Modeling principles

Exercise sheet – September 23, 2025

1 Heat equation

We wish to understand whether numerical schemes for the heat equation share the same qualitative properties as the exact equation. We consider a 1D situation, take homogeneous Dirichlet boundary conditions, and discretize the problem on $\Omega =]0,1[$ by the simplest method, a finite difference method. We thus consider (1), with $\alpha > 0$, the time-step Δt and the grid size Δx, with $1 = (N+1)\,\Delta x$ for some $N \in \mathbb{N}$. In what follows, u_j^n is the approximation of $u(n\,\Delta t, j\,\Delta x)$.

Exercice 1 (Explicit Euler scheme)**.** *The explicit Euler scheme is (2). We also assume that the initial condition satisfies $u_{j=0}^{n=0} = u_{j=N+1}^{n=0} = 0$. We are going to show that (2) is stable in L^∞ norm if and only if (3) is satisfied.*

1. *Recast (2) in the form (4).*
2. *Assuming that the CFL condition is satisfied and that the initial condition satisfies $0 \le u_j^0 \le M$ for all $0 \le j \le N+1$ (for some constant $M > 0$), show by recursion that the same inequalities hold for all subsequent times.*
3. *Deduce that $\max\limits_{0 \le j \le N+1} |u_j^n| \le \|U^0\|_\infty$. What about the L^∞ stability?*
4. *Assume now that the CFL condition does not hold. Recast (2) as $U^n = M\,U^{n-1}$, where the matrix M is given in the french version, and where $U^n \in \mathbb{R}^N$ collects the internal degrees of freedom.*
5. *Consider the vector*
$$\forall 1 \le j \le N, \quad \xi_j = (-1)^j.$$
Show that $\xi^T\xi = N$, that $M\,\xi = (3c-1, 1-4c, 4c-1, \dots)^T$ and deduce that $\xi^T M\,\xi = 2(1-3c) + (N-2)(1-4c)$.
6. *Show that $-\dfrac{\xi^T M\,\xi}{\xi^T\xi} = (4c-1)\dfrac{N-2+2(3c-1)/(4c-1)}{N}$.*
7. *Since $2c > 1$, we have $4c-1 > 1$. Show that there exists $N_0(c)$ such that, if $N \ge N_0(c)$, then $-\dfrac{\xi^T M\,\xi}{\xi^T\xi} > 1$, which means that the smallest eigenvalue of M satisfies*
$$\lambda_{\min} = \inf_{U \in \mathbb{R}^N} \frac{U^T M\,U}{U^T U} \le \frac{\xi^T M\,\xi}{\xi^T\xi} < -1.$$
8. *For the particular choice $U^0 = U_{\min}$ (eigenvector associated to the eigenvalue $\lambda_{\min}$), show that $U^n = \lambda_{\min}^n U^0$ and hence that $|u_j^n| = |\lambda_{\min}|^n\,|u_j^0| \underset{n\to+\infty}{\longrightarrow} +\infty$ for any $1 \le j \le N$.*
9. *Conclude on the L^∞ stability.*

2 Transport equation

Consider the transport equation (5) for some $V > 0$.

Exercice 2 (Upwind scheme stability)**.** *We discretize (5) with the* upwind scheme *(6).*

1. *Using that*
$$\frac{u_j^n - u_{j-1}^n}{\Delta x} = \partial_x v - \frac{\Delta x}{2} \partial_{xx} v + O(\Delta x^2),$$
and neglecting the quadratic remainder terms, show that the equivalent equation is (7).

2. *Under which condition (on the sign of the prefactor in front of $\partial_{xx} v$) does the equation lead to a stable solution? Deduce the CFL condition.*

3. *Consider (5) on $\Omega = (0, 2\pi)$ with periodic boundary conditions. Take the initial condition $u(t = 0, x) = \exp(ikx)$ for some $k \in \mathbb{N}^\star$ (which is indeed periodic).*

   - *Identify the solution $u(t, x)$.*
   - *What can you say about $\|u(t, \cdot)\|_{L^2(\Omega)}$ as a function of time?*

4. *Consider (7) on $\Omega = (0, 2\pi)$ with periodic boundary conditions. Take the initial condition $v(t = 0, x) = \exp(ikx)$ for some $k \in \mathbb{N}^\star$ (which is indeed periodic).*

   - *Look for a solution to (7) in the form $v(t, x) = \exp(p(t) + ikx)$ and identify $p(t)$;*
   - *What can you say about $\|v(t, \cdot)\|_{L^2(\Omega)}$ as a function of time?*