p spaces. For more details, we refer to the first year ENPC course *Analyse et Calcul Scientifique*.

In what follows, Ω is an open subset (which may or may not be bounded) of $\mathbb{R}^d$ (with $d \geq 1$) and p is a real number, $p \geq 1$.

We recall that two functions f and g defined on Ω are said to be equal almost everywhere on Ω if the measure of the set $\{x \in \Omega, \ f(x) \neq g(x)\}$ vanishes.

A.1 The space $L^1(\Omega)$

A.1.1 Definition

We first recall that the space $\mathcal{L}^1(\Omega)$ is defined by

$$\mathcal{L}^1(\Omega) = \left\{ f : \Omega \to \overline{\mathbb{R}}, \ f \text{ is measurable}, \ \int_\Omega |f(x)| \, dx < +\infty \right\}.$$

The vector space $L^1(\Omega)$ is obtained as the quotient of the space $\mathcal{L}^1(\Omega)$ with respect to the equivalence relation "$f \sim g$ if $f = g$ almost everywhere". When endowed with the norm

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)| \, dx,$$

the vector space $L^1(\Omega)$ is a Banach space.

A.1.2 Dominated convergence theorem

The following result is of paramount importance.

Theorem A.1. *Let f_n be a sequence of integrable functions in Ω (meaning $f_n \in L^1(\Omega)$) and let g be a non-negative function which is integrable on Ω. We assume that*

- *for almost every $x \in \Omega$, the sequence $(f_n(x))_n$ converges to some $f(x)$ when $n \to \infty$;*
- *for any $n \in \mathbb{N}$, and for almost every $x \in \Omega$, we have $|f_n(x)| \leq g(x)$.*

Then the function f is integrable on Ω and $\lim_{n\to\infty} \int_\Omega |f(x) - f_n(x)|\, dx = 0$, *which implies that*

$$\lim_{n\to\infty} \int_\Omega f_n(x)\, dx = \int_\Omega f(x)\, dx.$$

We admit the following result:

Theorem A.2. *Consider a sequence of functions $f_n \in L^1(\Omega)$ that converge to some f in $L^1(\Omega)$ when $n \to \infty$. Then there exists a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ that converges to f almost everywhere when $n \to \infty$.*

A.2 The space $L^2(\Omega)$

Definition A.3. *The space $\mathcal{L}^2(\Omega)$ is defined by*

$$\mathcal{L}^2(\Omega) = \left\{ f : \Omega \to \overline{\mathbb{R}},\ f \text{ is measurable, } \int_\Omega |f(x)|^2\, dx < +\infty \right\}.$$

The vector space $L^2(\Omega)$ is obtained as the quotient of the space $\mathcal{L}^2(\Omega)$ with respect to the equivalence relation "$f \sim g$ if $f = g$ almost everywhere".

Theorem A.4. *When endowed with the scalar product*

$$(f, g)_{L^2} = \int_\Omega f(x)\, g(x)\, dx,$$

the vector space $L^2(\Omega)$ is a Hilbert space.

A.3 The spaces $L^p(\Omega)$ and $L^p_{\text{loc}}(\Omega)$

Let p be a (finite) real number, $p \geq 1$.

Definition A.5. *The space $\mathcal{L}^p(\Omega)$ is defined as*

$$\mathcal{L}^p(\Omega) = \left\{ f : \Omega \to \overline{\mathbb{R}},\ f \text{ is measurable, } \int_\Omega |f(x)|^p\, dx < +\infty \right\}.$$

The vector space $L^p(\Omega)$ is obtained as the quotient of the space $\mathcal{L}^p(\Omega)$ with respect to the equivalence relation "$f \sim g$ if $f = g$ almost everywhere".

Theorem A.6. *When endowed with the norm*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p\, dx \right)^{1/p},$$

the vector space $L^p(\Omega)$ is a Banach space.

Theorem A.7 (Hölder inequality)**.** *Let $p \in \mathbb{R}$ with $1 < p < +\infty$, and let $q \in \mathbb{R}$ with $1/p + 1/q = 1$ (note that $1 < q < +\infty$). Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $f\, g \in L^1(\Omega)$ and we have the following Hölder inequality:*

$$\|f\, g\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)}\ \|g\|_{L^q(\Omega)}.$$

Lemma A.8. *Assume that the open set Ω is bounded. Let p and q two real numbers with $q > p \geq 1$. Then $L^q(\Omega) \subset L^p(\Omega)$.*

Definition A.9. *We say that f is locally integrable on Ω if, for any compact set K contained in Ω, the function f is integrable on K. We denote*

$$L^1_{\text{loc}}(\Omega) = \big\{f;\ f \in L^1(K) \text{ for any compact subset } K \subset \Omega\big\}$$

the vector space of locally integrable functions. Likewise,

$$L^p_{\text{loc}}(\Omega) = \{f;\ f \in L^p(K) \text{ for any compact subset } K \subset \Omega\}\,.$$

Lemma A.10. *We have that $L^p(\Omega) \subset L^p_{\text{loc}}(\Omega)$. For any real numbers p and q such that $q > p \geq 1$, we have $L^q_{\text{loc}}(\Omega) \subset L^p_{\text{loc}}(\Omega)$.*

Definition A.11. *Let f_n be a sequence of functions in $L^p_{\text{loc}}(\Omega)$ and $f \in L^p_{\text{loc}}(\Omega)$. We say that f_n converges to f in $L^p_{\text{loc}}(\Omega)$ when $n \to \infty$ if, for any compact subset $K \subset \Omega$, the sequence $f_n|_K$ converges to $f|_K$ in $L^p(K)$ when $n \to \infty$.*

A.4 The space $L^\infty(\Omega)$

Definition A.12. *A function f is said to be essentially bounded on Ω if there exists a non-negative real number M such that*

$$\mu\Big(\{x \in \Omega;\ |f(x)| \geq M\}\Big) = 0,$$

where μ is the Lebesgue measure. Hence, except on a null set (that is, a set the measure of which vanishes), we have $|f(x)| < M$.

The set of functions that are essentially bounded on Ω is denoted $\mathcal{L}^\infty(\Omega)$.

Definition A.13. *The vector space $L^\infty(\Omega)$ is obtained as the quotient of the space $\mathcal{L}^\infty(\Omega)$ with respect to the equivalence relation "$f \sim g$ if $f = g$ almost everywhere".*

Theorem A.14. *When endowed with the norm*

$$\|f\|_{L^\infty(\Omega)} = \inf\Big\{M \geq 0;\ \mu\Big(\{x \in \Omega;\ |f(x)| \geq M\}\Big) = 0\Big\},$$

the vector space $L^\infty(\Omega)$ is a Banach space.

Lemma A.15. *If f is essentially bounded on Ω, then $|f(x)| \leq \|f\|_{L^\infty(\Omega)}$ almost everywhere on Ω.*

If in addition f is continuous on Ω, then $|f(x)| \leq \|f\|_{L^\infty(\Omega)}$ for any $x \in \Omega$.

Theorem A.16. *If $f \in L^1(\Omega)$ and $g \in L^\infty(\Omega)$, then the product $f\,g$ belongs to $L^1(\Omega)$ and*

$$\|f\,g\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}\ \|g\|_{L^\infty(\Omega)}.$$

A.5 Other properties

Theorem A.17 (Interpolation). *Let $f \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \leq p \leq q \leq \infty$. Then, for any r such that $p \leq r \leq q$, we have that $f \in L^r(\Omega)$ and*

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{\alpha} \, \|f\|_{L^q}^{1-\alpha}$$

with $\alpha \in (0,1)$ such that $\dfrac{1}{r} = \dfrac{\alpha}{p} + \dfrac{1-\alpha}{q}$.

Definition - Theorem A.18. *Let f and g in $L^1(\mathbb{R}^d)$. The function $f \star g$, defined by*

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

is called the convolution *of f and g. It belongs to $L^1(\mathbb{R}^d)$ and*

$$\|f \star g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \, \|g\|_{L^1(\mathbb{R}^d)}.$$

Appendix B

Glossary of mathematical terms

English	French
arithmetic (operation)	(opération) arithmétique
assumption	hypothèse
axiom of choice	axiome du choix
almost everywhere (a.e.)	presque partout (p.p.)
ball	boule
basis	base
Banach space	espace de Banach
bilinear	bilinéaire
binomial coefficient	coefficient binomial
Borel set	ensemble borélien
Borel sigma-algebra	tribu borélienne
boundary value problem	problème aux limites
bounded set	ensemble borné
bounded function	fonction bornée
bounded above / bounded from above	majoré
bounded below / bounded from below	minoré
boundary	bord
to cancel	s'annuler
canonical	canonique
Cauchy sequence	suite de Cauchy
Cauchy-Schwarz inequality	inégalité de Cauchy-Schwarz
celestial motion	dynamique céleste
chain rule	dérivation des fonctions composées
characterisation	caractérisation
characteristic function	fonction charactéristique
closed set	ensemble fermé
closure	fermeture
compact set	ensemble compact
compactness	compacité
complement	complémentaire
conformal	conforme
consistency	consistance
concavity	concavité
completeness	complétude

English	French
continuity	continuité
contraction	contraction, fonction contractante
convolution (of functions)	(fonctions) convolées
convolution (as an operation)	convolution (comme opération)
to convolve	convoluer
coordinates	coordonnées
corollary	corollaire
countable	dénombrable
counter-example	contre-exemple
critical point	point critique
to deduce	déduire
determinant	déterminant
derivative	différentielle, dérivée
differentiable	différentiable
differentiation	dérivation
diffeomorphism	difféomorphisme
Dirac mass, Dirac delta	masse de Dirac
Dirichlet problem	problème de Dirichlet
discretisation	discrétisation
distribution	distribution
dominated convergence	convergence dominée
dynamics	dynamique
dynamical	dynamique
error analysis	analyse d'erreur
essentially bounded	essentiellement bornée
to establish	établir
Euclidean space	espace euclidien
even	pair
explicit (scheme)	(schéma) explicite
exponent	exposant
exponentially	de manière exponentielle
extended	prolongé
factor	facteur
field (algebraic object)	corps
field (function)	champ
finite element method	méthode des éléments finis
finiteness	finitude
finite-dimensional	de dimension finie
fixed point	point fixe
form	forme
Fubini's theorem	théorème de Fubini
function	application, fonction
Galerkin method	méthode de Galerkin
Hardy inequality	inégalité de Hardy
Hamiltonian	Hamiltonienne
Hermitian space	espace hermitien
Heaviside function	fonction de Heaviside
Hessian (matrix)	(matrice) hessienne
Hilbert space	espace de Hilbert
Hilbertian	hilbertien
Hölder inequality	inégalité de Hölder
homeomorphism	homéomorphisme
hyperplane	hyperplan

English	French
hypothesis	hypothèse
implicit (scheme)	(schéma) implicite
to imply	impliquer
in the sense of	au sens de
induced	engendré, induit
inner product	produit scalaire
integral	intégrale
integer	entier
integrable	intégrable
integration by parts	intégration par parties
isomorphism	isomorphisme
isometry	isométrie
index, indices	indice, indices
Jacobian matrix	matrice jacobienne
Jacobian	jacobien
Ker	Ker (le noyau d'une application)
Lax-Milgram (theorem)	théorème de Lax-Milgram
lemma	lemme
Let ... be	Soit ...
linear	linéaire
Lipschitz	Lipschitzien
lower bound	minoration
Lyapunov function	fonction de Lyapunov
mapping	application, fonction
mean value theorem	théorème des accroissements finis
measurable set	ensemble mesurable
measure	mesure
measure space	espace mesuré
mesh	maillage
Minkowski inequality	inégalité de Minkowski
monotonicity	monotonie
neighbourhood	voisinage
negative	strictement négatif
non-negative	positif ou nul
node	noeud
norm	norme
normed	normé
null (set)	(ensemble) négligeable
odd	impair
open set	ensemble ouvert
ordinary differential equation	équation différentielle ordinaire
order (of approximation)	ordre (d'approximation)
orthogonal	orthogonal
orthonormal	orthonormé
outward normal vector	vecteur normal sortant
parallelogram law/identity	formule de la médiane
partial derivative	dérivée partielle
partial differential equation	équation aux dérivées partielles
partial ordering	order partiel
piecewise	par morceaux
piecewise constant	constant par morceaux
Poincaré inequality	inégalité de Poincaré
Poisson problem	problème de Poisson

English	French
positive	strictement positif
non-positive	négatif ou nul
positivity	positivité
potential energy	énergie potentielle
power(s)	puissance(s)
preceding	précédant
probability measure	probabilité
probability space	espace probabilisé
principal value	valeur principale
quadratic	quadratique
to quotient	quotienter
Ran	Im (l'image d'une application)
real number, real	réel
regular	régulier
regularity	régularité
remainder	reste
Riesz representation theorem	Théorème de Riesz
robustness	robustesse
roots	racines
rounding error	erreur d'arrondi
scalar product	produit scalaire
scale	échelle
(numerical) scheme	schéma (numérique)
sequence	suite
(a) series	(une) série
sesquilinear	sesquilinéaire
set	ensemble
sigma-algebra	tribu
singularity	singularité
smooth	régulier, lisse
Sobolev space	espace de Sobolev
Span	Vect (espace vectoriel engendré par)
square (matrix)	(matrice) carrée
square-integrable	de carré intégrable
stability	stabilité
(time)step	pas (de temps)
step function	fonction étagée
stiffness matrix	matrice de rigidité
subset	sous-ensemble
such that (s.t.)	tel que (t.q.)
to suffice	suffire
sufficiently	suffisament
summable	sommable
symmetric	symétrique
symmetry	symétrie
to tend (to/towards)	tendre, converger (vers)
test function	fonction test
topology	topologie
trajectory	trajectoire
trapezium rule	méthode des trapèzes
triangle inequality	inégalité triangulaire
truncation	troncature

English	French
uniform	uniforme
uniqueness	unicité
units	unités
upper bound	majoration
to vanish	s'annuler
variational problem	problème variationnel
vector space	espace vectoriel
vertex	sommet
vertices	sommets
with values in	à valeurs dans
with respect to	par rapport à
Young's inequality	inégalité de Young

Bibliography

[1] H. Brézis, **Analyse fonctionnelle: théorie et applications**, Dunod, 1999.

[2] E. Cancès and A. Ern, **Analyse**, ENPC lecture notes, Sept. 2011.

[3] E. Cancès and A. Ern, **Analyse: Distributions, transformée de Fourier et équations aux dérivées partielles**, in preparation.

[4] A. Ern and G. Stoltz, **Calcul Scientifique**, ENPC lecture notes, Dec. 2014.