Tensor-based algorithms for the model reduction of high dimensional problems: application to stochastic fluid problems

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General context

High dimensional problem in tensor spaces:

Given $\boldsymbol{b} \in Y'$, seek $\boldsymbol{u} \in X$ solution of $L\boldsymbol{u} = \boldsymbol{b}$.

- $L : X \to Y'$ a linear and continuous isomorphism
- $X = \overline{{}_a \bigotimes_{\mu=1}^{s} X_\mu}^{||\cdot||_X}$ (resp. Y) a tensor Hilbert space of dual X' (resp. Y').
- X (resp. Y) is equipped with the norm $||\cdot||_X$ (resp. $||\cdot||_Y$)

Typical problems:

- Stochastic partial differential equations (SPDE)
- Parametric partial differential equations
- High dimensional algebraic systems in tensor format arising from discretization

Application: Stochastic PDEs arising from fluids

Problem: Find $\boldsymbol{u} : (\boldsymbol{x}, \boldsymbol{\xi}) \in \Omega \times \boldsymbol{\Xi} \to \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\xi})$ in $X = L^2(\boldsymbol{\Xi}, dP_{\boldsymbol{\xi}}; V)$ solution of

$$L(\boldsymbol{u}(\cdot, \boldsymbol{\xi}); \boldsymbol{\xi}) = \boldsymbol{b}(\boldsymbol{\xi}), \text{ a.s. } .$$

with uncertainties represented by $m \in \mathbb{N}$ random variables on $(\boldsymbol{\Xi}, \mathcal{B}, P_{\boldsymbol{\xi}})$: $\boldsymbol{\xi} \in \mathbb{R}^m$, and V an Hilbert space of functions on $\Omega \subset \mathbb{R}^d$.

Considered examples:

- Reaction-Advection-Diffusion problem: non-symetric problem

$$L = -\nu \triangle + \mathbf{c}(\boldsymbol{\xi}) \cdot \nabla + a(\boldsymbol{\xi}) \text{ with } X = L^2(\boldsymbol{\Xi}, dP_\xi; H^1_0(\Omega))$$

- Oseen problem: non-symetric saddle point problem

$$L = \begin{pmatrix} -\nu(\boldsymbol{\xi})\triangle + \mathrm{a}(\boldsymbol{\xi}) \cdot \nabla & \nabla \\ \nabla \cdot & 0 \end{pmatrix} \text{ with } X = L^2(\boldsymbol{\Xi}, dP_\xi; H^1_0(\Omega)) \times L^2(\boldsymbol{\Xi}, dP_\xi; L^2(\Omega))$$

Difficulty: Curse of dimensionality $\rightsquigarrow$ Model reduction

- Reduced basis approaches **[Rozza]**
- Low rank tensor approximation (Proper Generalized Decomposition) **[Nouy]**

Low rank approximation

❶ Approximation in tensor subset for $u \in X \approx \widetilde{u} \in \mathcal{S}_X \subset X$

Rank-r canonical tensors:

$$\mathcal{R}_r(X) = \left\{ \sum_{i=1}^{r} \bigotimes_{\mu=1}^{s} \phi_i^\mu ; \phi_i^\mu \in X_\mu \right\} \text{ with } X = \overline{{}_a\bigotimes_{\mu=1}^{s} X_\mu}^{\|\cdot\|_X}$$

Other: Tucker tensors, Tensor train tensors, Hierarchical Tucker tensors **[Khoromskij]**

❷ Best approximation in $\mathcal{S}_X$

$$\widetilde{\boldsymbol{u}} \in \Pi_{\mathcal{S}_X}(\boldsymbol{u}) = \arg\min_{\boldsymbol{v}\in\mathcal{S}_X} \|\boldsymbol{v} - \boldsymbol{u}\| \quad \rightsquigarrow \quad \widetilde{\boldsymbol{u}} \in \arg\min_{\boldsymbol{v}\in\mathcal{S}_X} \|L\boldsymbol{v} - \boldsymbol{b}\|_*$$

❸ Progressive constructions of approximations with Greedy approach **[Temlyakov]**

Limitation of the classical approach:

- × Bad convergence rate for usual norm $\|\cdot\|_*$ (ex.: $\|\cdot\|_2$ for non symmetric operator L)
- × Weakly coercive problems

Low rank approximation

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Rank-r canonical tensors:

$$\mathcal{R}_r(X) = \left\{ \sum_{i=1}^{r} \phi_i \otimes \psi_i ; \phi_i \in V, \psi_i \in S \right\} \text{ with } X = L^2(\Xi, dP_\xi) \otimes V = S \otimes V$$

$\rightsquigarrow$ Deterministic/Stochastic separation $s = 2$

Other: Tucker tensors, Tensor train tensors, Hierarchical Tucker tensors **[Khoromskij]**

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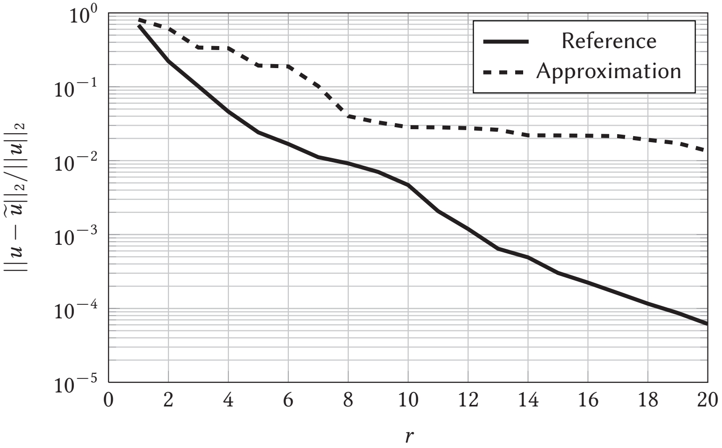
Fig. Reaction-diffusion-advection problem : comparison of convergence error for $||\cdot||_* = ||\cdot||_2$ for a $\mathcal{R}_{20}$ approximation



Main goal of the talk

Goal:

Present an approximation strategy to solve high dimensional PDEs (ex.: stochastic) in tensor subsets relying on best approximation problem formulated using ideal norms.

Outline

Ideal norm

Problem:

Given $\boldsymbol{b} \in Y'$ find the solution $\boldsymbol{u}$ of

$$L\boldsymbol{u} = \boldsymbol{b}$$

- $L : X \to Y'$ linear operator of adjoint $L^* : Y \to X'$, $\boldsymbol{b} \in Y'$
- Riesz operators $R_X : X \to X'$ (resp. $R_Y : Y \to Y'$)

$$\forall u, w \in X \quad \langle u, w \rangle_X = \langle u, R_X w \rangle_{X,X'} = \langle R_X u, w \rangle_{X',X} = \langle R_X u, R_X u \rangle_{X'}$$

- L is continuous

$$\sup_{v\in X} \sup_{w\in Y} \frac{\langle Lv, w \rangle_{Y',Y}}{||v||_X ||w||_Y} = \beta > 0.$$

- L is weakly coercive

$$\inf_{v\in X} \sup_{w\in Y} \frac{\langle Lv, w \rangle_{Y',Y}}{||v||_X ||w||_Y} = \alpha > 0.$$

- We have the stability condition for L

$$\alpha ||Lu||_{Y'} \leq ||u||_X \leq \beta ||Lu||_{Y'}$$

➥ Under these assumptions L is an isomorphism [Ern].

How to choose the norms $||\cdot||_X$ and $||\cdot||_{Y'}$? [Cohen,Dahmen]

$$||\cdot||_X = ||L\cdot||_{Y'} \Leftrightarrow ||\cdot||_{X'} = ||L^*\cdot||_Y$$

- This choice leads to a problem ideally conditioned i.e. $\alpha = \beta = 1$.
- Possibility to choose a priori and arbitrary $||\cdot||_X \rightsquigarrow$ "Goal oriented approximations"

Interpretation: Such a choice implies $\forall v, w \in X$

$$\langle v, w\rangle_X = \langle Lv, Lw\rangle_{Y'} = \langle Lv, R_Y^{-1}Lw\rangle_{Y',Y} = \langle v, R_X^{-1}L^*R_Y^{-1}Lw\rangle_X$$
$$\Rightarrow I_X = R_X^{-1}L^*R_Y^{-1}L \Leftrightarrow R_Y = LR_X^{-1}L^* \Leftrightarrow R_X = L^*R_Y^{-1}L$$

Example: algebraic system

- $L \in \mathbb{R}^{n\times n}$, $\boldsymbol{u}, \boldsymbol{b} \in \mathbb{R}^n$
- $X = Y = \mathbb{R}^n$
- $R_X = I$, $R_Y = LL^*$
- $||u||_Y = ||L^*u||_2 = ||L^*u||_X$

Best approximation problem

❶ Best approximation problem $\boldsymbol{u} \approx \widetilde{\boldsymbol{u}}$ in $\mathcal{S}_X \subset X$

$$\widetilde{\boldsymbol{u}} \in \Pi_{\mathcal{S}_X}(\boldsymbol{u}) = \arg\min_{\boldsymbol{v}\in\mathcal{S}_X} ||\boldsymbol{v}-\boldsymbol{u}||_X \Leftrightarrow ||L\widetilde{\boldsymbol{u}}-\boldsymbol{b}||_{Y'} = \min_{\boldsymbol{v}\in\mathcal{S}_X} ||L\boldsymbol{v}-\boldsymbol{b}||_{Y'}$$

❷ Equivalent problem:

$$\min_{\boldsymbol{v}\in\mathcal{S}_X} ||L\boldsymbol{v}-\boldsymbol{b}||_{Y'}$$

- Non computable norm $||\cdot||_{Y'}$

❸ Exact gradient type algorithm

We seek $\{\boldsymbol{u}^k, \boldsymbol{y}^k\}_{k\geq 0} \subset \mathcal{S}_X \times Y$ given $u_h^0 = 0$ s.t.

$$\begin{cases} \boldsymbol{y}^k &= R_Y^{-1}(L\boldsymbol{u}^k - \boldsymbol{b})), \\ \boldsymbol{u}^{k+1} &\in \Pi_{\mathcal{S}_X}(\boldsymbol{u}^k - R_X^{-1}L^*\boldsymbol{y}^k). \end{cases}$$

- This ideal gradient type algorithm converges in one iteration.
- $R_Y^{-1}(L\boldsymbol{v}-b)$ not affordable in practice !
- How to compute practically $\Pi_{\mathcal{S}_X}$?

Best approximation problem

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❷ Equivalent problem:

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Outline

First step

To compute:

$$\text{Find } \boldsymbol{y}^k \in Y \text{ s.t. } \boldsymbol{y}^k = R_Y^{-1}(L\boldsymbol{u}^k - \boldsymbol{b})$$

- $\Lambda^\delta : Y \to Y$ is a non linear mapping s.t $\forall \boldsymbol{y} \in \{L(\mathcal{S}_X - \boldsymbol{b}); \boldsymbol{v} \in \mathcal{S}_X\}$ we have

$$||\Lambda^\delta(\boldsymbol{y}) - \boldsymbol{y}||_{Y'} \leq \delta||\boldsymbol{y}||_{Y'}, \quad \delta \in (0,1)$$

- $\boldsymbol{y}^k$ is an approximation of $R_Y^{-1}L(\boldsymbol{u}^k - \boldsymbol{b})$ "with" a precision δ

How ?

- Preconditionned iterative solver **[Powell & al.]**
- Greedy construction in a fixed small low-rank subset (ex.: $\mathcal{R}_1$) **[Temlyakov]**

First step

To compute:

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How ?

- Preconditionned iterative solver **[Powell & al.]**
- Greedy construction in a fixed small low-rank subset (ex.: $\mathcal{R}_1$) **[Temlyakov]**

Second step

To compute:

$$\text{Find } \boldsymbol{u}^k \in \mathcal{S}_X^k \text{ s.t. } \boldsymbol{u}^{k+1} \in C^\varepsilon(\boldsymbol{u}^k - R_X^{-1}(L^*\boldsymbol{y}^k)).$$

- $C^\varepsilon : X \to X$ is a non linear mapping giving an approx. of $\boldsymbol{u}^k - R_X^{-1}(L^*\boldsymbol{y}^k)$
  - either with a fixed rank r,
  $$C^\varepsilon = \Pi_{\mathcal{S}_X}.$$
  ➡ In that case $\mathcal{S}_X^k = \mathcal{S}_X$ is fixed at each iteration.
  - or with a fixed precision $\varepsilon \in (0,1)$ s.t.
  $$C^\varepsilon = \Pi_{\mathcal{S}_X^k} \text{ and } ||C^\varepsilon(\boldsymbol{v}) - \boldsymbol{v}||_X \leq \varepsilon||\boldsymbol{v}||_X, \quad \forall \boldsymbol{v} \in X$$
  ➡ In that case $\mathcal{S}_X^k = \mathcal{R}_{r_k}(X)$ may change at each iteration.

How ?

1. $s = 2$: Rank-r SVD, $s > 2$: HOSVD, Alternating Minimization Algorithm
2. Greedy construction in a fixed small low-rank subset: adaptativity, large rank r_k, fixed precision

Summary of the algorithm

Set $\boldsymbol{u}^0 = 0$, $Z_0^0 = 0$ and $\mathcal{S}_Y$ a low rank tensor subset.

Gradient loop: for $k = 0$ to K do

1. Projection: $y_0^k = \arg\min_{y \in Z_0^k} ||\boldsymbol{y} - \boldsymbol{r}^k||_Y$; with $\boldsymbol{r}^k = R_Y^{-1}(L\boldsymbol{u}^k - \boldsymbol{b})$
2. Set $m = 0$;

Loop for Λ^d: while $err(\boldsymbol{y}_m^k, \boldsymbol{r}^k) \leq \delta$ do

a) $m = m + 1$;

b) Correction: $\boldsymbol{w}_m^k = \arg\min_{y \in \mathcal{S}_Y} ||\boldsymbol{y}_{m-1}^k + \boldsymbol{w} - \boldsymbol{r}^k||_Y$;

c) Set $Z_m^k = Z_{m-1}^k + \mathrm{span}\{\boldsymbol{w}_m^k\}$;

d) Projection: $\boldsymbol{y}_m^k = \arg\min_{y \in Z_m^k} ||\boldsymbol{y} - \boldsymbol{r}^k||_Y$;

3. Compute $\boldsymbol{u}^{k+1} \in C^{\varepsilon}(\boldsymbol{u}^k - R_X^{-1} L^* \boldsymbol{y}_m^k)$

Remark: Stopping criterion based on

$$err(\boldsymbol{y}_m^k, \boldsymbol{r}^k) = ||\boldsymbol{r}^k - \boldsymbol{y}_m^k||_Y / ||\boldsymbol{r}^k||_Y \leq \delta \rightsquigarrow ||\boldsymbol{y}_m^k - \boldsymbol{y}_{m+p}^k||_Y / ||\boldsymbol{y}_{m+p}^k||_Y \leq \delta$$

Greedy approximations for C^{ε}:

Weak greedy algorithm: [Temlyakov]

Let $\mathcal{S}_X$ be a low rank tensor subset, find $\{\boldsymbol{u}_m\}_{m\geq 1}$, with $\boldsymbol{u}_m \in \mathcal{S}_X + \cdots + \mathcal{S}_X = \mathcal{S}_X^m \subset X$ for $u_0 = 0$

1) Correction $\widetilde{\boldsymbol{w}}_m$: $\widetilde{\boldsymbol{w}}_m \in \arg\min_{\boldsymbol{v}\in\mathcal{S}_X} ||\Lambda^{\delta} R_Y^{-1}(L\boldsymbol{v} - \boldsymbol{b} + L\boldsymbol{u}_{m-1})||_Y$

2) Update: $\boldsymbol{u}_m = \boldsymbol{u}_{m-1} + \widetilde{\boldsymbol{w}}_m \in \mathcal{S}_X^m$

3) Stopping criterion: If $m = r$ or $||C^{\varepsilon}(\boldsymbol{v}) - \boldsymbol{v}||_X \leq \varepsilon ||\boldsymbol{v}||_X$ then $\boldsymbol{u}^k = \boldsymbol{u}_m$

Convergence result: [Billaud,Nouy,Zahm]

The weak greedy algorithm for a fixed rank subset $\mathcal{S}_X$ converges under some criterions depending on δ.

Convergence of the overall algorithm ...

... with fixed rank:

Proposition: [Billaud,Nouy,Zahm]

Assuming $0 < 2\delta < 1$ then the sequence $\{u^k\}_{k\geq 0}$ is s.t.

$$||E^k||_X \leq (2\delta)^k ||E^0||_X + \frac{1}{1-2\delta}||E^\Pi||_X, \forall k > 0$$

with $E^k = (u^k - u)$ and $E^\Pi = (u - \Pi_{\mathcal{S}_X}(u))$

... with fixed precision:

Proposition:

Assuming $0 < \delta(1+\varepsilon) < 1$ then the sequence $\{u^k\}_{k\geq 0}$ is s.t.

$$||E^k||_X \leq ((1+\varepsilon)\delta)^k ||E^0||_X + \frac{\varepsilon}{1-(1+\varepsilon)\delta}||u||_X, \forall k > 0.$$

Remarks:

- Convergence in two phases : decreasing and stagnation for both kind of algorithms.
- Practical interest of the second algorithm

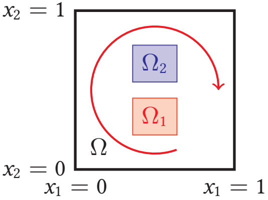
Outline

Test case

Stochastic reaction-advection-diffusion problem:

We seek $u(\boldsymbol{x}, \boldsymbol{\xi})$ satisfying a.e. in $\boldsymbol{\Xi}$

$$\begin{cases} -\triangle u + \boldsymbol{c}\nabla u + au & = f, \quad \text{in } \Omega \\ u & = 0, \quad \text{on } \partial\Omega \end{cases}$$



- $f(\boldsymbol{x}) = \mathbb{1}_{\Omega_1}(\boldsymbol{x}) - \mathbb{1}_{\Omega_2}(\boldsymbol{x})$
- $\boldsymbol{c}(\boldsymbol{x}, \boldsymbol{\xi}) = \xi_1(\frac{1}{2} - x_2, x_1 - \frac{1}{2}), \xi_1 \in \mathcal{U}(-350, 350)$
- $a(\boldsymbol{x}, \xi) = \xi_2, \xi_2 \in \ln \mathcal{U}(0.1, 100)$

Discretization:

- Approx. $\boldsymbol{x} = (x_1, x_2)$: $\mathbb{Q}_1$ with $N = 1521$ nodes $\rightsquigarrow V_N \subset V$
- Approx. $\boldsymbol{\xi}$: piecewise polynomial chaos of degree 5 with the dimension $S = 72 \rightsquigarrow S_P \subset S$

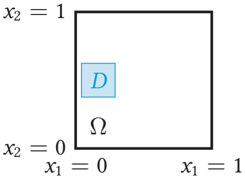
Algebraic system: dimension $s = 2$

$$L\boldsymbol{u} = \boldsymbol{b}$$

with $\boldsymbol{u} \in X = \mathbb{R}^N \otimes \mathbb{R}^P$, $L \in \mathbb{R}^{N\times N} \otimes \mathbb{R}^{P\times P}$ and $\boldsymbol{b} \in \mathbb{R}^N \otimes \mathbb{R}^P$.

Tested norms:

1. Canonical norm: $||v||_2^2 = \sum_{i=1}^{P}\sum_{i=1}^{N}(v_{ij})^2$: $R_X = I_N \otimes I_P$.
2. Weighted norm: $||v||_w^2 = \sum_{i=1}^{P}\sum_{i=1}^{N}(w(x_i)v_{ij})^2$: $R_X = \text{diag}(w(x_i))^2 \otimes I_P$,
   $w(\boldsymbol{x}) = 1000 \cdot \mathbb{1}_D(\boldsymbol{x}) + \mathbb{1}_{\Omega\setminus D}(\boldsymbol{x})$.



$\rightsquigarrow$ "Goal oriented": measure a quantity of interest localized in D:

$$Q(v) = \frac{1}{|D|}\int_D v d\boldsymbol{x}.$$

Confronted approaches: Approximation in $\mathcal{R}_r(X)$, with r fixed.

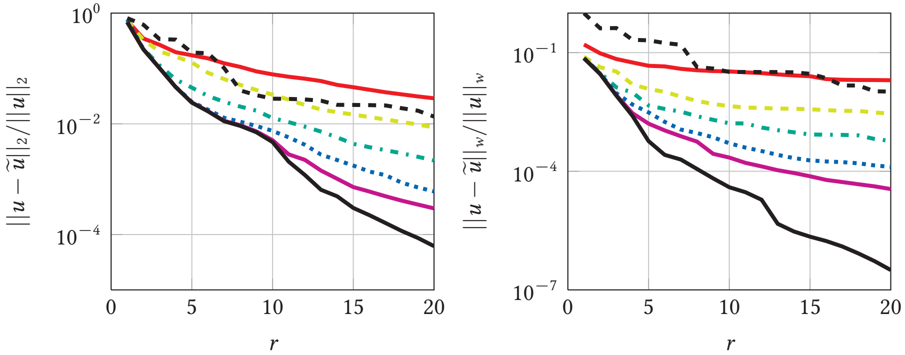
- ✓ Singular Value Decomposition (SVD): Ideal rank-r reference approximation
- ✓ Classical algorithm (CA):

$$\min_{v \in \mathcal{S}_X} ||L\boldsymbol{v} - \boldsymbol{b}||_2$$

- ✓ Perturbed algorithm (PA) : with $\delta \in \{0.01, 0.05, 0.2, 0.5, 0.9\}$ with fixed rank r

First results

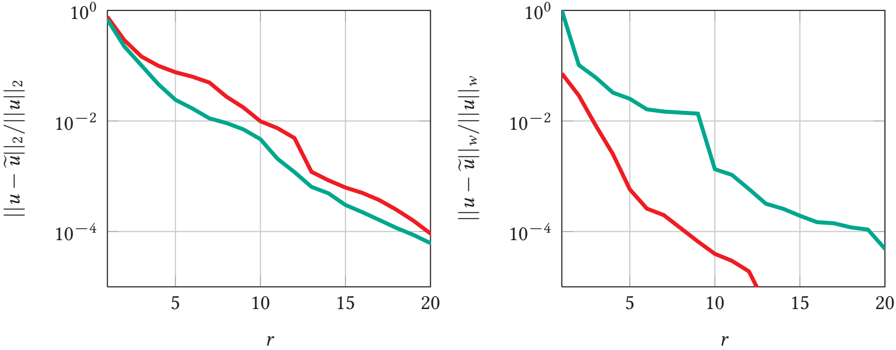
Fig.1 Convergence error for $||\cdot||_2$ and $||\cdot||_w$ with $\mathcal{S}_X = \mathcal{R}_{20}(X)$, — $\delta = 0.9$, — $\delta = 0.5$, — $\delta = 0.2$, — $\delta = 0.05$, — $\delta = 0.01$, — *SVD*, - - - *CA*



- Better result with PA then for CA for **both** norms
- Convergence closer to SVD for decreasing $\delta \to 0$
- Convergence slightly deteriorated for the w-norm

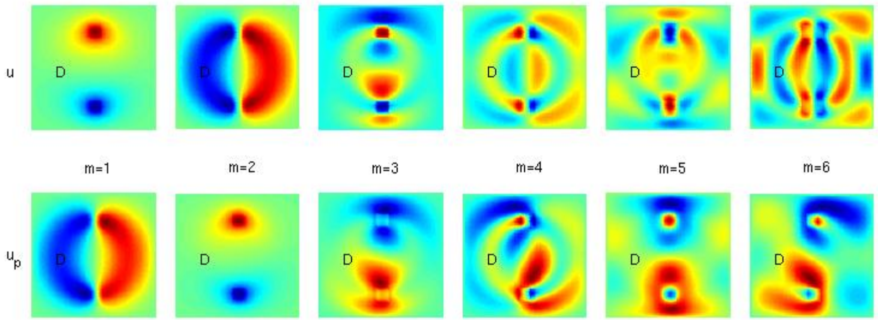
Interest of a weighted norm

Fig.2 Convergence error with rank for both $||\cdot||_2$ and $||\cdot||_w$ with — $\widetilde{u}_w$, — $\widetilde{u}_2$



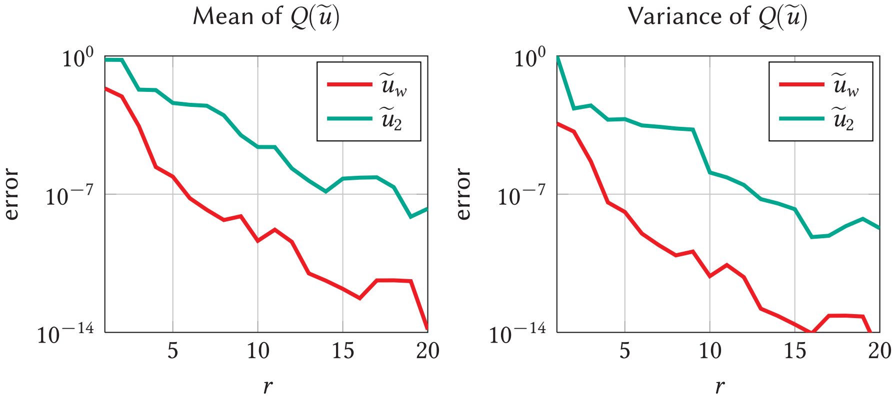
- Similar approximations when compared to the reference with the 2-norm
- $\widetilde{u}_w$ is a better approximation when computed and compared to reference with w-norm.

Fig.3 Comparison of first spatial modes of $\widetilde{u}_2$ and $\widetilde{u}_w$



- Different spatial modes of $\widetilde{u}_w$ and $\widetilde{u}_2$
- Features differently captured

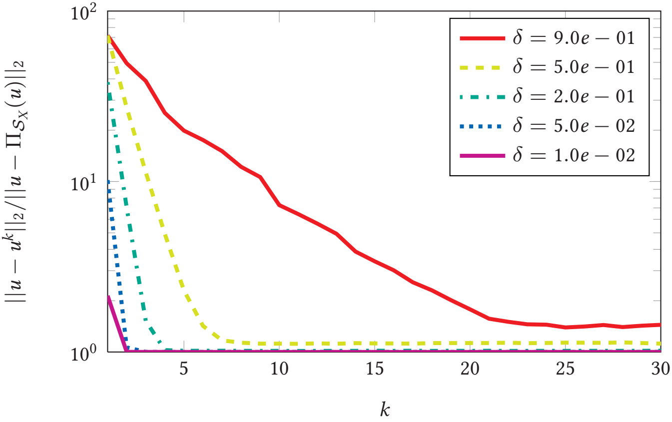
Fig.4 Comparison of the mean value and variance of $\widetilde{u}_2$ and $\widetilde{u}_w$



- Better mean and variance for $\widetilde{u}_w$ than $\widetilde{u}_2$

Gradient algorithm

Fig.5 Convergence study of the gradient type algorithm for different δ for the 2-norm and $\mathcal{S}_X = \mathcal{R}_{10}(X)$



- Convergence behavior consistent with theoric result (decreasing +stagnation)
- Good result, either for "large" δ (near 0.5)
- Similar result for both 2-norm and w-norm

Tab.1 Measured convergence rates of the linear phase for gradient type algorithm.

r \ δ	0.90	0.50	0.20	0.05	0.01
4	0.78	0.36	≈ 0	≈ 0	≈ 0
10	0.82	0.42	0.183	≈ 0	≈ 0
20	0.86	0.48	0.197	0.051	0.011

Tab.2 Measured stagnation values for gradient type algorithm.

δ	0.90	0.50	0.20	0.05	0.01
$2\delta/(1-2\delta)$	-	-	6.6e-1	1.1e-1	2.1e-2
4	3.3e-1	5.6e-2	4.9e-3	3.5e-4	3.0e-5
10	5.2e-1	1.3e-1	1.7e-2	1.8e-3	3.3e-5
20	6.4e-1	1.5e-1	1.9e-2	1.2e-3	7.3e-5

- Convergence rate near δ than 2δ
- Stagnation values overestimated: smaller than theoretical bound $2\delta/(1-2\delta)$

Outline

General framework

Problem:

Given $\boldsymbol{b} = (f, g) \in Y' = X_1' \times M_2'$, find $(u, p) \in X = X_2 \times M_1$ solution of

$$\left\{ \begin{array}{lcl} Au + B_1^* p & = & f, \\ B_2 u & = & g, \end{array} \right. \Leftrightarrow L\boldsymbol{u} = \boldsymbol{b}.$$

- Continous and linear operators $A : X_2 \to X_1'$, $B_i : X_i \to M_i'$ with $A^* : X_1 \to X_2'$, $B_i^* : M_i \to X_i$
- A is weakly coercive:

$$\inf_{u \in K_2} \sup_{v \in K_1} \frac{\langle Av, w \rangle_{X_1', X_1}}{||u||_{X_2} ||v||_{X_1}} \geq \alpha > 0.$$

  with $K_i = \left\{ v \in X_i; \forall q \in M_i;\ \langle B_i v, q \rangle_{M_i', M_i} = 0 \right\}$
- B_i satisfies the inf-sup condition:

$$\inf_{q \in M_i} \sup_{v \in X_i} \frac{\langle B_i v, q \rangle_{M_i', M_i}}{||v||_{X_i} ||q||_{M_i}} = \beta_i > 0.$$

- The product space X is equipped with $\langle \cdot, \cdot \rangle_X = \langle \cdot, \cdot \rangle_{X_2} + \langle \cdot, \cdot \rangle_{M_1}$.

➥ Under these assumptions L is an isomorphism **[Bernardi,Ern]**.

Best approximation problem

❶ Best approximation problem in $\mathcal{S}_{X_2} \subset X_2$, and in $\mathcal{S}_{M_1} \subset M_1$

$$\Pi_{\mathcal{S}_{X_2}}(u) \times \Pi_{\mathcal{S}_{M_1}}(p) = \arg \min_{(v,q)\in \mathcal{S}_{X_2}\times \mathcal{S}_{M_1}} ||v-u||^2_{X_2} + ||q-p||^2_{M_1}$$

❷ Equivalent approximation problem:

$$\arg \min_{(v,q)\in \mathcal{S}_{X_2}\times \mathcal{S}_{M_1}} ||R_Y^{-1}(L(v,q)-(f,g))||_Y$$

❸ Perturbed gradient type algorithm

Seek $\{u^k, p^k, y^k\}_{k\geq 0} \subset \mathcal{S}_{X_2} \times \mathcal{S}_{M_1} \times Y$ given u^0, p^0 s.t.

$$\begin{cases} y^k = (v^k, q^k) & = & \Lambda^\delta R_Y^{-1}(L\boldsymbol{u}^k - \boldsymbol{b}), \\ u^{k+1} & \in & \Pi_{\mathcal{S}_{X_2}}(u^k - R_{X_2}^{-1}(A^* v^k + B_2^* q^k)), \\ p^{k+1} & \in & \Pi_{\mathcal{S}_{M_1}}(p^k - R_{M_1}^{-1}(B_1 v^k)). \end{cases}$$

- Different approximations are constructed for p and u.
- Problem fully coupled when computing y^k.
- Perturbed algorithm:
  1. approximation with "δ-precision" of y^k
  2. approximation with a fixed rank (r_u, r_p) <u>or</u> fixed precision (η, μ) for u and p separately.

Convergence results

Proposition 1: fixed rank

Assuming $0 < \delta < 1/\sqrt{8}$ then the sequence $\{u^k, p^k\}_{k\geq 0}$ is s.t.

$$||E^k||_X \leq (\sqrt{8}\delta)^k ||E^0||_X + \frac{1}{1-\sqrt{8}\delta} ||E_\Pi||_X, \forall k > 0$$

with $E^k = (u^k - u, p^k - p), E_\Pi = (u - \Pi_{\mathcal{S}_{X_2}}(u), p - \Pi_{\mathcal{S}_{M_1}}(p))$

Proposition 2: fixed precision η, μ

Assuming $0 < \delta(1+\epsilon) < 1/\sqrt{2}$ then the sequence $\{u^k, p^k\}_{k\geq 0}$ is s.t.

$$||E^k||_X \leq (\sqrt{2}\delta(1+\epsilon))^k ||E^0||_X + \frac{\varepsilon}{1-\sqrt{2}\delta(1+\varepsilon)} ||\boldsymbol{u}||_X, \forall k > 0$$

with $\epsilon = \max(\eta, \mu), \boldsymbol{u} = (u, p)$

- Two phases of convergence : decreasing, stagnation
- A priori estimate when iterates are computed with fixed precision depend on the more coarse approximation.
- Global error estimation due to R_Y^{-1}

Test case

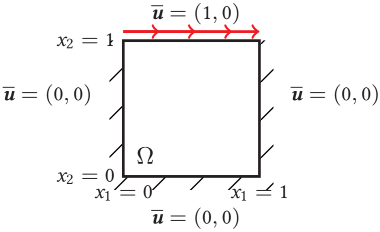
Oseen equations with uncertain parameters:

Given $(\Xi, \mathcal{B}(\Xi), P_{\xi})$ a probability space, find

$$u \in V = L^2(\Xi; [H^1(\Omega)]^d), p \in P = L^2(\Xi; L^2(\Omega))$$

satisfying a.e. in Ξ

$$\begin{cases} -\nu\triangle u + \boldsymbol{a}\nabla u + p & = & 0, & \text{in } \Omega \\ \nabla \cdot u & = & 0, & \text{in } \Omega \\ u & = & \overline{u}, & \text{on } \partial\Omega \end{cases}$$



Uncertainties on a and ν are represented by random variables

- $\nu(\boldsymbol{\xi}) = \nu_0 + \xi_1, \xi_1 \sim \mathcal{U}(-1,1)$
- $\boldsymbol{a}(\boldsymbol{\xi}) = \boldsymbol{a}_0(1+\xi_2), \xi_2 \sim \mathcal{N}(0,1)$

Discretization: Stochastic Galerkin approach

- Space: $\mathbb{P}_2 - \mathbb{P}_1 \rightsquigarrow n = 2189, m = 568$ nodes
- Uncertainties: polynomial chaos of degree 3 $\rightsquigarrow p = 10$

➥ $V_n \otimes S_p \subset V$ and $V_m \otimes S_p \subset P$

Test case

Oseen equations with uncertain parameters:

Given $(\Xi, \mathcal{B}(\Xi), P_{\boldsymbol{\xi}})$ a probability space, find

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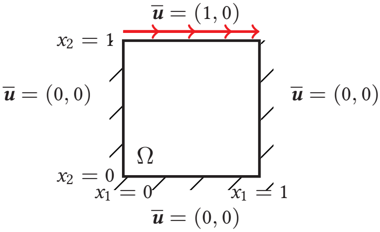

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➡ $V_n \otimes S_p \subset V$ and $V_m \otimes S_p \subset P$

Algebraic system: dimension $s = 2$

Find $u \in X_2 = \mathbb{R}^n \otimes \mathbb{R}^p$ and $p \in M_1 = \mathbb{R}^n \otimes \mathbb{R}^P$ s.t.

$$\begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

with $A \in \mathbb{R}^{n\times n} \otimes \mathbb{R}^{P\times P}$, $B \in \mathbb{R}^{m\times n} \otimes \mathbb{R}^{P\times P}$ and $f \in \mathbb{R}^n \otimes \mathbb{R}^P$, $g \in \mathbb{R}^m \otimes \mathbb{R}^P$.

Canonical norm: $||v||_2^2 = \sum_{i=1}^{P} \sum_{i=1}^{N} (v_{ij})^2$

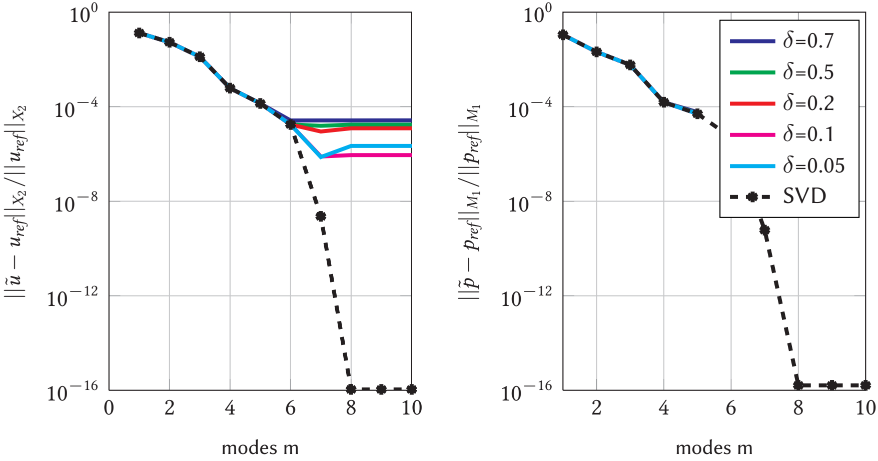
$$\Rightarrow R_{X_2} = I_n \otimes I_P, R_{M_1} = I_m \otimes I_P \Rightarrow R_Y = \begin{pmatrix} AR_{X_2}^{-1}A^* + B^*R_{M_1}^{-1}B & AR_{X_2}^{-1}B^* \\ BR_{X_2}^{-1}A^* & BR_{X_2}^{-1}B^* \end{pmatrix}.$$

Confronted approaches: Approximations with fixed precisions η, μ

- Singular Value Decomposition (SVD): Ideal rank-r reference approximation since $s = 2$
- Perturbed Algorithm (PA): $\delta \in \{0.05, 0.1, 0.2, 0.5, 0.7\}$ with $\eta = 1.e-6$ and $\mu = 1.e-4$

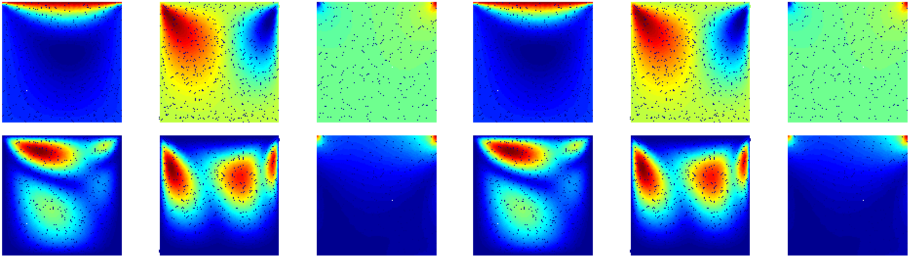
First results

Fig.6 Comparison of the relative convergence to reference in $||\cdot||_2$ for both pressure and velocity



- Good agreement with the reference
- Influence of the pressure approximation on the velocity

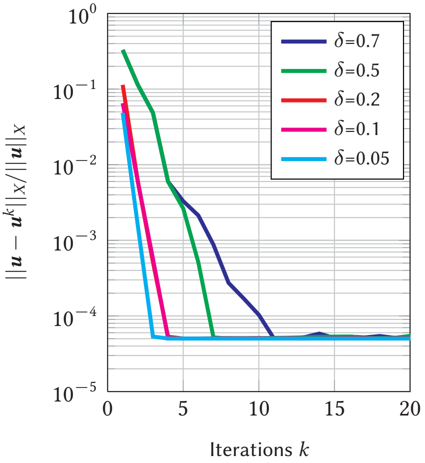
Fig.7 Comparison of both mean and expectations of velocities and pressure for the SVD reference (left) and the computed approximation PA (right)



- Mean profiles in good agreement with the deterministic profiles
- Good agreement between the two approaches

Gradient algorithm

Fig.8 Convergence study of the gradient type algorithm for different δ for the 2-norm and with the resp. precisions $\eta = 10^{-6}$, $\mu = 10^{-4}$



Tab.3 Measured convergence rates of the linear phase for gradient type algorithm.

δ	0.70	0.50	0.20	0.1	0.05
Th.	0.99	0.71	0.28	0.14	0.07
Me.	0.37	0.27	0.06	0.09	0.03

Tab.4 Measured stagnation values for gradient type algorithm.

δ	0.70	0.50	0.20	0.1	0.05
Th. (10^{-4})	100	3.41	1.39	1.16	1.07
Me. (10^{-5})	5.53	5.28	5.08	5.04	5.03

➥ Both rate of convergence and stagnation value overestimated

Outline

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Conclusion: Robust model reduction approach

- ✓ Ideal minimal residual formulation with ideal norms ⇝ Convergence improvement
- ✓ Validation on SPDEs with application arising from fluids mechanics
- ✓ Arbitrary choice of the norm associated to a specific problem ⇝ Goal oriented

Need of improvement: Algorithm for R_Y^{-1}

- Preconditioned solvers **[Powell]**
- Approximation of R_Y^{-1} in low tensor subsets **[Giraldi,Nouy,Legrain]**
- Stopping criterion for the dual problem ⇝ Still an open problem

Futher exploration:

- "Real" goal oriented explorations ⇝ PhD of O.Zahm, in collaboration with A. Nouy
- Uzawa algorithm for HD saddle point problems in collaboration with V. Erlacher

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Thank you for your attention

Bibliography

[Bernardi] C. Bernardi, C. Canuto, Y. Maday: *Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem*, SIAM J. Numer. Anal., Vol.25, Issue. 6, pp. 1237–1271, 1988.

[Billaud] Billaud Friess, M., Nouy, A. and Zahm, O., *A tensor approximation method based on ideal minimal residual formulations for the solution of high dimensional problems*, In revision, 2013

[Cohen] Cohen, A., Dahmen, W. and Welper, G., *Adaptivity and Variational Stabilization for Convection-Diffusion Equations*, Preprint, 2011

[Dahmen] Dahmen, W., Huang, C. and Schwab, C., *Adaptative Petrov-Galerkin methods for first order transport equations*, IGPM Report 321, RWTH Aachen, Volume 150, Pages 425–467, 2011

[Ern] A. Ern and J.-L. Guermond, *Theory and practice of finite elements*, volume 159 of applied mathematical sciences, 2004.

[Giraldi] L. Giraldi, A. Nouy, G. Legrain: *Low-rank approximate inverse for preconditioning tensor-structured linear systems*, arXiv:1304.6004, 2013.

[Nouy] Nouy, A., *Proper Generalized Decompositions and Separated Representations for the Numerical Solution of High Dimensional Stochastic Problems*, Archives of Computational Methods in Engineering, Volume 17, Number 4, 2010

[Powell] C.E. Powell, D.J. Silvester: *Preconditionning steady-state Navier-Stokes equations with random data*, MIMS EPrint 2012.35, 2012.

[Temlyakov] Temlyakov, A., *Greedy approximation*, 2011