Variance reduction in stochastic homogenization using antithetic variables

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Abstract

Some theoretical issues related to the problem of variance reduction in numerical approaches for stochastic homogenization are examined. On some simple, yet representative cases, it is demonstrated theoretically that a technique based on antithetic variables can indeed reduce the variance of the output of the computation, and decrease the overall computational cost of such a multiscale problem. The theoretical considerations presented here are companion to numerical experiments presented in [14, 6] that corroborate the theoretical results enclosed.

1 Introduction

The present article examines some theoretical questions related to variance reduction techniques that can be successfully applied to some stochastic homogenization problems. It is a follow-up to an introductory article [14] where some one-dimensional settings are considered theoretically and some two-dimensional numerical experiments are presented. In particular because of space limitation, the present contribution will be supplemented by another publication [6] presenting numerical results on a broad set of test cases. More details, both theoretical and numerical, will also be presented in [12].

The stochastic homogenization problem we consider here writes as follows. We consider the elliptic boundary value problem

$$\left\{ \begin{array}{rcll} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon},\omega\right)\nabla u^\varepsilon\right) & = & f & \text{in} \quad \mathcal{D}, \\ u^\varepsilon & = & 0 & \text{on} \quad \partial\mathcal{D}, \end{array} \right. \tag{1.1}$$

set on a domain $\mathcal{D}$ in $\mathbb{R}^d$. Here, ε denotes a supposedly small, positive constant that models the smallest possible scale present in the problem. The matrix A is assumed random and stationary in a sense that will be made precise below. Somewhat loosely stated, A typically models a material that has a periodic pattern (with a basic unit cell Q) and for which, in each cell, some stationary random structure is present. For ε small, it is almost impossible, practically, to directly attack (1.1) with a numerical discretization. A useful practical approach is to *first* transform (1.1) in the associated homogenized problem:

$$\left\{ \begin{array}{rcll} -\operatorname{div}\left(A^\star\nabla u^\star\right) & = & f & \text{in} \quad \mathcal{D}, \\ u^\star & = & 0 & \text{on} \quad \partial\mathcal{D}, \end{array} \right. \tag{1.2}$$

and *next* numerically solve the latter problem. The two-fold advantage of (1.2) as compared to (1.1) is that *it is deterministic* and *it does not involve the small scale* ε. This simplification comes at a price. The homogenized matrix $A^\star$ in (1.2) is given by an average of an integral involving the corrector function (a solution to an (random) auxiliary problem, reminiscent of (1.1), and set at the scale of the fine structure of the material). All this will be made precise below.

Now, practically computing the corrector function and the homogenized matrix $A^\star$ requires to generate several realizations of the material over a finite, supposedly large volume at the microscale, and approach the matrix by some empirical means. Although the *theoretical* value of $A^\star$ is deterministic (and this is the whole point and the definite success of homogenization theory to obtain this), it is because of the numerical approximation process itself that randomness again comes into the picture. Generating different configurations of the material and then efficiently averaging over these realizations require to understand how variance affects the result. This is the purpose of the present article to investigate some theoretical questions in this direction. Before proceeding and for the sake of consistency, we now present in more details some basic elements of stochastic homogenization, and situate the questions under consideration in a more general existing literature.

1.1 Homogenization theoretical setting

To begin with, we introduce the basic setting of stochastic homogenization we will employ. We refer to [15] for a general, numerically oriented presentation, and to [5, 11, 19] for classical textbooks. We also refer to [7, 8] or [21] for a presentation of our particular setting. Throughout this article, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and we denote by $\mathbb{E}(X) = \int_\Omega X(\omega) d\mathbb{P}(\omega)$ the expectation value of any random variable $X \in L^1(\Omega, d\mathbb{P})$. We next fix $d \in \mathbb{N}^*$ (the ambient physical dimension), and assume that the group $(\mathbb{Z}^d, +)$ acts on Ω. We denote by $(\tau_k)_{k\in\mathbb{Z}^d}$ this action, and assume that it preserves the measure $\mathbb{P}$, that is, for all $k \in \mathbb{Z}^d$ and all $A \in \mathcal{F}$, $\mathbb{P}(\tau_k A) = \mathbb{P}(A)$. We assume that the action τ is *ergodic*, that is, if $A \in \mathcal{F}$ is such that $\tau_k A = A$ for any $k \in \mathbb{Z}^d$, then $\mathbb{P}(A) = 0$ or 1. In addition, we define the following notion of stationarity (see [7, 8]): any $F \in L^1_{\rm loc}\left(\mathbb{R}^d, L^1(\Omega)\right)$ is said to be *stationary* if, for all $k \in \mathbb{Z}^d$,

$$F(x+k, \omega) = F(x, \tau_k\omega), \tag{1.3}$$

almost everywhere in x and almost surely. In this setting, the ergodic theorem [20, 25] can be stated as follows: *Let $F \in L^\infty\left(\mathbb{R}^d, L^1(\Omega)\right)$ be a stationary random variable in the above sense. For $k = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d$, we set $|k|_\infty = \sup_{1\le i\le d} |k_i|$. Then*

$$\frac{1}{(2N+1)^d} \sum_{|k|_\infty \le N} F(x, \tau_k\omega) \underset{N\to\infty}{\longrightarrow} \mathbb{E}\left(F(x,\cdot)\right) \quad \textit{in } L^\infty(\mathbb{R}^d), \textit{ almost surely.}$$

This implies that (denoting by Q the unit cube in $\mathbb{R}^d$)

$$F\left(\frac{x}{\varepsilon}, \omega\right) \underset{\varepsilon\to 0}{\overset{*}{\longrightarrow}} \mathbb{E}\left(\int_Q F(x,\cdot)dx\right) \quad \textit{in } L^\infty(\mathbb{R}^d), \textit{ almost surely.}$$

Besides technicalities, the purpose of the above setting is simply to formalize that, even though realizations may vary, the function F at point $x \in \mathbb{R}^d$ and the function F at point $x+k$, $k \in \mathbb{Z}^d$, share the same law. In the homogenization context we now turn to, this means that the local, microscopic environment (encoded in the matrix A) is everywhere the same *on average*. From this, homogenized, macroscopic properties will follow. In addition, and this is evident reading the above setting, the microscopic environment considered has a relation to an underlying *periodic* structure (thus the integer shifts k in (1.3)).

As briefly introduced above, we now wish to solve the *multiscale random* elliptic problem (1.1). Let us formalize this. The domain $\mathcal{D}$ is an open, regular, bounded subset of $\mathbb{R}^d$. The right-hand side is an L^2 function f on $\mathcal{D}$. The random symmetric matrix A is assumed stationary in the sense (1.3). We also assume that A is bounded and that, in the sense of quadratic forms, A is positive and almost surely bounded away from zero.

In this specific setting, the homogenized matrix $A^\star$, that appears in the homogenized problem (1.2) obtained in the limit of small ε, reads

$$A^\star_{ij} = \int_Q \mathbb{E}\left[(\nabla w_{e_j}(x,\cdot) + e_j)^T \ A(x,\cdot)\ (\nabla w_{e_i}(x,\cdot) + e_i)\right] dx, \tag{1.4}$$

where, for any vector $p \in \mathbb{R}^d$, the *corrector* w_p is the solution (unique up to the addition of a random constant) in $\left\{w \in L^2_{\rm loc}(\mathbb{R}^d, L^2(\Omega)),\ \nabla w \in L^2_{\rm unif}(\mathbb{R}^d, L^2(\Omega))\right\}$

to

$$\begin{cases} -\operatorname{div}\left[A(\nabla w_p + p)\right] = 0 \quad \text{on } \mathbb{R}^d \quad \text{a.s.,} \\ \nabla w_p \text{ is stationary in the sense of (1.3),} \\ \displaystyle\int_Q \mathbb{E}(\nabla w_p) = 0. \end{cases} \tag{1.5}$$

We have used the notation L^2_{unif} for the *uniform* L^2 space, that is the space of functions for which, say, the L^2 norm on a ball of unit size is bounded above independently from the center of the ball.

1.2 The questions we consider

Now that the theory has been briefly presented, we turn to practice. The homogenized matrix $A^\star$ needs to be computed, so that in a second step the homogenized solution $u^\star$ may be approximated. By classical results in homogenization theory, we know $u^\star$ is a good approximation of u^ε, in a sense made precise in the literature (see *e.g.* [11]). The matrix $A^\star$ is approximated by the matrix

$$\left[A^\star_N\right]_{ij}(\omega) = \frac{1}{|Q_N|}\int_{Q_N} \left(e_i + \nabla w^N_{e_i}(y,\omega)\right)^T A(y,\omega)\left(e_j + \nabla w^N_{e_j}(y,\omega)\right) dy \tag{1.6}$$

which is in turn obtained by solving the corrector problem on a *truncated* domain, say the cube $Q_N \subset \mathbb{R}^d$ of size $(2N+1)^d$ centered at the origin:

$$\begin{cases} -\operatorname{div}\left(A(\cdot,\omega)\left(p + \nabla w^N_p(\cdot,\omega)\right)\right) = 0 \quad \text{on} \quad \mathbb{R}^d, \\ w^N_p(\cdot,\omega) \text{ is } Q_N\text{-periodic.} \end{cases} \tag{1.7}$$

As briefly explained above, although $A^\star$ itself is a deterministic object, its practical approximation $A^\star_N$ is random. It is only in the limit of infinitely large domains Q_N that the deterministic value is attained. Our aim is to design a numerical technique that, for finite N, allows to compute $A^\star_N$ more effectively, that is, with a smaller variance.

The issue of variance in stochastic homogenization is not new. It has seen lately a definite revival, mainly motivated by numerical concerns. It is not our purpose here to review in details the important contributions existing in the literature. We however wish to cite some results particularly relevant to our own study:

- the original contribution [27] by Yurinski, where the convergence of some truncated approximation of $A^\star$ is established, along with an estimate of the rate of convergence (in short, problem (1.5) is regularized and then $A^\star$ is approximated on a bounded domain),
- a similar study [10] by Bourgeat and Piatnitsky for a specific approximation more relevant to actual numerical practice (in short, both problem (1.5) and the integral in (1.4) are truncated as in (1.6)–(1.7)),

- the work [23] by Naddaf and Spencer on a discrete ("lattice-type") approximation of the differential operator present in the original problem (1.1),
- and the enterprise by Gloria and Otto (see [17] for homogenization problems set on random lattices and publications announced in preparation for some problems for differential operators) to establish sharp estimates of the convergence of the numerical approximation in terms of size of the truncation domain and other discretization parameters.

In all the above works, the convergence and the rate of convergence are studied. We take here the problem from a slightly different perspective: we are interested in *basic practical* issues. Can we improve the prefactor in the convergence of $A_N^\star$ to $A^\star$ as $N \to +\infty$ (loosely stated, the variance in a Central Limit Theorem type result)? or, even more practically, can we reduce the confidence interval for empirical means approximating $\mathbb{E}(A_N^\star)$? and similar issues.

To better understand the issue of reducing variance in stochastic homogenization, we consider a specific, well known variance reduction technique, the technique of *antithetic variables* [22, page 27]. In the sequel of this article, we consider two specific cases.

Our first setting (in Section 2) is a "genuinely" random setting (this terminology will be clear when we introduce our second setting shortly below). We consider a random matrix A constructed with independent, identically distributed random variables on the cells of our periodic lattice (although A does not need to be constant on each cell and equal to these random variables; see *e.g.* example (2.6) below). Since solving problem (1.5) and directly computing $A^\star$ is out of reach practically, our numerical approach considers the truncations (1.6)–(1.7) on a finite domain Q_N, solved for a set of realizations of the random matrix. Empirical means of the truncated homogenized matrix $A_N^\star(\omega)$ are obtained, along with a (approximation of a) confidence interval involving the variance. The consideration of antithetic variables allows to improve the approximation. This is experimentally observed, and documented in [14, 6, 12]. We establish here theoretically that the variance of the homogenized objects is indeed diminished by the technique (we are, unfortunately, unable to explicitly estimate the gain). In our study, the matrix $A_N^\star$, its eigenvalues, its trace and determinant, and the eigenvalues of the elliptic operator associated to $A_N^\star$ are considered, but other objects could be studied: the eigenvectors, the differential operator itself, the approximation of the homogenized solution $u^\star$, the residual $u^\varepsilon - u^\star$ (somewhat as a follow-up to the studies [9, 3]), etc.

Our second setting is a "weakly" random setting. By this we mean that the random matrix A is a small perturbation of a deterministic, periodic matrix. Consequently, the solution of the problem is only seeked *at the first order in the size of the perturbation*. The setting has been introduced in [8] (and is recalled in Section 3.1 below). Its practical interest is that the computation comes down to a set of *fully deterministic* computations. So in practice, no variance issue is relevant. We however consider this case *pretending* not to exploit the simplification: we treat the problem stochastically and prove that the technique of variance reduction still works. As we can compute everything deterministically "in the backroom", the setting, although clearly particular and not general, is an appropriate test-bed to get some insight on some of the generically relevant issues.

It is important to note that some of the results we establish are limited to the technique of antithetic variables. Some others are not. They can therefore be useful for other variance reduction techiques. This is the case for our estimates of variance of the output of the computations in terms of the variance of the original parameters (see *e.g.* estimate (3.14)).

Let us conclude this introduction mentioning that of course there exists many other settings where similar questions can be asked. We treat here the very specific case of a *linear*, *elliptic* second order equation *in divergence form.* The coefficient is assumed to be constructed with independent, identically distributed random variables set on a simple underlying periodic structure. The technique used for variance reduction is that of antithetic variables. Many more difficult situations could be addressed: other types of stationary ergodic coefficients, other types of equations, other techniques of truncations and regularizations of the original problems, other techniques for variance reduction, other numerical approaches, ...

Some of these issues (but clearly not all!) will be addressed in [6].

2 A "fully" stochastic case

We consider in this section a "genuinely" random setting, in contrast to the setting of Section 3, where randomness will come as a *small perturbation* of a deterministic periodic setting.

2.1 Mathematical setting and statement of our main result

In this section, we make the following two assumptions on the matrix A of (1.1). First, we assume that, for any N, there exists an integer n (possibly $n = |Q_N|$, but not necessarily) and a function $\mathcal{A}$, defined on $Q_N \times \mathbb{R}^n$, such that the tensor $A(x, \omega)$ writes

$$\forall x \in Q_N, \quad A(x, \omega) = \mathcal{A}(x, X_1(\omega), \dots, X_n(\omega)) \quad \text{a.s.,} \tag{2.1}$$

where $\{X_k(\omega)\}_{1 \le k \le n}$ are independent scalar random variables, which are all distributed according to the uniform law $\mathcal{U}[0,1]$. In general, the function $\mathcal{A}$, as well as the number n of independent, identically distributed variables involved in (2.1), depend on N, the size of Q_N, although this dependency is not made explicit in (2.1).

Second, we assume that the function $\mathcal{A}$ in (2.1) is such that, for all $x \in Q_N$, and any vector $\xi \in \mathbb{R}^d$, the map

$$(x_1, \dots, x_n) \in \mathbb{R}^n \mapsto \xi^T \mathcal{A}(x, x_1, \dots, x_n)\xi \tag{2.2}$$

is non-decreasing with respect to each of its arguments.

Before proceeding, we now give a set of specific examples of matrices A that satisfy the above assumptions. Consider a family $(a_k(\omega))_{k\in\mathbb{Z}^d}$ of independent, identically distributed random variables having a continuous positive probability density $p(x)$, and set

$$A(x,\omega) = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{Q+k}(x) a_k(\omega)\,\mathrm{Id}, \tag{2.3}$$

where Q is the unit cube of $\mathbb{R}^d$, centered at the origin, and $Q+k$ is the cube Q translated by the vector $k \in \mathbb{Z}^d$. We assume that there exist $\alpha > 0$ and $\beta < \infty$ such that, for all k, $0 < \alpha \le a_k \le \beta < +\infty$ almost surely. Consequently, A is uniformly coercive and bounded. Example (2.3) corresponds to a spherical matrix $A(x,\omega)$ that is constant in each cube $Q+k$, with independent, identically distributed values $a_k(\omega)$.

Introduce now the cumulative distribution function $P(x) = \displaystyle\int_{-\infty}^{x} p(s)\,ds$, where p is the common density of all a_k, and its reciprocal function f: for any $x \in \mathbb{R}$, $x = f(P(x))$. Introduce the random variables $X_k(\omega) = P(a_k(\omega))$. Then $X_k \sim \mathcal{U}(0,1)$, and $a_k(\omega) = f(X_k(\omega))$. In addition, since P is an increasing function, the function f is also increasing. We have

$$A(x,\omega) = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{Q+k}(x) f(X_k(\omega))\,\mathrm{Id}, \tag{2.4}$$

where $(X_k(\omega))_{k\in\mathbb{Z}^d}$ is a family of independent random variables that are all uniformly distributed in $[0,1]$, and f is non-decreasing. This yields an example falling in our framework (2.1)–(2.2).

Remark 1 *Assume that, in* (2.3)*, the variables a_k are all distributed according to a Bernoulli law of parameter $r \in (0,1)$, that is, $a_0 \sim \mathcal{B}(r)$, $\mathbb{P}(a_0 = \alpha) = r$ and $\mathbb{P}(a_0 = \beta) = 1-r$, for some $0 < \alpha < \beta$. Such random variables do not have a* continuous *probability density. Yet, without loss of generality, we may write $a_0(\omega) = f(X_0(\omega))$ with $X_0 \sim \mathcal{U}([0,1])$, where f is the non-decreasing function*

$$f(x) = \alpha + (\beta-\alpha)\mathbf{1}_{\{r\le x\le 1\}}.$$

The example (2.3)*, with such a_k, can thus be written in the form* (2.4)*, and hence falls again in our framework.*

Of course, Example (2.3) can be readily extended to the case of non-spherical matrices. Consider a function F, defined on $[0,1]$, such that, for each $x \in [0,1]$,

$F(x)$ is a symmetric matrix. We also assume that $F(x)$ is uniformly coercive and bounded, and that, for any $\xi \in \mathbb{R}^d$, the function $x \in [0,1] \mapsto \xi^T F(x)\xi$ is non-decreasing. Then

$$A(x,\omega) = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{Q+k}(x) F(X_k(\omega)) \tag{2.5}$$

also satisfies our assumptions.

We eventually give an example of a matrix A satisfying our assumptions and that, on each cell, is not equal to independent, identically distributed variables. For this purpose, define positive coefficients κ_p for $|p|_\infty \leq 1$, and consider a non-decreasing function f. We then set

$$A(x,\omega) = \sum_{k\in\mathbb{Z}^d} \left[\mathbf{1}_{Q+k}(x) \sum_{j\in\mathbb{Z}^d,\ |j-k|_\infty \leq 1} \kappa_{j-k}\ f(X_j(\omega)) \ \mathrm{Id} \right], \tag{2.6}$$

which clearly satisfies (2.1)–(2.2). In (2.6), it is immediately seen that the value of $A(x,\omega)$ in the cell $Q+k$ is a local average of the values $f(X_j(\omega))$ Id, for $|j-k|_\infty \leq 1$.

The main result of this section is the following:

Proposition 1 *We assume* (2.1)–(2.2)*. Let $A^\star_N(\omega)$ be the approximated homogenized matrix, obtained by solving the corrector problem* (1.7) *on the truncated domain $Q_N \subset \mathbb{R}^d$. We define on Q_N the field*

$$B(x,\omega) := \mathcal{A}(x, 1 - X_1(\omega), \dots, 1 - X_n(\omega)), \tag{2.7}$$

antithetic to $A(\cdot,\omega)$ defined by (2.1)*. We associate to this field the corrector problem* (1.7) *(replacing A by B), the solution of which is denoted by v_p^N, and the matrix $B^\star_N(\omega)$, defined from v_p^N using* (1.6)*. Set*

$$\widetilde{A}^\star_N(\omega) := \frac{1}{2}\left(A^\star_N(\omega) + B^\star_N(\omega)\right). \tag{2.8}$$

Then

$$\mathbb{E}\left(\widetilde{A}^\star_N\right) = \mathbb{E}\left(A^\star_N\right), \tag{2.9}$$

and, for any $\xi \in \mathbb{R}^d$,

$$\mathbb{V}\mathrm{ar}\left(\xi^T \widetilde{A}^\star_N \xi\right) \leq \frac{1}{2}\mathbb{V}\mathrm{ar}\left(\xi^T A^\star_N \xi\right). \tag{2.10}$$

Otherwise stated, $\widetilde{A}^\star_N$ is an unbiased estimator of $\mathbb{E}\left(A^\star_N\right)$, and its variance is smaller than half of that of $A^\star_N$, in the sense of (2.10)*.*

The practical consequences of this result for variance reduction techniques are discussed below.

Taking $\xi = e_i$ in (2.10) implies that the variance of the diagonal coefficient $\left[\widetilde{A}_N^\star\right]_{ii}$ is reduced by a factor two. Similar results can be obtained, first on the eigenvalues of the matrix $A_N^\star$, and second on the eigenvalues of the (approximate) homogenized elliptic operator $L_A = -\operatorname{div}\left[A_N^\star(\omega)\nabla\,\cdot\right]$. This is the purpose of the following two corollaries.

Corollary 1 *We assume* (2.1)–(2.2)*. Denote by* $\{\lambda_k(A,\omega)\}_{1\le k\le d}$ *and* $\{\lambda_k(B,\omega)\}_{1\le k\le d}$ *the eigenvalues of* $A_N^\star(\omega)$ *and* $B_N^\star(\omega)$ *respectively, sorted in non-decreasing order, where* $A_N^\star(\omega)$ *and* $B_N^\star(\omega)$ *are defined as in Proposition 1. Define*

$$\widetilde{\lambda}_k(\omega) := \frac{1}{2}\big[\lambda_k(A,\omega) + \lambda_k(B,\omega)\big].$$

Then, for all $1 \le k \le d$,

$$\mathbb{E}\left(\widetilde{\lambda}_k\right) = \mathbb{E}\left(\lambda_k(A,\cdot)\right) \quad \text{and} \quad \mathbb{V}\mathrm{ar}\left(\widetilde{\lambda}_k\right) \le \frac{1}{2}\mathbb{V}\mathrm{ar}\left(\lambda_k(A,\cdot)\right). \tag{2.11}$$

Remark 2 *The above corollary shows variance reduction for each eigenvalue of the homogenized matrix. It is easily seen that the proof of this result carries over to the case when the quantity of interest is a function* $\mathcal{F}\left(\lambda_1(A,\omega),\dots,\lambda_d(A,\omega)\right)$ *of these eigenvalues, provided* $\mathcal{F}$ *is a real-valued function that is non-decreasing with respect to each of its arguments. We indeed have*

$$\mathbb{E}\left(\widetilde{Z}\right) = \mathbb{E}\left(Z\right) \quad \text{and} \quad \mathbb{V}\mathrm{ar}\left(\widetilde{Z}\right) \le \frac{1}{2}\mathbb{V}\mathrm{ar}\left(Z\right),$$

where $Z(\omega) = \mathcal{F}\left(\lambda_1(A,\omega),\dots,\lambda_d(A,\omega)\right)$ *and*

$$\widetilde{Z}(\omega) = \frac{1}{2}\big[\mathcal{F}\left(\lambda_1(A,\omega),\dots,\lambda_d(A,\omega)\right) + \mathcal{F}\left(\lambda_1(B,\omega),\dots,\lambda_d(B,\omega)\right)\big].$$

Typical examples for such functions are

$$\begin{aligned} \mathcal{F}\left(\lambda_1(A,\omega),\dots,\lambda_d(A,\omega)\right) &= \sum_{k=1}^d \lambda_k(A,\omega) = \operatorname{Tr} A_N^\star(\omega), \\ \mathcal{F}\left(\lambda_1(A,\omega),\dots,\lambda_d(A,\omega)\right) &= \Pi_{k=1}^d \lambda_k(A,\omega) = \det A_N^\star(\omega). \end{aligned}$$

Using the technique of antithetic variables, we hence achieve variance reduction for any diagonal coefficient of the matrix $A_N^\star(\omega)$, as well as for its trace and its determinant, and for any quantity of the form $\xi^T A_N^\star \xi$, for any given vector $\xi \in \mathbb{R}^d$. Note however that our argument does not cover the case of the off-diagonal coefficients of $A_N^\star(\omega)$, although numerical results on several test cases (see [6]) indicate that the variance of these coefficients is also reduced by the present method.

Corollary 2 *We assume* (2.1)–(2.2)*. Let* $(\lambda_k(L_A,\omega), u_k(L_A,\omega))_{k\in\mathbb{N}}$ *be the eigenelements of the operator* $L_A = -\operatorname{div}[A_N^\star(\omega)\nabla\cdot]$ *with homogeneous Dirichlet boundary conditions, i.e.*

$$-\operatorname{div}[A_N^\star(\omega)\nabla u_k(L_A,\omega)] = \lambda_k(L_A,\omega)\, u_k(L_A,\omega)$$

with $u_k(L_A,\omega) \in H_0^1(\mathcal{D})$ *and* $\|u_k(L_A,\omega)\|_{L^2(\mathcal{D})} = 1$*. We similarly consider the eigenelements of* $L_B = -\operatorname{div}[B_N^\star(\omega)\nabla\cdot]$*:*

$$-\operatorname{div}[B_N^\star(\omega)\nabla u_k(L_B,\omega)] = \lambda_k(L_B,\omega)\, u_k(L_B,\omega).$$

We assume that, almost surely, $\lambda_k(L_A,\omega)$ *and* $\lambda_k(L_B,\omega)$ *are sorted in non-decreasing order. Define*

$$\widetilde{\lambda}_k(L,\omega) := \frac{1}{2}\left(\lambda_k(L_A,\omega) + \lambda_k(L_B,\omega)\right).$$

Then, for all $k \in \mathbb{N}$*,*

$$\mathbb{E}\left(\widetilde{\lambda}_k(L,\cdot)\right) = \mathbb{E}\left(\lambda_k(L_A,\cdot)\right) \quad \textit{and} \quad \mathbb{V}\mathrm{ar}\left(\widetilde{\lambda}_k(L,\cdot)\right) \leq \frac{1}{2}\mathbb{V}\mathrm{ar}\left(\lambda_k(L_A,\cdot)\right). \quad (2.12)$$

The proofs of the above results (Proposition 1 and Corollaries 1 and 2) are given in Section 2.4. They are obtained combining some classical results on variance reduction using antithetic variables [22, page 27] and some monotonicity results from the theory of homogenization. For consistency, these results are recalled in Sections 2.2 and 2.3, respectively.

The proof goes as follows. First, we recall that the technique of antithetic variables reduces variance for the computation of $\mathbb{E}(f(X_1,\ldots,X_n))$, when f is a real-valued function, that is *non-decreasing* of each of its argument, and $X = (X_1,\ldots,X_n)$ is a vector of independent random variables. This is made precise in Section 2.2. Second, we assume that the tensor field $A(x,\omega)$ of (1.1) writes as a non-decreasing function (in the sense of symmetric positive matrices) of independent random variables $X_k(\omega)$ (these are assumptions (2.1) and (2.2)). Then, as recalled in Section 2.3, we use that the homogenization process *preserves the order of symmetric matrices* to claim that $A_N^\star(\omega)$ is likewise a non-decreasing function of the random variables $X_k(\omega)$. Consequently, we obtain variance reduction for $A_N^\star$. In the same vein, since the map that associates to a symmetric matrix its eigenvalues is increasing, we obtain variance reduction for the eigenvalues of $A_N^\star$. This argument is formalized in Sections 2.2, 2.3 and 2.4.

Before proceeding, we briefly explain the usefulness of the above results for variance reduction techniques. Assume we want to compute the expectation of $\xi^T A_N^\star(\omega)\xi$, for some fixed vector $\xi \in \mathbb{R}^d$ (similar arguments hold for the computation of the expectation of any quantity considered above: the eigenvalues of the matrix $A_N^\star(\omega)$, its trace, its determinant, or the eigenvalues of the associated elliptic operator). Following a classical Monte-Carlo method, we estimate $\mathbb{E}\left(\xi^T A_N^\star \xi\right)$ by its empirical mean. To this end, we consider $2M$ independent, identically distributed copies $\{A_m(x,\omega)\}_{1\leq m\leq 2M}$ of the random field

$A(x,\omega)$ on Q_N. To each copy A_m, we associate an approximate homogenized matrix $A_N^{\star;m}(\omega)$, obtained using the solution to the corrector problem (1.7) in the average (1.6). We next introduce the empirical mean

$$\mu_{2M}\left(\xi^T A_N^\star \xi\right)(\omega) = \frac{1}{2M}\sum_{m=1}^{2M} \xi^T A_N^{\star,m}(\omega)\xi, \tag{2.13}$$

and consider that, in practice, the mean $\mathbb{E}\left(\xi^T A_N^\star \xi\right)$ is equal to its estimator $\mu_{2M}\left(\xi^T A_N^\star \xi\right)$ within an approximate margin of error $1.96\dfrac{\sqrt{\mathbb{V}\mathrm{ar}\left(\xi^T A_N^\star \xi\right)}}{\sqrt{2M}}$.

Alternate to considering (2.13), we may consider

$$\mu_M\left(\xi^T \widetilde{A}_N^\star \xi\right)(\omega) = \frac{1}{M}\sum_{m=1}^{M} \xi^T \widetilde{A}_N^{\star;m}(\omega)\xi, \tag{2.14}$$

where $\widetilde{A}_N^{\star,m}$ is defined by (2.8). Again, in practice, the mean $\mathbb{E}\left(\xi^T A_N^\star \xi\right) = \mathbb{E}\left(\xi^T \widetilde{A}_N^\star \xi\right)$ is equal to $\mu_M\left(\xi^T \widetilde{A}_N^\star \xi\right)$ within an approximate margin of error $1.96\dfrac{\sqrt{\mathbb{V}\mathrm{ar}\left(\xi^T \widetilde{A}_N^\star \xi\right)}}{\sqrt{M}}$. Observe now that both estimators (2.13) and (2.14) are of equal cost, since they require the same number $2M$ of corrector problems to be solved. The accuracy of the latter is better if and only if $\mathbb{V}\mathrm{ar}\left(\xi^T \widetilde{A}_N^\star \xi\right) \le \dfrac{1}{2}\mathbb{V}\mathrm{ar}\left(\xi^T A_N^\star \xi\right)$, which is exactly the bound (2.10) of Proposition 1.

We conclude this discussion by describing for illustration a typical numerical result (see [14, 6, 12] for more comprehensive numerical experiments). We consider the case (2.3), with $(a_k(\omega))_{k\in\mathbb{Z}^d}$ a family of independent random variables that are all distributed according to a Bernoulli law of parameter $r \in (0,1)$: $\mathbb{P}(a_0 = \alpha) = r$ and $\mathbb{P}(a_0 = \beta) = 1 - r$, for some $0 < \alpha < \beta$. As noted in Remark 1, this case falls in our framework (2.1)–(2.2). Following (2.7), we introduce the antithetic field

$$B(x,\omega) = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{Q+k}(x) b_k(\omega)\,\mathrm{Id},$$

where the variable b_k, antithetic to a_k, is defined as follows. We recall from Remark 1 that $a_k(\omega) = \alpha + (\beta-\alpha)\mathbf{1}_{\{r\le X_k(\omega)\le 1\}}$, with $X_k \sim \mathcal{U}([0,1])$. We then set $b_k(\omega) = \alpha + (\beta-\alpha)\mathbf{1}_{\{0\le X_k(\omega)\le 1-r\}}$.

Focusing, to fix the ideas, on the computation of $\mathbb{E}\left([A_N^\star]_{11}\right)$, we introduce the effectivity ratio

$$R = \frac{\mathbb{V}\mathrm{ar}\left([A_N^\star]_{11}\right)}{2\mathbb{V}\mathrm{ar}\left(\left[\widetilde{A}_N^\star\right]_{11}\right)},$$

where in practice the above variances are replaced by empirical variances. This ratio quantifies the gain in computational time (at fixed accuracy). Results are reported in Table 1, for Bernoulli variables of parameter $\alpha = 3$, $\beta = 20$ and $r = 1/2$ (numerical tests have been performed using the finite elements software FreeFem++, see `http://www.freefem.org`). On Figure 1, we plot the estimated means (2.13) and (2.14) along with their confidence intervals. These results indeed show the efficiency of the approach.

N	5	10	20	40	60	80	100
R	5.34	3.91	5.41	3.07	4.41	4.49	4.28

Table 1: Representative effectivity ratio R, in the case of a Bernoulli variable of parameter $\alpha = 3$, $\beta = 20$ and $r = 1/2$ (results extracted from [14]).

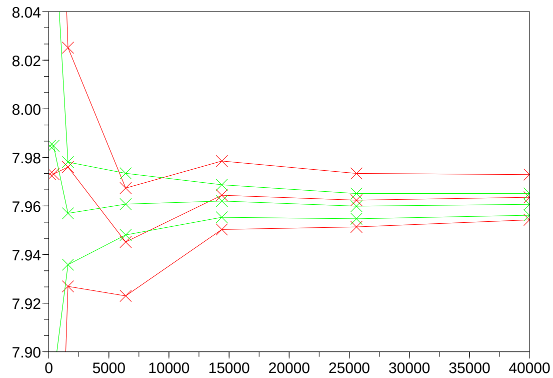

Figure 1: Estimated means (with confidence intervals) for $[A^\star_N]_{11}$ (red) and $\left[\widetilde{A}^\star_N\right]_{11}$ (green), in the case of a Bernoulli variable of parameter $\alpha = 3$, $\beta = 20$ and $r = 1/2$. Results are extracted from [14] (in each case, $2M = 100$ corrector problems have been solved).

Note that the above test case is a challenging one, as the ratio β/α is large. When the normalized variance of the field is smaller, even larger effectivity ratios R are obtained. Consider again the case (2.3), namely

$$A(x,\omega) = \sum_{k \in \mathbb{Z}^d} \mathbf{1}_{Q+k}(x) a_k(\omega) \,\mathrm{Id},$$

where now $(a_k(\omega))_{k\in\mathbb{Z}^d}$ is a family of independent random variables that are all uniformly distributed between $\alpha_0 = 3$ and $\beta_0 = 5$. In that case, we obtain effectivity ratios R of the order of 50, as shown in Table 2.

N	40	60	80	100
R	52.9	59.2	55.7	71.5

Table 2: Representative effectivity ratio R, in the case of a random variable uniformly distributed between $\alpha_0 = 3$ and $\beta_0 = 5$ (results extracted from [6]).

2.2 Classical results on antithetic variables

We first recall the following lemma, and provide its proof for consistency. This result is crucial for our proof of variance reduction using the technique of antithetic variables.

Lemma 1 ([22], page 27) *Let f and g be two real-valued functions defined on $\mathbb{R}^n$, which are non-decreasing with respect to each of their arguments. Consider $X = (X_1, \dots, X_n)$ a vector of random variables, which are all independent from one another. Then*

$$\mathbb{C}\text{ov}(f(X), g(X)) \geq 0. \tag{2.15}$$

Proof: We prove the lemma by induction. Consider X and Y two independent scalar random variables, identically distributed. Both functions f and g are non-decreasing, so

$$(f(X) - f(Y))\ (g(X) - g(Y)) \geq 0.$$

We now take the expectation of the above inequality:

$$\mathbb{E}(f(X)\ g(X)) + \mathbb{E}(f(Y)\ g(Y)) \geq \mathbb{E}(f(Y)\ g(X)) + \mathbb{E}(f(X)\ g(Y)).$$

As X and Y share the same law, and are independent, this yields

$$\mathbb{E}(f(X)\ g(X)) \geq \mathbb{E}(f(X))\ \mathbb{E}(g(X)),$$

and (2.15) follows for $n = 1$.

Assume now that, for some N, we have proved the result for any random vector X of any dimension $n \leq N - 1$. Let us now prove the result for a vector X of dimension N. For any fixed x_N, the functions $(x_1, \dots, x_{N-1}) \mapsto f(x_1, \dots, x_{N-1}, x_N)$ and $(x_1, \dots, x_{N-1}) \mapsto g(x_1, \dots, x_{N-1}, x_N)$ are non-decreasing with respect to each of their arguments. It follows from the induction assumption that, for any x_N,

$$\mathbb{E}(f(X_1, \dots, X_{N-1}, x_N)\ g(X_1, \dots, X_{N-1}, x_N)) \geq \\ \mathbb{E}(f(X_1, \dots, X_{N-1}, x_N))\ \mathbb{E}(g(X_1, \dots, X_{N-1}, x_N)). \tag{2.16}$$

Introducing

$$F(x) = \mathbb{E}(f(X_1, \dots, X_{N-1}, x)) \quad \text{and} \quad G(x) = \mathbb{E}(g(X_1, \dots, X_{N-1}, x)),$$

inequality (2.16) reads

$$\mathbb{E}(f(X_1,\ldots,X_{N-1},x_N)\ g(X_1,\ldots,X_{N-1},x_N)) \geq F(x_N)\ G(x_N).$$

We now set $x_N = X_N(\omega)$ and take expectation to obtain

$$\mathbb{E}(f(X)\ g(X)) \geq \mathbb{E}(F(X_N)\ G(X_N)). \tag{2.17}$$

By construction, F and G are non-decreasing functions. The result for $n = 1$ applies, and yields

$$\mathbb{E}(F(X_N)\ G(X_N)) \geq \mathbb{E}(F(X_N))\ \mathbb{E}(G(X_N)) = \mathbb{E}(f(X))\ \mathbb{E}(g(X)). \tag{2.18}$$

Collecting (2.17) and (2.18), we conclude the proof. □

Remark 3 *The proof clearly shows that the result also holds if, for each variable, f and g are either both non-decreasing or both non-increasing.*

The following result is a simple consequence of the above lemma.

Corollary 3 *Let f be a function defined on $\mathbb{R}^n$, which is non-decreasing with respect to each of its arguments. Consider $X = (X_1,\ldots,X_n)$ a vector of random variables, which are all independent from one another, and distributed according to the uniform law $\mathcal{U}[0,1]$. Then*

$$\mathbb{V}\mathrm{ar}\left(\frac{1}{2}\left(f(X) + f(1-X)\right)\right) \leq \frac{1}{2}\mathbb{V}\mathrm{ar}\left(f(X)\right),$$

where we denote $1 - X = (1 - X_1,\ldots,1 - X_n) \in \mathbb{R}^n$.

Proof: Choosing $g(x_1,\ldots,x_n) = -f(1-x_1,\ldots,1-x_n)$ in Lemma 1, we obtain that

$$\mathbb{C}\mathrm{ov}(f(X), f(1-X)) = \mathbb{C}\mathrm{ov}(f(X_1,\ldots,X_n), f(1-X_1,\ldots,1-X_n)) \leq 0.$$

We next observe that

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left(\frac{1}{2}\left(f(X) + f(1-X)\right)\right) &= \frac{1}{2}\mathbb{V}\mathrm{ar}(f(X)) + \frac{1}{2}\mathbb{C}\mathrm{ov}\left(f(X), f(1-X)\right)\\ &\leq \frac{1}{2}\mathbb{V}\mathrm{ar}(f(X)),\end{aligned}$$

where we have used that $\mathbb{V}\mathrm{ar}(f(X)) = \mathbb{V}\mathrm{ar}(f(1-X))$. □

2.3 Monotonicity in periodic homogenization

We first recall a classical result of periodic homogenization, which is useful in the sequel. We provide its proof for consistency.

Lemma 2 ([26], page 12) *Consider, for all x in $\mathbb{R}^d$, a symmetric matrix $A(x) \in \mathbb{R}^{d\times d}$. Assume that A is $\overline{Q}$-periodic, uniformly coercive and bounded, for $\overline{Q}$ a cube of $\mathbb{R}^d$. We denote by $A^\star$ the matrix obtained from A by homogenization of the operator $L_\varepsilon = -\operatorname{div}\left[A\left(\frac{x}{\varepsilon}\right)\nabla\cdot\right]$.*

We now consider a matrix $B(x)$ enjoying the same properties as $A(x)$, and such that

$$\forall \xi \in \mathbb{R}^d, \quad \xi^T B(x)\xi \geq \xi^T A(x)\xi \quad \text{a.e. on } \overline{Q}.$$

We correspondingly denote by $B^\star$ the matrix obtained from B by homogenization of L_ε, replacing A by B. Then

$$\forall \xi \in \mathbb{R}^d, \quad \xi^T B^\star \xi \geq \xi^T A^\star \xi.$$

Proof: For any $p \in \mathbb{R}^d$, let w_p be the corrector function associated to the matrix $A(x)$, i.e. the function defined on $\mathbb{R}^d$, $\overline{Q}$-periodic, and solution to

$$-\operatorname{div}\left(A\left(p+\nabla w_p\right)\right) = 0 \quad \text{on} \quad \mathbb{R}^d.$$

By definition, $w_p \in H^1_\square(\overline{Q})$ satisfies

$$\forall \theta \in H^1_\square(\overline{Q}), \quad \int_{\overline{Q}} \nabla\theta^T A(x)(p+\nabla w_p)\,dx = 0, \tag{2.19}$$

where $H^1_\square(\overline{Q})$ denotes the set of functions that belong to $H^1(\overline{Q})$ and are $\overline{Q}$-periodic. On the other hand, by definition of the homogenized matrix $A^\star$,

$$p^T A^\star p = \frac{1}{|\overline{Q}|}\int_{\overline{Q}} (p+\nabla w_p)^T A(x)(p+\nabla w_p).$$

Similar assertions hold for the matrix B. Denoting by v_p the corrector function associated to that matrix, we have

$$\begin{aligned}
p^T B^\star p &= \frac{1}{|\overline{Q}|}\int_{\overline{Q}} (p+\nabla v_p)^T B(x)(p+\nabla v_p) \\
&\geq \frac{1}{|\overline{Q}|}\int_{\overline{Q}} (p+\nabla v_p)^T A(x)(p+\nabla v_p) \\
&= \frac{1}{|\overline{Q}|}\int_{\overline{Q}} (p+\nabla w_p+\nabla(v_p-w_p))^T A(x)(p+\nabla w_p+\nabla(v_p-w_p)) \\
&= \frac{1}{|\overline{Q}|}\int_{\overline{Q}} (p+\nabla w_p)^T A(x)(p+\nabla w_p) \\
&\quad +\frac{2}{|\overline{Q}|}\int_{\overline{Q}} (p+\nabla w_p)^T A(x)\nabla(v_p-w_p) \\
&\quad +\frac{1}{|\overline{Q}|}\int_{\overline{Q}} (\nabla(v_p-w_p))^T A(x)\nabla(v_p-w_p).
\end{aligned}$$

Since $A(x)$ is coercive, the third term is non-negative. Using (2.19), we see that the second term vanishes. We are left with

$$p^T B^\star p \geq \frac{1}{|\overline{Q}|}\int_{\overline{Q}} (p+\nabla w_p)^T A(x)(p+\nabla w_p) = p^T A^\star p.$$

This concludes the proof. □

We now recall the following elementary result on the monotonicity of eigenvalues of symmetric matrices. Consider two symmetric matrices A and B of size $d \times d$, such that $\xi^T B \xi \geq \xi^T A \xi$ for any $\xi \in \mathbb{R}^d$. Then, for any $1 \leq k \leq d$, $\lambda_k(B) \geq \lambda_k(A)$, where $\lambda_k(A)$ and $\lambda_k(B)$ are the eigenvalues of A and B respectively, sorted in non-decreasing order.

This result can be readily extended to the eigenvalues of the corresponding elliptic operators, as stated in the following lemma:

Lemma 3 *Consider two symmetric matrices $A(x)$ and $B(x)$, of size $d \times d$, defined on a bounded domain $\mathcal{D} \subset \mathbb{R}^d$. We assume that these two matrices are uniformly coercive and bounded, and that*

$$\forall \xi \in \mathbb{R}^d, \quad \xi^T B(x) \xi \geq \xi^T A(x) \xi \quad \text{a.e. on } \mathcal{D}. \tag{2.20}$$

Consider the eigenelements $(\lambda_k(L_A), u_k(L_A))_{k \in \mathbb{N}}$ of the operator $L_A = -\operatorname{div}[A(x)\nabla \cdot]$ with homogeneous Dirichlet boundary conditions, i.e.

$$-\operatorname{div}[A(x)\nabla u_k(L_A)] = \lambda_k(L_A) u_k(L_A),$$

with $u_k(L_A) \in H^1_0(\mathcal{D})$ and $\|u_k(L_A)\|_{L^2(\mathcal{D})} = 1$. We assume that $\lambda_k(L_A)$ are sorted in non-decreasing order. Let $(\lambda_k(L_B), u_k(L_B))_{k \in \mathbb{N}}$ be the eigenelements of L_B. Then

$$\forall k \in \mathbb{N}, \ \lambda_k(L_B) \geq \lambda_k(L_A). \tag{2.21}$$

This result is an immediate consequence of the Courant-Fisher characterization of the eigenvalues.

2.4 Proof of Proposition 1 and Corollaries 1 and 2

Now that we have collected all the necessary ingredients, we may turn here to the proof of our main results.

Proof of Proposition 1: As $1 - X_k(\omega)$ and $X_k(\omega)$ share the same law, so do the fields $A(x,\omega)$ and $B(x,\omega)$, on Q_N. Hence, the homogenized matrices $A^\star_N(\omega)$ and $B^\star_N(\omega)$ have the same law, and we obtain (2.9).

We now prove (2.10). Let $\mathcal{P}_N$ be the operator that associates to a given Q_N-periodic tensor field A the effective tensor $\mathcal{P}_N(A)$ obtained by periodic homogenization. Then, by construction, the approximation $A^\star_N(\omega)$ defined by (1.6)–(1.7) is the effective matrix obtained by periodic homogenization of $A_{|Q_N}(\cdot,\omega)$:

$$A^\star_N(\omega) = \mathcal{P}_N \left[A_{|Q_N}(\cdot,\omega) \right] \quad \text{almost surely.}$$

In the framework of our Assumption (2.1), we have

$$\forall x \in Q_N, \ A(x,\omega) = \mathcal{A}(x, X_1(\omega), \ldots, X_n(\omega)),$$

where $\{X_k(\omega)\}_{1\leq k\leq n}$ are independent scalar random variables that are all uniformly distributed in $[0,1]$. Setting $X(\omega) = (X_1(\omega), \dots, X_n(\omega))$, we have

$$A^\star_N(\omega) = \mathcal{P}_N\left[\mathcal{A}(\cdot, X(\omega))\right] \in \mathbb{R}^{d\times d}. \tag{2.22}$$

We now choose a vector $\xi \in \mathbb{R}^d$, and introduce the function

$$\begin{array}{rcl} f : \mathbb{R}^n & \rightarrow & \mathbb{R} \\ x & \mapsto & \xi^T \mathcal{P}_N\left[\mathcal{A}(\cdot, x)\right]\xi. \end{array}$$

By construction, we have

$$f(X(\omega)) = \xi^T A^\star_N(\omega)\xi \tag{2.23}$$

and, using the definition (2.8) of $\widetilde{A}^\star_N(\omega)$, we have

$$\frac{1}{2}\left(f(X(\omega)) + f(1 - X(\omega))\right) = \frac{1}{2}\xi^T\left(A^\star_N(\omega) + B^\star_N(\omega)\right)\xi = \xi^T \widetilde{A}^\star_N(\omega)\xi. \tag{2.24}$$

We now show that f is non-decreasing with respect to each of its arguments. We infer from Assumption (2.2) that, for any $\eta \in \mathbb{R}^d$ and any $y \in Q_N$, the function $x \in \mathbb{R}^n \mapsto \eta^T \mathcal{A}(y, x)\eta$ is non-decreasing with respect to each of its arguments. In view of Lemma 2, we thus obtain that, for any $\eta \in \mathbb{R}^d$, the function $x \in \mathbb{R}^n \mapsto \eta^T \mathcal{P}_N\left(\mathcal{A}(\cdot, x)\right)\eta$ is non-decreasing with respect to each of its arguments. As a consequence, f is non-decreasing.

We are now in position to use Corollary 3, which yields

$$\mathbb{V}\text{ar}\left(\frac{1}{2}\left(f(X) + f(1 - X)\right)\right) \leq \frac{1}{2}\mathbb{V}\text{ar}\left(f(X)\right).$$

Using (2.24) and (2.23), we obtain

$$\mathbb{V}\text{ar}\left(\xi^T \widetilde{A}^\star_N\xi\right) = \mathbb{V}\text{ar}\left[\frac{1}{2}\left(f(X) + f(1 - X)\right)\right] \leq \frac{1}{2}\mathbb{V}\text{ar}\left(f(X)\right) = \frac{1}{2}\mathbb{V}\text{ar}\left(\xi^T A^\star_N\xi\right),$$

which concludes the proof of (2.10). □

Proof of Corollary 1: As shown in the proof of Proposition 1, $A^\star_N(\omega)$ and $B^\star_N(\omega)$ share the same law, so their eigenvalues also share the same law, and we thus obtain the first assertion in (2.11).

We now prove the second assertion. As in the proof of Proposition 1, we make use of the operator $\mathcal{P}_N$ that associates to any Q_N-periodic tensor field the effective tensor obtained by periodic homogenization. Expression (2.22) holds.

We next fix some k, $1 \leq k \leq d$, and consider the function

$$\begin{array}{rcl} \Lambda : \mathbb{R}^n & \rightarrow & \mathbb{R} \\ x & \mapsto & k\text{-th eigenvalue of the matrix } \mathcal{P}_N\left[\mathcal{A}(\cdot, x)\right]. \end{array}$$

We clearly have

$$\lambda_k(A,\omega) = \Lambda(X(\omega)) \quad \text{and} \quad \lambda_k(B,\omega) = \Lambda(1 - X(\omega)), \tag{2.25}$$

where $\lambda_k(A,\omega)$ and $\lambda_k(B,\omega)$ denote the k-th eigenvalue (sorted increasingly) of $A^\star_N(\omega)$ and $B^\star_N(\omega)$, respectively.

The function Λ is non-decreasing with respect to each of its arguments. It is an immediate consequence of the proof of Proposition 1 and of the monotonicity of eigenvalues of symmetric matrices recalled in Section 2.3.

Corollary 3 then yields

$$\mathbb{V}\text{ar}\left(\frac{1}{2}\left(\Lambda(X) + \Lambda(1-X)\right)\right) \le \frac{1}{2}\mathbb{V}\text{ar}\left(\Lambda(X)\right).$$

By definition of $\widetilde{\lambda}_k$, and using (2.25), we obtain

$$\begin{aligned}
\mathbb{V}\text{ar}\left(\widetilde{\lambda}_k\right) &= \mathbb{V}\text{ar}\left(\frac{1}{2}\left(\lambda_k(A,\cdot) + \lambda_k(B,\cdot)\right)\right) \\
&= \mathbb{V}\text{ar}\left(\frac{1}{2}\left(\Lambda(X) + \Lambda(1-X)\right)\right) \\
&\le \frac{1}{2}\mathbb{V}\text{ar}\left(\Lambda(X)\right) \\
&\le \frac{1}{2}\mathbb{V}\text{ar}\left(\lambda_k(A,\cdot)\right),
\end{aligned}$$

which concludes the proof of (2.11). □

The proof of Corollary 2 follows the same pattern as that of Corollary 1.

3 A “weakly” stochastic case

In this section, we assume that the matrix A in (1.1) is a *perturbation* of a periodic matrix:

$$A(x,\omega) = A_{\text{per}}(x) + \eta A_1(x,\omega), \tag{3.1}$$

where A_{per} is Q-periodic and δ-Hölder continuous (for some $0 < \delta < 1$), A_1 is stationary in the sense (1.3), and η is a deterministic, supposedly small, parameter. We assume that A, A_{per} and A_1 are all symmetric, uniformly coercive and bounded matrices. We also assume the following special structure for A_1:

$$A_1(x,\omega) = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{Q+k}(x)\, X_k(\omega)\ \text{Id}, \tag{3.2}$$

where Id denotes the identity matrix and $(X_k)_{k\in\mathbb{Z}^d}$ is a sequence of (scalar) independent, identically distributed random variables, satisfying, for some $\alpha > 0$ and $\beta < \infty$, the bound $0 < \alpha \le X_k \le \beta < +\infty$ almost surely and for all k. Some extensions of the above setting will be considered in Section 3.4 below.

Since, in view of (3.1), A is a perturbation of A_{per}, we may expand the homogenized matrix $A^\star$ in powers of η, as shown in [8]. Let us first recall this expansion, at first order.

3.1 Main result

In [8], an expansion of the homogenized matrix (and of all the relevant quantities, such as the corrector function) in terms of a series in powers of η is shown to exist. The setting in [8] is slightly different from the present setting. It is however straightforward to check that our arguments carry over to the present case.

First, the corrector ∇w_p, solution to (1.5), is easily proved to have an expansion in powers of η:

$$\nabla w_p = \nabla w_p^0 + \eta \nabla w_p^1 + \dots, \tag{3.3}$$

where w_p^0 is the unique (up to the addition of a constant) solution to

$$\begin{cases} -\operatorname{div}\left[A_{\rm per}(\nabla w_p^0 + p)\right] = 0, \\ w_p^0 \text{ is } Q\text{-periodic}, \end{cases} \tag{3.4}$$

and w_p^1 is the unique (up to the addition of a random constant) solution to

$$\begin{cases} -\operatorname{div}\left[A_{\rm per}\nabla w_p^1\right] = \operatorname{div}\left[A_1\left(\nabla w_p^0 + p\right)\right] \quad \text{a.s. on } \mathbb{R}^d, \\ \nabla w_p^1 \text{ is stationary}, \\ \displaystyle\int_Q \mathbb{E}(\nabla w_p^1) = 0. \end{cases} \tag{3.5}$$

Second, the homogenized matrix (1.4) satisfies

$$A^\star = A^\star_{\rm per} + \eta A^\star_1 + O(\eta^2), \tag{3.6}$$

where

$$\left[A^\star_{\rm per}\right]_{ij} = \int_Q (\nabla w^0_{e_i} + e_i)^T A_{\rm per}(\nabla w^0_{e_j} + e_j), \tag{3.7}$$

and

$$\begin{aligned} [A^\star_1]_{ij} \quad = \quad & \int_Q \mathbb{E}(\nabla w^1_{e_i})^T A_{\rm per}(\nabla w^0_{e_j} + e_j) + \int_Q (\nabla w^0_{e_i} + e_i)^T A_{\rm per}\mathbb{E}(\nabla w^1_{e_j}) \\ & + \int_Q (\nabla w^0_{e_i} + e_i)^T \mathbb{E}(A_1)(\nabla w^0_{e_j} + e_j). \end{aligned} \tag{3.8}$$

In essence, this specific setting does not give rise to any variance concerns, for two reasons at least. First, as observed in [8], and as evident on (3.8), the knowledge of $\overline{w}_p^1 := \mathbb{E}(\nabla w_p^1)$ is actually sufficient to compute $A_1^\star$. Taking expectations in (3.5), we indeed see that $\overline{w}_p^1$ is solution to the cell problem

$$\begin{cases} -\operatorname{div}\left[A_{\rm per}\nabla \overline{w}_p^1\right] = \operatorname{div}\left[\mathbb{E}(A_1)\left(\nabla w_p^0 + p\right)\right] \quad \text{on } \mathbb{R}^d, \\ \overline{w}_p^1 \text{ is } Q\text{-periodic}, \end{cases} \tag{3.9}$$

which is much easier to solve than the corrector problem (1.5) or its truncated version (1.7), since it is a *deterministic* problem set on a *single* cell. Hence, as pointed out in the introduction, the determination of $A^\star$ comes down, in practice, to solving the deterministic problems (3.4) and (3.9).

Second, in practice, the *exact* corrector w_p, which solves (1.5), is approximated by the solution w_p^N to the truncated problem (1.7), which also has an expansion in powers of η, as shown in [13]: similarly to (3.3), we have

$$\nabla w_p^N = \nabla w_p^0 + \eta \nabla w_p^{1,N} + \dots,$$

where $w_p^{1,N}$ solves the *truncated* problem

$$\begin{cases} -\operatorname{div}\left[A_{\rm per} \nabla w_p^{1,N}\right] = \operatorname{div}\left[A_1 \left(\nabla w_p^0 + p\right)\right], \\ w_p^{1,N} \text{ is } Q_N - \text{periodic.} \end{cases} \tag{3.10}$$

In turn, the approximate homogenized matrix (1.6) satisfies

$$A_N^\star(\omega) = A_{\rm per}^\star + \eta A_{1,N}^\star(\omega) + O(\eta^2),$$

where

$$\begin{aligned} \left[A_{1,N}^\star\right]_{ij}(\omega) = \frac{1}{|Q_N|} \Bigg[\int_{Q_N} & \nabla w_{e_i}^{1,N}(x,\omega)^T A_{\rm per}(x)(\nabla w_{e_j}^0(x) + e_j) dx \\ + \int_{Q_N} & (\nabla w_{e_i}^0(x) + e_i)^T A_1(x,\omega)(\nabla w_{e_j}^0(x) + e_j) dx \\ & + \int_{Q_N} (\nabla w_{e_i}^0(x) + e_i)^T A_{\rm per}(x) \nabla w_{e_j}^{1,N}(x,\omega) dx \Bigg]. \end{aligned} \tag{3.11}$$

Observe now that the first term of (3.11) reads

$$\begin{aligned} \int_{Q_N} \left(\nabla w_{e_i}^{1,N}\right)^T A_{\rm per} \left(\nabla w_{e_j}^0 + e_j\right) &= -\int_{Q_N} w_{e_i}^{1,N} \operatorname{div}\left[A_{\rm per}(\nabla w_{e_j}^0 + e_j)\right] \\ &+ \int_{\partial Q_N} w_{e_i}^{1,N} \left[A_{\rm per}(\nabla w_{e_j}^0 + e_j)\right] \cdot n. \end{aligned}$$

Using (3.4), we see that the first term of the right hand side vanishes. Since $w_{e_i}^{1,N}$, $A_{\rm per}$ and $w_{e_j}^0$ are Q_N-periodic, the second term vanishes as well. Hence (3.11) reads

$$\left[A_{1,N}^\star\right]_{ij}(\omega) = \frac{1}{|Q_N|} \int_{Q_N} (\nabla w_{e_i}^0(x) + e_i)^T A_1(x,\omega)(\nabla w_{e_j}^0(x) + e_j) dx. \tag{3.12}$$

In the specific case (3.2), we thus have

$$\left[A_{1,N}^\star\right]_{ij}(\omega) = \frac{\gamma_{ij}}{|Q_N|} \sum_{|k|_\infty \le N} X_k(\omega),$$

with $\gamma_{ij} = \displaystyle\int_Q \left(\nabla w^0_{e_i} + e_i\right)^T \left(\nabla w^0_{e_j} + e_j\right)$. The variables X_k being independent and identically distributed, we obtain that $\left[A^\star_{1,N}\right]_{ij}$ converges almost surely to $\gamma_{ij}\mathbb{E}(X_0)$, the rate of convergence being given by the central limit theorem. Of course, using (3.12), this argument is not restricted to the form (3.2) of A_1, and can be extended to more general cases.

Observe yet that the fact that the first and third terms of (3.11) vanish strongly relies on the specific equation (and boundary conditions) used to build a numerical approximation of the corrector, and may not be expected in a more general setting.

As announced in the introduction, with a view to use (3.1)-(3.2) to test and further understand our variance reduction approach, we *pretend* in this section not to exploit the various simplifications, and we thus treat the problem entirely stochastically. We make use of this opportunity to derive a series of technical lemmas (see Lemmas 4 and 5) that we believe might be useful for a similar study in a more general setting.

Our aim is to show that, for any fixed N, applying the variance reduction strategy described in Section 2.1, we obtain a better estimate of the approximate homogenized matrix $\mathbb{E}(A^\star_N)$ *using empirical means*, in the spirit of (2.13) and (2.14). When the number of independent realizations M increases to $+\infty$, the rates at which the empirical means (2.13) and (2.14) converge to the expectation are identical, but the prefactor is better in the latter case. In addition, using the specificities of this setting, we are also able to analyze the convergence of the approximation procedure when N goes to $+\infty$.

The main result of this section is the following proposition.

Proposition 2 *Let A be defined by* (3.1)*, where $A_{\rm per}$ is periodic, Hölder continuous, and A_1 satisfies* (3.2)*, with $(X_k)_{k\in\mathbb{Z}^d}$ a sequence of independent, identically distributed scalar random variables. Assume in addition that A, $A_{\rm per}$ and A_1 are symmetric, uniformly coercive and bounded matrices. For any $N \in \mathbb{N}$ and $1 \le i, j \le d$, define*

$$\left[A^{\star,\rm exact}_{1,N}\right]_{ij}(\omega) = \frac{1}{|Q_N|}\left[\int_{Q_N} \nabla w^1_{e_i}(x,\omega)^T A_{\rm per}(x)(\nabla w^0_{e_j}(x) + e_j)dx \right. + \int_{Q_N} (\nabla w^0_{e_i}(x) + e_i)^T A_1(x,\omega)(\nabla w^0_{e_j}(x) + e_j)dx \left. + \int_{Q_N} (\nabla w^0_{e_i}(x) + e_i)^T A_{\rm per}(x)\nabla w^1_{e_j}(x,\omega)dx\right], \quad (3.13)$$

where w^0 and w^1 are solution to (3.4) *and* (3.5) *respectively, and $Q_N = \cup_{|k|_\infty \le N}(Q+k)$ is the cube of size $(2N+1)^d$ centered at the origin. Then, there exist d^2 coefficients $C^{ij}_N > 0$, independent of $(X_k)_{k\in\mathbb{Z}^d}$, such that*

$$\mathbb{V}\mathrm{ar}\left(\left[A^{\star,\rm exact}_{1,N}\right]_{ij}\right) = C^{ij}_N\ \mathbb{V}\mathrm{ar}(X_0). \quad (3.14)$$

In addition, we have

$$C_N^{ij} \leq \frac{C}{|Q_N|}, \tag{3.15}$$

where C does not depend on i, j, and N, and only depends on A_{per}.

The estimate (3.14) above shows that reducing the variance of X_0 (for instance using the technique described in Proposition 1) reduces the variance on $A_{1,N}^{\star,\mathrm{exact}}$. This also gives a quantitative estimate of the variance reduction.

Note that, in the above Proposition, we have used the first order term w_p^1 in the expansion of the *exact* corrector. If this term is replaced by the first order term $w_p^{1,N}$ of the *approximate* corrector (solution to a truncated problem), then we recover $A_{1,N}^\star$ defined by (3.11).

Corollary 4 *Under the assumptions of Proposition 2, we have that*

$$\lim_{N\to\infty} A_{1,N}^{\star,\mathrm{exact}} = A_1^\star \quad a.s.,$$

where $A_1^\star$ is defined by (3.8). Assume in addition that

$$\lim_{N\to\infty} |Q_N|^2 \ \mathbb{V}\mathrm{ar}\left(\left[A_{1,N}^{\star,\mathrm{exact}}\right]_{ij}\right) = +\infty. \tag{3.16}$$

Then, $\dfrac{\left[A_{1,N}^{\star,\mathrm{exact}}\right]_{ij} - [A_1^\star]_{ij}}{\sqrt{\mathbb{V}\mathrm{ar}\left(\left[A_{1,N}^{\star,\mathrm{exact}}\right]_{ij}\right)}}$ *is a random variable that converges in law to a normal Gaussian random variable.*

The proof of Proposition 2 (and of Corollary 4) is the purpose of Section 3.3 below. It makes use of some preliminary results established in Section 3.2. Section 3.4 presents some extensions.

3.2 Decomposition of ∇w^1

In this section, we study the first order term in the expansion (3.3) of the corrector function. Our purpose is to prove the following two lemmas.

Lemma 4 *Let $p \in \mathbb{R}^d$. The problem*

$$\left\{ \begin{array}{l} -\operatorname{div}\left[A_{\mathrm{per}}\nabla\phi_p\right] = \operatorname{div}\left[\mathbf{1}_Q\left(\nabla w_p^0 + p\right)\right], \\ \phi_p \in L^2_{\mathrm{loc}}(\mathbb{R}^d), \quad \nabla\phi_p \in \left(L^2(\mathbb{R}^d)\right)^d, \end{array}\right. \tag{3.17}$$

has a solution, which is unique up to the addition of a constant. Moreover, there exists a constant $C > 0$ such that

$$\forall x \in \mathbb{R}^d \text{ with } |x| \geq 1, \quad |\nabla\phi_p(x)| \leq \frac{C}{|x|^d}. \tag{3.18}$$

Lemma 5 *Let $p \in \mathbb{R}^d$, and let w_p^1 be defined by* (3.5)*. Let $\overline{w}_p^1$ be the unique solution (up to the addition of a constant) to*

$$\left\{ \begin{array}{l} -\operatorname{div}\left[A_{\text{per}} \nabla \overline{w}_p^1\right] = \operatorname{div}\left[\mathbb{E}(A_1)\left(\nabla w_p^0 + p\right)\right], \\ \overline{w}_p^1 \text{ is } Q-\text{periodic}. \end{array} \right. \tag{3.19}$$

Then, we have

$$\nabla w_p^1(x,\omega) = \nabla \overline{w}_p^1(x) + \sum_{k \in \mathbb{Z}^d} \nabla \phi_p(x-k)\left(X_k(\omega) - \mathbb{E}(X_k)\right), \tag{3.20}$$

and the sum in (3.20) *is a convergent series in $L^2(Q \times \Omega)$.*

Proof of Lemma 4: We first prove the existence of a solution to (3.17). The argument is standard and we provide it here for consistency. To begin with, we define a regularized version of the equation: given $\delta > 0$, we consider the problem

$$\left\{ \begin{array}{l} -\operatorname{div}\left[A_{\text{per}} \nabla \phi_{p,\delta}\right] + \delta \phi_{p,\delta} = \operatorname{div}\left[\mathbf{1}_Q\left(\nabla w_p^0 + p\right)\right], \\ \phi_{p,\delta} \in H^1(\mathbb{R}^d). \end{array} \right. \tag{3.21}$$

Applying the Lax-Milgram lemma, it is clear that (3.21) has a unique solution in $H^1(\mathbb{R}^d)$. Next, multiplying the equation by $\phi_{p,\delta}$ and integrating over $\mathbb{R}^d$, we find

$$\int_{\mathbb{R}^d} \nabla \phi_{p,\delta}^T A_{\text{per}} \nabla \phi_{p,\delta} + \delta \int_{\mathbb{R}^d} \phi_{p,\delta}^2 = -\int_Q \nabla \phi_{p,\delta}^T \left(\nabla w_p^0 + p\right),$$

hence, using Cauchy-Schwarz inequality and elementary calculus,

$$\int_{\mathbb{R}^d} |\nabla \phi_{p,\delta}|^2 \leq C \quad \text{and} \quad \int_{\mathbb{R}^d} \phi_{p,\delta}^2 \leq \frac{C}{\delta} \tag{3.22}$$

for some constant $C > 0$ independent of δ. One can thus define a sequence $\delta_n \to 0$ such that

$$\nabla \phi_{p,\delta_n} \rightharpoonup T \in \left(L^2(\mathbb{R}^d)\right)^d$$

as $n \to \infty$, weakly in $\left(L^2(\mathbb{R}^d)\right)^d$. Hence, the equality $\partial_i \partial_j \phi_{p,\delta_n} = \partial_j \partial_i \phi_{p,\delta_n}$ passes to the limit in the sense of distributions and implies $\partial_i T_j = \partial_j T_i$. This implies that $T = \nabla \phi_p$ for some $\phi_p \in L^2_{\text{loc}}(\mathbb{R}^d)$, with $\nabla \phi_p \in \left(L^2(\mathbb{R}^d)\right)^d$. Moreover, for any $\xi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \nabla \phi_{p,\delta_n}^T A_{\text{per}} \nabla \xi + \delta \int_{\mathbb{R}^d} \phi_{p,\delta_n} \xi = -\int_Q \nabla \xi^T \left(\nabla w_p^0 + p\right).$$

Passing to the limit $n \to \infty$ in this equation, we find that ϕ_p is a solution to (3.17), in the appropriate functional spaces. This concludes the proof of existence.

Proving uniqueness (up to the addition of a constant) of the solution to (3.17) amounts to proving that

$$\left\{ \begin{array}{l} -\operatorname{div}\left[A_{\text{per}} \nabla \phi\right] = 0, \\ \phi \in L^2_{\text{loc}}(\mathbb{R}^d), \quad \nabla \phi \in \left(L^2(\mathbb{R}^d)\right)^d, \end{array} \right. \tag{3.23}$$

implies $\nabla\phi \equiv 0$. For this purpose, we consider $\chi \in C^\infty(\mathbb{R}^d)$ such that $\chi = 1$ in B_R, $\chi = 0$ in B_{R+1}^c, $\chi \geq 0$ on $\mathbb{R}^d$ and $|\nabla\chi| \leq 1$. Multiplying (3.23) by $\chi\phi$, we find that

$$\int_{\mathbb{R}^d} \left(\nabla\phi^T A_{\rm per} \nabla\phi\right) \chi + \int_{\mathbb{R}^d} \nabla\chi^T A_{\rm per} \nabla\phi\phi = 0,$$

hence

$$m_{\rm per} \int_{B_R} |\nabla\phi|^2 \leq M_{\rm per} \int_{B_{R+1}\setminus B_R} |\nabla\phi|\ |\phi|,$$

where $M_{\rm per} = \|A_{\rm per}\|_{L^\infty}$ and $m_{\rm per} > 0$ is the coercivity constant of $A_{\rm per}$. Next, we point out that the above computations are also valid if we replace ϕ by $\phi - \phi_R$, where ϕ_R is the constant

$$\phi_R = \frac{1}{|B_{R+1} \setminus B_R|} \int_{B_{R+1}\setminus B_R} \phi.$$

Hence, using the Cauchy-Schwarz inequality,

$$\int_{B_R} |\nabla\phi|^2 \leq \frac{M_{\rm per}}{m_{\rm per}} \left(\int_{B_{R+1}\setminus B_R} |\nabla\phi|^2\right)^{1/2} \left(\int_{B_{R+1}\setminus B_R} (\phi - \phi_R)^2\right)^{1/2}.$$

We next make use of the Poincaré-Wirtinger inequality [16, page 164] on the bounded domain $B_{R+1} \setminus B_R$:

$$\|\phi - \phi_R\|_{L^2(B_{R+1}\setminus B_R)} \leq C\|\nabla\phi\|_{L^2(B_{R+1}\setminus B_R)},$$

for some constant C independent of R and ϕ. We thus obtain

$$\int_{B_R} |\nabla\phi|^2 \leq C\, \frac{M_{\rm per}}{m_{\rm per}} \int_{B_{R+1}\setminus B_R} |\nabla\phi|^2.$$

As $\nabla\phi \in \left(L^2(\mathbb{R}^d)\right)^d$, we have that $\displaystyle\lim_{R\to\infty} \int_{B_{R+1}\setminus B_R} |\nabla\phi|^2 = 0$. The above bound thus yields $\displaystyle\lim_{R\to\infty} \int_{B_R} |\nabla\phi|^2 = 0$. As a consequence, $\displaystyle\int_{\mathbb{R}^d} |\nabla\phi|^2 = 0$, which implies $\nabla\phi \equiv 0$. This concludes the proof of uniqueness.

It now remains to prove (3.18). Let G be the Green function associated with the operator $-\operatorname{div}[A_{\rm per}\nabla\cdot]$, that is, the solution of

$$-\operatorname{div}_x \left(A_{\rm per}(x)\nabla_x G(x,y)\right) = \delta_y(x), \tag{3.24}$$

with $\nabla_x G(\cdot, y) \in L^p(\mathbb{R}^d \setminus B_r(y))$, for all $p > d/(d-1)$ and $r > 0$. Such a solution is unique according to [18, Theorem 1.1] (actually, the results of [18] are cited only for $d \geq 3$, but the existence proof there carries through to the case $d = 2$).

We then have

$$\begin{aligned} \phi_p(x) &= \int_{\mathbb{R}^d} G(x,y) \operatorname{div}_y \left[\mathbf{1}_Q(y) \left(\nabla w_p^0(y) + p\right)\right] dy \\ &= -\int_Q \nabla_y G(x,y) \left(\nabla w_p^0(y) + p\right) dy. \end{aligned} \tag{3.25}$$

We now recall that, if $d \geq 3$, according to [18, Theorem 3.3] (see also [4, Sec. 2] and [2, Theorem 13]), G satisfies the estimate:

$$\forall |x-y| \geq 1, \quad |G(x,y)| \leq C|x-y|^{2-d}.$$

In addition (see [2, Theorem 13]), we have, in the case $d = 2$,

$$\forall |x-y| \geq 1, \quad |G(x,y)| \leq C\left(1 + \log|x-y|\right).$$

We also recall (see [1]) that, for any $d \geq 2$,

$$\forall |x-y| \geq 1, \quad |\nabla_x G(x,y)| + |\nabla_y G(x,y)| \leq C|x-y|^{1-d} \tag{3.26}$$

and

$$\forall |x-y| \geq 1, \quad |\nabla_x \nabla_y G(x,y)| \leq C|x-y|^{-d}, \tag{3.27}$$

for some constant $C > 0$. We now write

$$\nabla \phi_p(x) = -\int_Q \nabla_x \nabla_y G(x,y) \left(\nabla w_p^0(y) + p\right) \, dy,$$

which, using (3.27), gives (3.18). This concludes the proof of Lemma 4. □

Proof of Lemma 5: We first point out that (3.19) admits a solution in $H^1_\square(Q)$. It is a simple consequence of the Lax-Milgram lemma. Next, we prove that the sum in (3.20) is a convergent series in $L^2(Q \times \Omega)$. For this purpose, we compute the norm of the remainder of the series, using the independence of the X_k:

$$\left\| \sum_{|k| \geq N+1} \nabla \phi_p(\cdot - k)\left(X_k - \mathbb{E}(X_k)\right) \right\|^2_{L^2(Q\times\Omega)} = \mathbb{V}\mathrm{ar}(X_0) \sum_{|k| \geq N+1} \int_{Q+k} |\nabla \phi_p|^2,$$

which converges to 0 as $N \to \infty$ since $\nabla \phi_p \in \left(L^2(\mathbb{R}^d)\right)^d$. Hence, the right-hand side of (3.20) defines a function $T \in \left(L^2(Q \times \Omega)\right)^d$. As $\partial_i T_j = \partial_j T_i$, there exists a function $\widetilde{w}_p^1$ such that

$$\nabla \widetilde{w}_p^1 = T = \nabla \overline{w}_p^1 + \sum_{k \in \mathbb{Z}^d} \nabla \phi_p(x-k)\left(X_k(\omega) - \mathbb{E}(X_k)\right).$$

As $\overline{w}_p^1$ is Q-periodic, we infer from the above equality that

$$\nabla \widetilde{w}_p^1 \text{ is stationary and } \int_Q \mathbb{E}(\nabla \widetilde{w}_p^1) = 0. \tag{3.28}$$

Next, we compute

$$A_{\mathrm{per}} \nabla \widetilde{w}_p^1 = \sum_{k \in \mathbb{Z}^d} A_{\mathrm{per}} \nabla \phi_p(x-k)\left(X_k - \mathbb{E}(X_k)\right) + A_{\mathrm{per}} \nabla \overline{w}_p^1.$$

Taking the divergence of this equation, we thus find that

$$\begin{aligned}
-\operatorname{div}\left[A_{\mathrm{per}}\nabla \widetilde{w}_p^1\right] &= \sum_{k\in\mathbb{Z}^d} -\operatorname{div}\left[A_{\mathrm{per}}\nabla\phi_p(x-k)\right](X_k-\mathbb{E}(X_k)) \\
&\quad -\operatorname{div}\left[A_{\mathrm{per}}\nabla\overline{w}_p^1\right] \\
&= \sum_{k\in\mathbb{Z}^d} \operatorname{div}\left[\mathbf{1}_{Q+k}\left(\nabla w_p^0+p\right)\right](X_k-\mathbb{E}(X_k)) \\
&\quad +\operatorname{div}\left[\nabla w_p^0+p\right]\mathbb{E}(X_k) \\
&= \operatorname{div}\left[A_1\left(\nabla w_p^0+p\right)\right]. \qquad (3.29)
\end{aligned}$$

Collecting (3.28) and (3.29), we see that $\widetilde{w}_p^1$ solves (3.5). As the solution to this equation is unique up to the addition of a (possibly random) constant $C(\omega)$, we obtain that $\widetilde{w}_p^1 = w_p^1 + C(\omega)$, hence proving (3.20). □

3.3 Variance of $A_{1,N}^{\star,\mathrm{exact}}$

We are now in position to prove Proposition 2. Let us briefly explain the structure of the argument. Exploiting the fact that the randomness in $A(x,\omega)$ only appears as a small perturbation, it turns out that $A_{1,N}^{\star,\mathrm{exact}}$, which is the *first order* correction of the homogenized matrix, depends *linearly* of the random variables $X_k(\omega)$ involved in the matrix $A(x,\omega)$. This can be clearly seen on (3.13), (3.2) and (3.20). As a consequence, it is possible to explicitly, analytically, write how the random matrix $A_{1,N}^{\star,\mathrm{exact}}$ depends on the random variables X_k. This explicit expression yields the relation between the variance of $A_{1,N}^{\star,\mathrm{exact}}$ and that of X_k (see (3.14) above).

Proof of Proposition 2: Using (3.13) and the expression (3.20) of ∇w_p^1 provided by Lemma 5, we have

$$\begin{aligned}
\left[A_{1,N}^{\star,\mathrm{exact}}\right]_{ij} &= \frac{1}{|Q_N|}\sum_{|k|\le N}\Bigg[\int_Q \left(\nabla w_{e_i}^0+e_i\right)^T\left(\nabla w_{e_j}^0+e_j\right)X_k \\
&+ \sum_{\ell\in\mathbb{Z}^d}\int_{Q+k}\nabla\phi_{e_i}(x-\ell)^T A_{\mathrm{per}}(x)\left(\nabla w_{e_j}^0(x)+e_j\right)dx\,(X_\ell-\mathbb{E}(X_\ell)) \\
&+ \sum_{\ell\in\mathbb{Z}^d}\int_{Q+k}\left(\nabla w_{e_i}^0(x)+e_i\right)^T A_{\mathrm{per}}(x)\nabla\phi_{e_j}(x-\ell)dx\,(X_\ell-\mathbb{E}(X_\ell))\Bigg] \\
&+ \frac{1}{|Q_N|}\int_{Q_N}\left(\nabla\overline{w}_{e_i}^1\right)^T A_{\mathrm{per}}\left(\nabla w_{e_j}^0+e_j\right) \\
&+ \frac{1}{|Q_N|}\int_{Q_N}\left(\nabla w_{e_i}^0+e_i\right)^T A_{\mathrm{per}}\nabla\overline{w}_{e_j}^1.
\end{aligned}$$

Setting, for any $1 \leq i, j \leq d$ and $m \in \mathbb{Z}^d$,

$$\gamma_{ij} = \int_Q \left(\nabla w^0_{e_i} + e_i\right)^T \left(\nabla w^0_{e_j} + e_j\right), \tag{3.30}$$

$$\alpha_{ij}(m) = \int_Q \nabla \phi_{e_i}(x-m)^T A_{\rm per}(x) \left(\nabla w^0_{e_j}(x) + e_j\right) dx, \tag{3.31}$$

and using that, for any $p \in \mathbb{R}^d$, the functions w^0_p, $\overline{w}^1_p$ and $A_{\rm per}$ are Q-periodic, we obtain

$$\begin{aligned}\left[A^{\star,\rm exact}_{1,N}\right]_{ij} &= \frac{1}{|Q_N|} \sum_{|k| \leq N} \left[\gamma_{ij} X_k + \sum_{\ell \in \mathbb{Z}^d} \left(\alpha_{ij}(\ell) + \alpha_{ji}(\ell)\right)\left(X_{\ell+k} - \mathbb{E}(X_{\ell+k})\right)\right] \\ &+ \frac{1}{|Q|} \int_Q \left(\nabla \overline{w}^1_{e_i}\right)^T A_{\rm per} \left(\nabla w^0_{e_j} + e_j\right) \\ &+ \frac{1}{|Q|} \int_Q \left(\nabla w^0_{e_i} + e_i\right)^T A_{\rm per} \nabla \overline{w}^1_{e_j}.\end{aligned}$$

Introducing

$$\beta_{ij}(\ell) = \alpha_{ij}(\ell) + \alpha_{ji}(\ell), \quad Y_k = X_k - \mathbb{E}(X_k), \tag{3.32}$$

we next obtain, using (3.8), that

$$\left[A^{\star,\rm exact}_{1,N}\right]_{ij} = [A^\star_1]_{ij} + \frac{1}{|Q_N|} \sum_{|k| \leq N} \left[\gamma_{ij} Y_k + \sum_{\ell \in \mathbb{Z}^d} \beta_{ij}(\ell) Y_{\ell+k}\right]. \tag{3.33}$$

Note that

$$\begin{aligned}|\beta_{ij}(\ell)|^2 &\leq 2\left(\alpha^2_{ij}(\ell) + \alpha^2_{ji}(\ell)\right) \\ &\leq 2\|A_{\rm per}\|^2_{L^\infty}\; C(w^0) \int_{Q+\ell} \left(|\nabla \phi_{e_i}|^2 + |\nabla \phi_{e_j}|^2\right),\end{aligned} \tag{3.34}$$

where $C(w^0) = \max_j \|\nabla w^0_{e_j} + e_j\|_{L^2(Q)}$. As $\nabla \phi_p$ belongs to $\left(L^2(\mathbb{R}^d)\right)^d$ for any $p \in \mathbb{R}^d$, we first deduce that, for any $1 \leq i, j \leq d$,

$$\sum_{\ell \in \mathbb{Z}^d} |\beta_{ij}(\ell)|^2 < \infty. \tag{3.35}$$

Furthermore, we have

$$\mathbb{E}\left(|\beta_{ij}(\ell) Y_{k+\ell}|^2\right) = \mathbb{V}\mathrm{ar}\left(X_0\right)\; |\beta_{ij}(\ell)|^2.$$

Since Y_k are independent, identically distributed random variables of zero mean, we infer from (3.35) that the sum $\sum_{\ell \in \mathbb{Z}^d} \beta_{ij}(\ell) Y_{\ell+k}$ in (3.33) is convergent in $L^2(\Omega)$, for any k.

As $\mathbb{E}(Y_k) = 0$, we infer from (3.33) that $\mathbb{E}\left(A_{1,N}^{\star,\mathrm{exact}}\right) = A_1^\star$. Since $A_1^\star$, γ_{ij} and $\beta_{ij}(\ell)$ are deterministic and the variables $(Y_k)_{k\in\mathbb{Z}^d}$ are independent, identically distributed, and of mean zero, we deduce from (3.33) that

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left(\left[A_{1,N}^{\star,\mathrm{exact}}\right]_{ij}\right) &= \mathbb{V}\mathrm{ar}(X_0)\frac{\gamma_{ij}^2}{|Q_N|} \\
&+ \frac{1}{|Q_N|^2}\sum_{|k|\le N}\sum_{|k'|\le N}\mathbb{C}\mathrm{ov}\left(\sum_{\ell\in\mathbb{Z}^d}\beta_{ij}(\ell)Y_{\ell+k}, \sum_{\ell'\in\mathbb{Z}^d}\beta_{ij}(\ell')Y_{\ell'+k'}\right) \\
&+ \frac{2}{|Q_N|^2}\sum_{|k|\le N}\sum_{|k'|\le N}\mathbb{C}\mathrm{ov}\left(\gamma_{ij}Y_{k'}, \sum_{\ell\in\mathbb{Z}^d}\beta_{ij}(\ell)Y_{\ell+k}\right), \\
&= \mathbb{V}\mathrm{ar}(X_0)\frac{\gamma_{ij}^2}{|Q_N|} \\
&+ \frac{1}{|Q_N|^2}\sum_{|k|\le N}\sum_{|k'|\le N}\sum_{\ell\in\mathbb{Z}^d}\beta_{ij}(\ell)\beta_{ij}(\ell+k-k')\mathbb{V}\mathrm{ar}(X_0) \\
&+ \frac{2\gamma_{ij}}{|Q_N|^2}\sum_{|k|\le N}\sum_{|k'|\le N}\beta_{ij}(k'-k)\mathbb{V}\mathrm{ar}(X_0) \\
&= C_N^{ij}\ \mathbb{V}\mathrm{ar}(X_0),
\end{aligned}$$

where

$$C_N^{ij} = \frac{\gamma_{ij}^2}{|Q_N|} + \frac{1}{|Q_N|^2}\sum_{|k|\le N}\sum_{|k'|\le N}\sum_{\ell\in\mathbb{Z}^d}\beta_{ij}(\ell)\beta_{ij}(\ell+k-k') + \frac{2\gamma_{ij}}{|Q_N|^2}\sum_{|k|\le N}\sum_{|k'|\le N}\beta_{ij}(k'-k). \quad (3.36)$$

Note that the sum $\sum_{\ell\in\mathbb{Z}^d}\beta_{ij}(\ell)\beta_{ij}(\ell+k-k')$ is finite. This is a simple consequence of the Cauchy-Schwarz inequality and (3.35). Hence C_N^{ij} is finite and we thus have proved (3.14).

We next prove (3.15). For this purpose, we examine each term of the right-hand side of (3.36) separately. It is clear that the first one satisfies (3.15). The second one writes

$$\frac{1}{|Q_N|^2}\sum_{\ell\in\mathbb{Z}^d}\sum_{|k|\le N}\sum_{|k'|\le N}\beta_{ij}(\ell+k')\beta_{ij}(\ell+k) = \sum_{\ell\in\mathbb{Z}^d}\left[\frac{1}{|Q_N|}\sum_{|k|\le N}\beta_{ij}(k+\ell)\right]^2. \quad (3.37)$$

We first consider the large values of ℓ, using the fact that, according to (3.18), $|\beta_{ij}(k+\ell)| \le C|k+\ell|^{-d}$ for some constant C independent of ℓ and k. Hence, we have

$$
\begin{aligned}
\sum_{|\ell|\geq 2N}\left(\frac{1}{|Q_N|}\sum_{|k|\leq N}\beta_{ij}(k+\ell)\right)^2 &\leq \sum_{|\ell|\geq 2N}\left(\frac{1}{|Q_N|}\sum_{|k|\leq N}\frac{C}{|k+\ell|^d}\right)^2 \\
&\leq \sum_{|\ell|\geq 2N}\frac{C^2}{(|\ell|-N)^{2d}} \\
&\leq \sum_{|\ell|\geq 2N}\frac{2^{2d}C^2}{|\ell|^{2d}} \\
&\leq \frac{C'}{N^d}, \qquad (3.38)
\end{aligned}
$$

for some constant C' independent of N. Here, we have used the fact that if $|\ell| \geq 2N$, then $|\ell| - N \geq |\ell|/2$.

Next, we consider the small values of ℓ. For this purpose, we note that

$$
\begin{aligned}
\sum_{|k|\leq N}\beta_{ij}(k+\ell) &= \sum_{|k|\leq N}\left[\int_{Q+\ell+k}\left(\nabla w^0_{e_i}(x)+e_i\right)^T A_{\text{per}}(x)\nabla\phi_{e_j}(x)dx\right. \\
&\quad \left.+\int_{Q+\ell+k}\nabla\phi_{e_i}(x)^T A_{\text{per}}(x)\left(\nabla w^0_{e_j}(x)+e_j\right)dx\right] \\
&= \int_{Q_N+\ell}\left(\nabla w^0_{e_i}(x)+e_i\right)^T A_{\text{per}}(x)\nabla\phi_{e_j}(x)dx \\
&\quad +\int_{Q_N+\ell}\nabla\phi_{e_i}(x)^T A_{\text{per}}(x)\left(\nabla w^0_{e_j}(x)+e_j\right)dx.
\end{aligned}
$$

We use an integration by parts and (3.4) to write this as

$$
\begin{aligned}
\sum_{|k|\leq N}\beta_{ij}(k+\ell) = \int_{\partial(Q_N+\ell)}\phi_{e_j} n^T A_{\text{per}}\left(\nabla w^0_{e_i}+e_i\right) \\
+\int_{\partial(Q_N+\ell)}\phi_{e_i}\left(\nabla w^0_{e_j}+e_j\right)^T A_{\text{per}} n.
\end{aligned}
$$

Recalling (3.25) and (3.26), one easily proves that $|\phi_p(x)| \leq C/(1+|x|^{d-1})$, hence

$$
\left|\sum_{|k|\leq N}\beta_{ij}(k+\ell)\right| \leq C\int_{\partial(Q_N+\ell)}\frac{dx}{1+|x|^{d-1}},
$$

where C is a constant which does not depend on N and ℓ, and dx is the Lebesgue measure on the $(d-1)$-dimension surface $\partial(Q_N+\ell)$. In order to estimate the right-hand side of this inequality, we change variables in the integral, getting

$$\begin{aligned}\left|\sum_{|k|\le N}\beta_{ij}(k+\ell)\right| &\le C\int_{\partial(Q+\frac{\ell}{N})}\frac{N^{d-1}dy}{1+N^{d-1}|y|^{d-1}},\\ &= C\int_{\partial Q}\frac{dy}{\frac{1}{N^{d-1}}+\left|y-\frac{\ell}{N}\right|^{d-1}}. \qquad (3.39)\end{aligned}$$

We use a Riemann sum to write

$$\begin{aligned}\frac{1}{|Q_N|}\sum_{|\ell|\le 2N}\left(\sum_{|k|\le N}\beta_{ij}(k+\ell)\right)^2 &\le \frac{C}{|Q_N|}\sum_{|\ell|\le 2N}\left(\int_{\partial Q}\frac{dy}{\left|y-\frac{\ell}{N}\right|^{d-1}}\right)^2\\ &\to 2C\int_{2Q}\left(\int_{\partial Q}\frac{dy}{\left|y-z\right|^{d-1}}\right)^2 dz. \qquad (3.40)\end{aligned}$$

Note that the integrals in (3.40) converge since ∂Q is a $(d-1)$-dimensional manifold, hence

$$\int_{\partial Q}\frac{dy}{|y-z|^{d-1}}\le C\log\left(\mathrm{dist}(z,\partial Q)\right),$$

which, as a function of z, belongs to $L^2(2Q)$. Collecting (3.38) and (3.40), we find

$$\sum_{\ell\in\mathbb{Z}^d}\left(\frac{1}{|Q_N|}\sum_{|k|\le N}\beta_{ij}(k+\ell)\right)^2\le\frac{C}{|Q_N|},$$

thereby proving that the second term of the right-hand side of (3.36) satisfies (3.15).

For the last term of the right-hand side of (3.36), we return to (3.39), and use it exactly as in (3.40) to prove

$$\begin{aligned}\frac{1}{|Q_N|}\sum_{|k'|\le N}\left|\sum_{|k|\le N}\beta_{ij}(k+k')\right| &\le \frac{1}{|Q_N|}\sum_{|k'|\le N}\left|\int_{\partial Q}\frac{dy}{\left|y-\frac{k'}{N}\right|^{d-1}}\right|\\ &\longrightarrow \int_Q\left|\int_{\partial Q}\frac{dy}{\left|y-z\right|^{d-1}}\right|dz.\end{aligned}$$

Collecting all the above estimates concludes the proof of (3.15). □

Proof of Corollary 4: By construction (see (3.13)),

$$\left[A_{1,N}^{\star,\mathrm{exact}}\right]_{ij}(\omega)=\frac{1}{|Q_N|}\int_{Q_N}F_{ij}(x,\omega)\,dx,$$

where F_{ij} is a stationary function. The ergodic theorem thus applies:

$$\lim_{N\to\infty}\left[A_{1,N}^{\star,\mathrm{exact}}\right]_{ij}=\mathbb{E}\left(\int_Q F_{ij}(x,\cdot)\,dx\right)=[A_1^\star]_{ij}\quad\text{a.s.},$$

which proves the first assertion.

We now prove the second assertion. In view of (3.33), we have

$$S_N^{ij} := \left[A_{1,N}^{\star,\text{exact}}\right]_{ij} - [A_1^\star]_{ij} = \frac{1}{|Q_N|} \sum_{|k|\leq N} \theta_k^{ij} \tag{3.41}$$

with

$$\theta_k^{ij} = \gamma_{ij} Y_{-k} + \sum_{\ell\in\mathbb{Z}^d} \beta_{ij}(\ell) Y_{\ell-k} = \sum_{\ell\in\mathbb{Z}^d} \overline{\beta}_{ij}(\ell+k) Y_\ell, \tag{3.42}$$

where

$$\overline{\beta}_{ij}(\ell) = \beta_{ij}(\ell) \quad \text{if } \ell \neq 0, \quad \overline{\beta}_{ij}(0) = \beta_{ij}(0) + \gamma_{ij},$$

with Y_k and β_{ij} defined by (3.31)-(3.32), and γ_{ij} by (3.30). Thus, Y_k are independent, identically distributed random variables with $\mathbb{E}(Y_0) = 0$ and $\mathbb{E}(Y_0^2) < \infty$. We infer from (3.35) that $\displaystyle\sum_{\ell\in\mathbb{Z}^d} \left(\overline{\beta}_{ij}(\ell)\right)^2 < \infty$. Hence, equations (3.41) and (3.42) exactly correspond to the setting considered in [24].

Observe now that

$$\begin{aligned}
|Q_N|^2 \, \mathbb{V}\text{ar}\left(\left[A_{1,N}^{\star,\text{exact}}\right]_{ij}\right) &= \mathbb{E}\left[\left(\sum_{|k|\leq N} \theta_k^{ij}\right)^2\right] \\
&= \sum_{|k|\leq N} \sum_{|k'|\leq N} \mathbb{E}\left[\theta_k^{ij} \; \theta_{k'}^{ij}\right] \\
&= \sum_{|k|\leq N} \sum_{|k'|\leq N} \sum_{\ell\in\mathbb{Z}^d} \overline{\beta}_{ij}(\ell+k)\overline{\beta}_{ij}(\ell+k')\mathbb{E}(Y_0^2) \\
&= \mathbb{E}(Y_0^2) \sum_{\ell\in\mathbb{Z}^d} \left(\sum_{|k|\leq N} \overline{\beta}_{ij}(\ell+k)\right)^2 .
\end{aligned}$$

Our assumption (3.16) hence reads $\displaystyle\lim_{N\to\infty} \sum_{\ell\in\mathbb{Z}^d} \left(\sum_{|k|\leq N} \overline{\beta}_{ij}(\ell+k)\right)^2 = \infty$. We are thus in position to apply [24, Theorem 1], which yields the second assertion. □

3.4 Extensions

We briefly mention here some possible extensions of Proposition 2.

Our proof, performed in the setting (3.2), can be extended to the case

$$A_1(x,\omega) = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{Q+k}(x) B_k(\omega),$$

where $(B_k)_{k\in\mathbb{Z}^d}$ is a sequence of independent, identically distributed symmetric matrices, that are uniformly coercive and bounded. A result similar to (3.14)–(3.15) is then valid, in the following sense. Define the matrix $\Sigma \in \mathbb{R}^{d^2\times d^2}$ as the covariance matrix of A_1, that is

$$\Sigma_{mn} = \mathbb{C}\mathrm{ov}\left([A_1]_{ij}, [A_1]_{k\ell}\right), \quad \text{where} \quad \left\{\begin{array}{l} m = (i-1)d + j, \\ n = (k-1)d + \ell. \end{array}\right.$$

Similarly, define $\Sigma_N \in \mathbb{R}^{d^2\times d^2}$ as the covariance matrix of $A^{\star,\mathrm{exact}}_{1,N}$. Then we have

$$\Sigma_N = C_N^T \, \Sigma \, C_N, \tag{3.43}$$

where the matrix $C_N \in \mathbb{R}^{d^2\times d^2}$ does not depend on the random matrices B_k, and satisfies the bound

$$|C_N| \le \frac{C}{\sqrt{|Q_N|}}, \tag{3.44}$$

for C a constant independent of N. Note that in order to prove the above property, one needs to carry out computations similar to those in Sections 3.2 and 3.3, except that (3.17) is now a system of d^2 partial differential equations indexed by $1 \le n, m \le d$:

$$\left\{\begin{array}{l} -\operatorname{div}\left[A_{\mathrm{per}}\nabla\phi_p^{m,n}\right] = \operatorname{div}\left[E_{mn}\mathbf{1}_Q\left(\nabla w_p^0 + p\right)\right], \\ \phi_p^{m,n} \in L^2_{\mathrm{loc}}(\mathbb{R}^d), \quad \nabla\phi_p^{m,n} \in \left(L^2(\mathbb{R}^d)\right)^d, \end{array}\right. \tag{3.45}$$

where E_{mn} is the canonical basis of the space of matrices of size $d\times d$. This problem is easily shown to have a unique solution, by the same method as in the proof of Lemma 4, with the estimate (3.18). Then, Lemma 5 reads

$$\nabla w_p^1(x,\omega) = \nabla\overline{w}_p^1(x) + \sum_{k\in\mathbb{Z}^d}\sum_{1\le m,n\le d}\nabla\phi_p^{m,n}(x-k)\left(B_k^{mn}(\omega) - \mathbb{E}(B_k^{mn})\right),$$

with $\overline{w}_p^1$ defined by (3.19), and where $B_k^{mn}(\omega)$ is the coefficient mn of the matrix $B_k(\omega)$. This decomposition may then be used to prove (3.43) and (3.44) above, by the same method as in Section 3.3.

Another observation is the following. Return to the case (3.2), but assume now that X_k is a stationary sequence (we do not assume independence anymore). Then a similar result holds. In such a case, one needs to assume that

$$\mathcal{E} = \sum_{k\in\mathbb{Z}^d}|\mathbb{C}\mathrm{ov}(X_0, X_k)| < +\infty, \tag{3.46}$$

and equality (3.14) is then replaced by

$$\mathbb{V}\mathrm{ar}\left(\left[A^{\star,\mathrm{exact}}_{1,N}\right]_{ij}\right) \le \mathcal{E}\, C_N^{ij}, \tag{3.47}$$

where C_N^{ij} is independent from the random variables X_k and satisfies (3.15).

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