On the well-posedness of Stochastic Lagrangian Models

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talk at CEMRACS 2013
The equation of 9.00 am

For any arbitrary finite $T > 0$, we are interested in $((X_t, U_t); 0 \leq t \leq T)$, solving

\[
\begin{cases}
X_t = X_0 + \int_0^t U_s \, ds, \\
U_t = U_0 + \int_0^t B[X_s; \rho_s] \, ds + \sigma W_t - \sum_{0 < s \leq t} 2 (U_s - \cdot n_D(X_s)) n_D(X_s) 1_{\{X_s \in \partial D\}}, \\
\rho(t) \text{ is the probability density of } (X_t, U_t) \text{ for all } t \in (0, T],
\end{cases}
\]

(1)

$$(W_t, t \geq 0)$$ is a standard $\mathbb{R}^d$-Brownian motion ;
The diffusion $\sigma > 0$ is a constant ;
$\mathcal{D}$ is an open bounded domain of $\mathbb{R}^d$.

This study is a joint work with Jean Francois Jabir (University of Valparaiso) ;
Preprint on arXiv.
The nonlinear coefficient

The drift coefficient $B : \mathcal{D} \times L^1(\mathcal{D} \times \mathbb{R}^d) \to \mathbb{R}^d$

$$B[x; \psi] = \begin{cases} \frac{\int_{\mathbb{R}^d} b(v) \psi(t, x, v) dv}{\int_{\mathbb{R}^d} \psi(t, x, v) dv}, & \text{whenever } \int_{\mathbb{R}^d} \psi(t, x, v) dv \neq 0, \\
0 & \text{otherwise,} \end{cases}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a given measurable function.

Formally the function $(t, x) \mapsto B[x; \rho(t)]$ in (1) corresponds to the conditional expectation

$$(t, x) \mapsto \mathbb{E}[b(U_t) / X_t = x]$$

and the velocity equation in (1) rewrites

$$U_t = U_0 + \int_0^t \mathbb{E}[b(U_s) / X_s] ds + \sigma W_t + K_t.$$
Motivation

The law of \((X_t, U_t) = (x_t, y_t, z_t, u_t, v_t, w_t)\) is a local probability distribution of physical quantities (within the meaning of the statistical approach to turbulence)

\[
\rho(t, x, y, z, v, u, w) dx dy dz du dv dw
\]

Computation of local moment fields:

\[
\langle u \rangle(t, x, y, z) = \frac{\int_{D \times \mathbb{R}^3} u \rho(t, x, y, z, v, u, w) du dv dw}{\int_{D \times \mathbb{R}^3} \rho(t, x, y, z, v, u, w) du dv dw} = \mathbb{E}[u_t/X_t = (x, y, z)]
\]

\[
\langle uv \rangle(t, x, y, z) = \frac{\int_{D \times \mathbb{R}^3} uv \rho(t, x, y, z, v, u, w) du dv dw}{\int_{D \times \mathbb{R}^3} \rho(t, x, y, z, v, u, w) du dv dw} = \mathbb{E}[u_t v_t/X_t = (x, y, z)]
\]
Lagrangian modeling for turbulent flow

On a given probability space \((\Omega, \mathcal{F}, \mathbb{P})\), consider the fluid particle state vector \((X_t, U_t, \psi_t)\) satisfying

\[
\begin{align*}
    dX_t &= U_t dt, \\
    dU_t &= \left[-\frac{1}{\rho} \nabla_x \langle \mathcal{P} \rangle (t, X_t) + \nu \nabla_x \langle \mathcal{U} \rangle (t, X_t) \right] dt \\
    &\quad - G(t, X_t) (U_t - \langle \mathcal{U} \rangle (t, X_t)) dt + \sqrt{C(t, X_t) \varepsilon(t, X_t)} dW_t, \\
    d\psi_t &= D_1(t, X_t, \psi_t) dt + D_2(t, X_t, \psi_t) \tilde{d}W_t.
\end{align*}
\]

\((W, \tilde{W})\) is a 4D-Brownian motion.

\[
\begin{align*}
    \langle \mathcal{U}^{(i)} \rangle (t, x), \langle \mathcal{U}^{(i)} \mathcal{U}^{(j)} \rangle (t, x) \text{ are computed as conditional expectations.} \\
    \varepsilon, C, G, D_1, D_2 \text{ are determined by the RANS closure.}
\end{align*}
\]
Statistical approach of turbulent flows

The Reynolds averages (or ensemble averages) are expectations:

\[ \langle U \rangle(t, x) := \int_{\Omega} U(t, x, \omega) d\mathbb{P}(\omega). \]

Reynolds decomposition

\[ U(t, x, \omega) = \langle U \rangle(t, x) + u(t, x, \omega), \]

\[ P(t, x, \omega) = \langle P \rangle(t, x) + p(t, x, \omega). \]

The random field \( u(t, x, \omega) \) is the turbulent part of the velocity.

Incompressible Navier Stokes equation in \( \mathbb{R}^3 \), for the velocity field \( (U^{(1)}, U^{(2)}, U^{(3)}) \) and the pressure \( P \), with constant mass density \( \varrho \)

\[ \partial_t U + (U \cdot \nabla) U = \nu \Delta U - \frac{1}{\varrho} \nabla P, \quad t > 0, \quad x \in \mathbb{R}^3, \]

\[ \nabla \cdot U = 0, \quad t \geq 0, \quad x \in \mathbb{R}^3, \]

\[ U(0, x) = U_0(x), \quad x \in \mathbb{R}^3. \]
The Reynolds averaged NS Equation for the mean velocity: RANS Equation

Assuming Reynolds decomposition, we obtain the unclosed equation with constant mass density $\rho$

$$
\partial_t \langle \mathbf{U}^{(i)} \rangle + \sum_{j=1}^{3} \langle \mathbf{U}^{(j)} \rangle \partial_{x_j} \langle \mathbf{U}^{(i)} \rangle + \sum_{j=1}^{3} \partial_{x_j} \langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \nu \Delta \langle \mathbf{U}^{(i)} \rangle - \frac{1}{\rho} \partial_{x_i} \langle \mathbf{P} \rangle,
$$

$$\nabla \cdot \langle \mathbf{U} \rangle = 0, \ t \geq 0, \ x \in \mathbb{R}^3,$$

$$\langle \mathbf{U} \rangle(0,x) = \langle \mathbf{U}_0 \rangle(x), \ x \in \mathbb{R}^3,$$

where $\langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \langle \mathbf{U}^{(i)} \mathbf{U}^{(j)} \rangle - \langle \mathbf{U}^{(i)} \rangle \langle \mathbf{U}^{(j)} \rangle, \ i, j)$ is Reynolds stress tensor.

Depending of the physics, a turbulent closure of the Reynolds stress is associated to the RANS Equation.

Direct modeling of the Reynolds stress, or turbulent viscosity model, ...

kinetic turbulent energy $k(t,x) := \sum_{i=1}^{3} \frac{1}{2} \langle \mathbf{u}^{(i)} \mathbf{u}^{(i)} \rangle(t,x)$

and

pseudo-dissipation $\varepsilon(t,x) := \nu \sum_{i=1}^{3} \sum_{k} \langle \partial_{x_k} \mathbf{u}^{(i)} \partial_{x_k} \mathbf{u}^{(i)} \rangle(t,x).$
An alternative viewpoint to compute the Reynolds stress [by Stephen B. Pope]

Let $\rho_E(t, x; V)$ be the probability density function (PDF) of the random field $\mathcal{U}(t, x)$, then

$$
\langle \mathcal{U}^{(i)} \rangle(t, x) = \int_{\mathbb{R}^3} V^{(i)} \rho_E(t, x; V) dV,
$$

$$
\langle \mathcal{U}^{(i)} \mathcal{U}^{(j)} \rangle(t, x) = \int_{\mathbb{R}^3} V^{(i)} V^{(j)} \rho_E(t, x; V) dV.
$$

The closure problem is reported on the PDE satisfied by the probability density function $\rho_E$.

In a series of papers (see e.g. Pope 85), Stephen B. Pope propose to model the PDF $\rho_E$ with a Lagrangian description of the flow, or equivalently with the Lagrangian probability density function (the PDF method).
Fluid particle model family

On a given probability space \((\Omega, \mathcal{F}, \mathbb{P})\), consider the fluid particle state vector \((X_t, U_t, \psi_t)\) satisfying

\[
\begin{align*}
    dX_t &= U_t dt, \\
    dU_t &= \left[ -\frac{1}{\varrho} \nabla_x \langle P \rangle(t, X_t) + \nu \triangle_x \langle U \rangle(t, X_t) \right] dt \\
    &\quad - G(t, X_t) (U_t - \langle U \rangle(t, X_t)) dt + \sqrt{C(t, X_t) \varepsilon(t, X_t)} dW_t, \\
    d\psi_t &= D_1(t, X_t, \psi_t) dt + D_2(t, X_t, \psi_t) d\tilde{W}_t.
\end{align*}
\]

\((W, \tilde{W})\) is a 4D-Brownian motion.

One needs to

- compute de Eulerian fields \(\langle U^{(i)} \rangle(t, x), \langle U^{(i)} U^{(j)} \rangle(t, x)\).
- determine \(\varepsilon, C, G, D_1, D_2\) by the RANS closure.
Numerical Experiments: comparison with wind measures

In [Bernardin Bossy Chauvin Drobinski Rousseau Salameh 2009]

The MM5 model is run for 3 days between March 23rd and 25th, 1998 over the 3 nested domains with respective horizontal resolutions of 27, 9 and 3 km.

The initial and boundary conditions are taken from the ECMWF reanalyses.

◇ represents the location of the buoy ASIS.
The numerical framework

Our computational domain $\mathcal{D}$. For example, a given cell of the NWP solver.

Boundary condition:

$$\forall x \in \partial \mathcal{D}, \langle \mathcal{U} \rangle(t, x) = V_{\text{ext}}(t, x)$$

($V_{\text{ext}}$ is the $MM5$ forcing.)
The guidance with an external velocity field

The Downscaling method

Let $\mathcal{D}$ be an open set of $\mathbb{R}^3$, and a velocity $V_{\text{ext}}$ given at $\partial \mathcal{D}$ :

\[
\begin{align*}
dX_t &= U_t dt, \\
dU_t &= \left[ -\frac{1}{\varrho} \nabla \langle \mathcal{P} \rangle(t, X_t) \\
&\quad - \left( \frac{1}{2} + \frac{3}{4} C_0 \right) \frac{\varepsilon(t, X_t)}{k(t, X_t)} (U_t - \langle U \rangle(t, X_t)) \right] dt \\
&\quad + \sqrt{C_0 \varepsilon(t, X_t)} dW_t \\
&\quad + \sum_{0 < s \leq t} 2 (V_{\text{ext}}(s, X_s) - U_s) \mathbf{1}_{\{X_s \in \partial \mathcal{D}\}}.
\end{align*}
\]

The jump term should ensure that

\[
\langle U \rangle(t, x) = V_{\text{ext}}(t, x), \forall x \in \partial \mathcal{D}.
\]
The guidance with an external velocity field

Boundary condition

\[ \forall x \in \partial \mathcal{D}, \]

\[ \langle \mathcal{U} \rangle(t, x) = V_{\text{ext}}(t, x). \]

\[ \int_{\mathbb{R}^d} v \rho_{\ell}(t, x, v) dv \quad = \quad V_{\text{ext}}(t, x). \]

\[ \int_{\mathbb{R}^d} v \rho_{\ell}(t, x, v) dv = \int_{\mathbb{R}^d} V_{\text{ext}}(t, x) \rho_{\ell}(t, x, v) dv \]

\[ \iff \quad \int_{\mathbb{R}^d} v \rho_{\ell}(t, x, v) dv = \int_{\mathbb{R}^d} v \rho_{\ell}(t, x, v + 2(V_{\text{ext}}(t, x) - v)) dv \]

\[ \uparrow \]

\[ \rho_{\ell}(t, x, v) = \rho_{\ell}(t, x, v + 2(V_{\text{ext}}(t, x) - v)), \quad \forall v \in \mathbb{R}^d \]

leads to specular boundary condition with jump on \( \partial \mathcal{D} \) for \( \rho_{\ell} \)...
The equation of 9.15 am

\[
\begin{aligned}
X_t &= X_0 + \int_0^t U_s \, ds, \\
U_t &= U_0 + \int_0^t B[X_s; \rho_s] \, ds + \sigma W_t - \sum_{0 < t_s < t} 2 (U_{t_s} \cdot n_{\mathcal{D}}(X_s)) n_{\mathcal{D}}(X_s) \mathbf{1}_{\{X_s \in \partial \mathcal{D}\}},
\end{aligned}
\]

\(\rho(t)\) is the probability density of \((X_t, U_t)\) for all \(t \in (0, T]\),

\[Q_T = (0, T) \times \mathcal{D} \times \mathbb{R} \quad \text{and} \quad \Sigma_T = (0, T) \times \partial \mathcal{D} \times \mathbb{R}\]

**Definition 1. Trace of the density along \(\Sigma_T\)**

\(\gamma(\rho) : \Sigma_T \to \mathbb{R}\) is the trace of \((\rho(t); \ t \in [0, T])\) along \(\Sigma_T\) if it is nonnegative and satisfies, for all \(t\) in \((0, T]\), \(f\) in \(C_c^\infty(\overline{Q_t})\):

\[
\int_{\Sigma_t} (u \cdot n_{\mathcal{D}}(x)) \gamma(\rho)(s, x, u) f(s, x, u) \, ds \, d\sigma_{\partial \mathcal{D}}(x) \, du
= -\int_{\mathcal{D} \times \mathbb{R}^d} f(t, x, u) \rho_t(x, u) \, dx \, du + \int_{\mathcal{D} \times \mathbb{R}^d} f(0, x, u) \rho_0(x, u) \, dx \, du
+ \int_{Q_t} \left( \partial_s f + u \cdot \nabla_x f + B[\cdot; \rho_\cdot] \cdot \nabla u f + \frac{\sigma^2}{2} \triangle u f \right)(s, x, u) \rho_s(x, u) \, ds \, dx \, du
\]
and, for \(dt \otimes d\sigma_{\partial \mathcal{D}}\) a.e. \((t, x)\) in \((0, T) \times \partial \mathcal{D}\),

\[
\int_{\mathbb{R}^d} |(u \cdot n_{\mathcal{D}}(x))| \gamma(\rho)(t, x, u) \, du + \infty, \quad \int_{\mathbb{R}^d} \gamma(\rho)(t, x, u) \, du > 0.
\]
Main Theorem

Under \((H)\), there exists a unique solution in law to (1) in \(\Pi_\omega\).

\[
\Pi_\omega := \{ Q, \text{ probability measure on } C([0, T]; \overline{D}) \times \mathcal{D}([0, T]; \mathbb{R}^d), \text{ s.t.}, \]

\[
\text{for all } t \in [0, T], \quad \rho_t = Q \circ (x(t), u(t))^{-1} \in L^2(\omega; \mathcal{D} \times \mathbb{R}^d) \}
\]

Moreover the time-marginal densities \((\rho_t, t \in [0, T])\) is in \(V^1(\omega, Q_T)\) and admits a trace \(\gamma(\rho)\) in the sense of Definition 1 which satisfies the no-permeability boundary condition

\[
\mathbb{E}\{(U_t \cdot n_D(X_t))/X_t = x\} = \frac{\int_{\mathbb{R}^d} (u \cdot n_D(x)) \gamma(\rho)(t, x, u) \, du}{\int_{\mathbb{R}^d} \gamma(\rho)(t, x, u) \, du} = 0, \quad dt \otimes d\sigma_{\partial D} - \text{a.e. on } (0, T)
\]

or equivalently the specular boundary condition:

\[
\gamma(\rho)(t, x, u) = \gamma(\rho)(t, x, u - 2(u \cdot n_D(x)) n_D(x)), \quad dt \otimes d\sigma_{\partial D} \otimes du - \text{a.e. on } (0, T) \times \partial D \times \mathbb{R}^d.
\]
The hypotheses \((H)\)

\((H_{\text{Langevin}})\) for the construction of the linear Langevin process

\((H_{\text{MVFP}})\) for the well-posedness of the Vlasov-Fokker-Planck equation

- \((H_{\text{Langevin}})-(i)\) \((X_0, U_0)\) is distributed according to the initial law \(\mu_0\) having its support in \(D \times \mathbb{R}^d\) and such that \(\int_{D \times \mathbb{R}^d} (|x|^2 + |u|^2) \, \mu_0(dx, du) < +\infty\).

- \((H_{\text{Langevin}})-(ii)\) The boundary \(\partial D\) is a compact \(C^3\) submanifold of \(\mathbb{R}^d\).

- \((H_{\text{MVFP}})-(i)\) \(b : \mathbb{R}^d \to \mathbb{R}^d\) is a **bounded** measurable function.

- \((H_{\text{MVFP}})-(ii)\) The initial law \(\mu_0\) has a density \(\rho_0\) in the weighted space \(L^2(\omega, D \times \mathbb{R}^d)\) with \(\omega(u) := (1 + |u|^2)^{\frac{\alpha}{2}}\) for some \(\alpha > d \vee 2\).

- \((H_{\text{MVFP}})-(iii)\) There exist two measurable functions \(P_0, \overline{P}_0 : \mathbb{R}^+ \to \mathbb{R}^+\) such that

\[
0 < P_0(|u|) \leq \rho_0(x, u) \leq \overline{P}_0(|u|), \text{ a.e. on } D \times \mathbb{R}^d;
\]

and

\[
\int_{\mathbb{R}^d} (1 + |u|)\omega(u)\overline{P}_0^2(|u|)du < +\infty.
\]
Notation

\[ Q_t := (0, t) \times \mathcal{D} \times \mathbb{R}^d, \]
\[ \Sigma^+ := \{(x, u) \in \times \partial \mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_D(x)) > 0\}, \quad \Sigma^+_t := (0, t) \times \Sigma^+, \]
\[ \Sigma^- := \{(x, u) \in \times \partial \mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_D(x)) < 0\}, \quad \Sigma^-_t := (0, t) \times \Sigma^-, \]
\[ \Sigma^0 := \{(x, u) \in \partial \mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_D(x)) = 0\}, \quad \Sigma^0_t := (0, t) \times \Sigma^0, \]
\[ \Sigma_T := (0, T) \times \partial \mathcal{D} \times \mathbb{R}^d \text{ endowing with } d\lambda_{\Sigma_T} := dt \otimes d\sigma_{\partial \mathcal{D}}(x) \otimes du. \]

\(d\sigma_{\partial \mathcal{D}}\) is the surface measure on \(\partial \mathcal{D}\).

The weighted Lebesgue space (with \(\omega(u) := (1 + |u|^2)^{\alpha/2}\))

\[ L^2(\omega, Q_t) := \{\psi : Q_t \rightarrow \mathbb{R} ; \sqrt{\omega} \psi \in L^2(Q_t)\}, \text{ with } \|\psi\|^2_{L^2(\omega, Q_t)} = \|\sqrt{\omega} \psi\|_{L^2(Q_t)} \]

The weighted Sobolev space

\[ V_1(\omega, Q_T) = C([0, T]; L^2(\omega, \mathcal{D} \times \mathbb{R}^d)) \cap L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d)), \]

equipped with the norm

\[ \|\phi\|^2_{V_1(\omega, Q_T)} = \max_{t \in [0, T]} \left\{ \int_{\mathcal{D} \times \mathbb{R}^d} \omega(u) |\phi(t, x, u)|^2 \, dx \, du \right\} + \int_{Q_T} \omega(u) |\nabla_u \phi(t, x, u)|^2 \, dt \, dx \, du. \]
Plan of the proof

- Phase 1. Construct of a solution to the linear Confined Langevin equation.

- Phase 2. Construct a density \( \rho \) solution of the conditional McKean Vlasov Fokker Planck equation, with specular condition.

- Phase 3. Add the drift \( B[x; \rho] \) to the Langevin process constructed in Phase 1 and show that the result is the unique solution of our confined stochastic Lagrangian model.
Sketch of the Phase 1 -a)

Prove the well-posedness of the confined linear Langevin equation: there exists a unique solution, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \in [0, T]), \mathcal{P})$ endowed with a Brownian motion $W$, to

\[
\begin{aligned}
X_t &= x_0 + \int_0^t U_s \, ds, \\
U_t &= u_0 + \sigma W_t - 2 \sum_{0 \leq s \leq t} (U_s \cdot n_\mathcal{D}(X_s)) n_\mathcal{D}(X_s) \mathbf{1}_{\{X_s \in \partial \mathcal{D}\}}, \quad \forall t \in [0, T],
\end{aligned}
\]

for any $(x_0, u_0) \in (\mathcal{D} \times \mathbb{R}^d) \cup (\Sigma \setminus \Sigma^0)$. 

(5)
The confined Brownian motion primitive in the half line

Starting from \((X_0, U_0)\) with \(X_0 > 0\), and \((B_t)\) Brownian motion in \(\mathbb{R}\),

\[
Y_t = X_0 + \int_0^t V_s ds, \quad V_t = U_0 + B_t.
\]

Set \(X_t = |Y_t|,\)

\[
U_t = V_t S_t^+, \quad \text{with } S_t := \text{sign}(Y_t).
\]

Lemma B. & Jabir 2011

If \(\rho_0\) has its support in \((0, +\infty) \times \mathbb{R}\), then \(S_t\) jumps a countable number of times, and \(U_t\) solves

\[
U_t = U_0 + W_t - 2 \sum_{0 < s \leq t} U_s \mathbf{1}_{\{X_s = 0\}} \quad \text{a.s.}
\]

where \(W_t\) is a Brownian motion.

( Lachal 97 : Passage time of the Brownian motion primitive at 0)
Construction of the confined Langevin process, by local straightening of the boundary

This method impose that $\partial D$ is a compact submanifold of class $C^3$. The local straightening is then $C^2$, enough to apply the Ito formula to the process in the new system of coordinates.
Phase 1-b) On the semigroup of the confined Langevin process

For some test function $\psi : \mathcal{D} \times \mathbb{R}^d \to \mathbb{R}^+$, for all $(x, u) \in (\mathcal{D} \times \mathbb{R}^d) \cup (\Sigma \setminus \Sigma^0)$, we define

$$
\Gamma^\psi(t, x, u) := \mathbb{E}_\mathbb{P} \left[ \psi(X_t^{x,u}, U_t^{x,u}) \right],
$$

where $((X_t^{x,u}, U_t^{x,u}); t \in [0, T])$ is the solution of (5) starting from $(0, x, u)$.

Proposition

Assume $(H_{\text{Langevin}})$. For all nonnegative $\psi \in \mathcal{C}_c^\infty(\mathcal{D} \times \mathbb{R}^d)$, $\Gamma^\psi$ is a nonnegative function that belongs to $L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d))$ and satisfies the energy equality:

$$
\| \Gamma^\psi(t) \|^2_{L^2(\mathcal{D} \times \mathbb{R}^d)} + \sigma^2 \| \nabla_u \Gamma^\psi \|^2_{L^2(Q_t)} = \| \psi \|^2_{L^2(\mathcal{D} \times \mathbb{R}^d)}, \forall t \in (0, T)
$$

Furthermore, $\Gamma^\psi(t)$ is solution in the sense of distributions of

$$
\begin{cases}
\partial_t \Gamma^\psi - (u \cdot \nabla_x \Gamma^\psi) - \frac{\sigma^2}{2} \triangle_u \Gamma^\psi = 0, & \text{on } Q_T, \\
\Gamma^\psi(0, x, u) = \psi(x, u), & \text{on } \mathcal{D} \times \mathbb{R}^d, \\
\Gamma^\psi(t, x, u) = \Gamma^\psi(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), & \text{on } \Sigma_T^+. \\
\end{cases}
$$

and $\forall p \in (1, +\infty)$,

$$
\| \Gamma^\psi(t) \|^p_{L^p(\mathcal{D} \times \mathbb{R}^d)} \leq \| \psi \|^p_{L^p(\mathcal{D} \times \mathbb{R}^d)}, \forall t \in (0, T)
$$
Main ingredient

Theorem

Assume ($H_{\text{Langevin}}$). Given two nonnegative functions $f_0 \in L^2(D \times \mathbb{R}^d) \cap C_b(D \times \mathbb{R}^d)$ and $q \in L^2(\Sigma_T^+ \cap C_b(\Sigma_T^+))$, there exists a unique nonnegative function $f \in C^1_{b,1,2}(Q_T) \cap C(Q_T \setminus \Sigma_T^0) \cap L^2((0, T) \times D; H^1(\mathbb{R}^d))$ solution to

\[
\begin{cases}
\partial_t f(t, x, u) - (u \cdot \nabla_x f(t, x, u)) - \frac{\sigma^2}{2} \Delta u f(t, x, u) = 0, & \text{for all } (t, x, u) \in Q_T, \\
f(0, x, u) = f_0(x, u), & \text{for all } (x, u) \in D \times \mathbb{R}^d, \\
f(t, x, u) = q(t, x, u), & \text{for all } (t, x, u) \in \Sigma_T^+.
\end{cases}
\]

In addition, for the Langevin process $(x_t^{x,u}, u_t^{x,u}; t \in [0, T])$ starting from $(x, u) \in D \times \mathbb{R}^d$ at $t = 0$ and $\beta^{x,u} := \inf\{t > 0; x_t^{x,u} \in \partial D\}$, we have

\[
f(t, x, u) = \mathbb{E}_P \left[ f_0(x_t^{x,u}, u_t^{x,u}) 1\{t \leq \beta^{x,u}\} \right] + \mathbb{E}_P \left[ q(t - \beta^{x,u}, x_t^{x,u}, u_t^{x,u}) 1\{t > \beta^{x,u}\} \right]
\]

Furthermore, for all $t \in (0, T)$, $f$ satisfies

\[
\begin{align*}
\|f(t)\|^2_{L^2(D \times \mathbb{R}^d)} + \|f\|^2_{L^2(\Sigma_t^-)} + \sigma^2 \|\nabla u f\|^2_{L^2(Q_t)} &= \|f_0\|^2_{L^2(D \times \mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_T^+)} , \\
\|f(t)\|^p_{L^p(D \times \mathbb{R}^d)} + \|f\|^p_{L^p(\Sigma_t^-)} + \sigma^2 p(p - 1) \|\nabla u f\|^p_{L^p(Q_t)} &\leq \|f_0\|^p_{L^p(D \times \mathbb{R}^d)} + \|q\|^p_{L^p(\Sigma_T^+)} .
\end{align*}
\]
Interior regularity

or $\alpha \in (0, 1)$, we further denote by $D_x^\alpha$ the fractional derivative w.r.t. $x$-variables, defined as the fractional Laplace operator of order $\alpha$

\[ D_x^\alpha = (-\triangle_x)^{\alpha/2}. \]

Theorem Bouchut 2002

Let $h \in L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$. Assume that $\phi \in L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$, such that $\nabla_u \phi \in (L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d))^d$, satisfies (in the sense of distributions)

\[ \partial_t \phi + (u \cdot \nabla_x \phi) - \frac{\sigma^2}{2} \triangle_u \phi = h, \quad \text{in} \; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d. \quad (7) \]

Then there exists a positive constant $C$

\[ \| \partial_t \phi + (u \cdot \nabla_x \phi) \|_{L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)} + \frac{\sigma^2}{2} \| \triangle_u \phi \|_{L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)} \leq C(d) \| h \|_{L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)}, \]

\[ \| \nabla_u D_x^{1/3} \phi \|_{L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)} + \| D_x^{2/3} \phi \|_{L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)} \leq C(d) \| h \|_{L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)}. \]
Continuity up and along $\Sigma^-$

Feynman-Kac formula (by the interior regularity):

$$f(t, x, u) = \mathbb{E}_\mathbb{P} \left[ f_0(x_t^{x, u}, u_t^{x, u})1_{\{t \leq \beta^{x,u}\}} \right] + \mathbb{E}_\mathbb{P} \left[ q(t - \beta^{x,u}, x_{\beta^{x,u}}^{x,u}, u_{\beta^{x,u}}^{x,u})1_{\{t > \beta^{x,u}\}} \right].$$

The continuity of $f$ up to $\Sigma_T^-$ will follow from the continuity of $(y, v) \mapsto (\beta^{y,v}, x_t^{y,v}, u_t^{y,v})$.

$\mathbb{P}$-almost surely, for all $t \geq 0$, the flow $(y, v) \mapsto (x_t^{y,v}, u_t^{y,v})$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$.

As $(y, v) \notin \Sigma_T^0 \cup \Sigma_T^+$, we have $\beta^{y,v} = \tau^{y,v} := \inf\{t > 0; x_t^{y,v} \notin \overline{D}\}$.

To prove that $(y, v) \mapsto \tau^{y,v}$ is continuous up to $\Sigma_T^-$, follow the general argument of the continuity of exit time related to a flow of continuous processes given in Darling & Pardoux 1997.
Chained PDEs

We consider also the semigroup related to the stopped process:

$$\Gamma_n^\psi(t, x, u) = \mathbb{E}_\mathbb{P} \left[ \psi(X^x, u_{t \land \tau_n^x, u}, U^x, u_{t \land \tau_n^x, u}) \right],$$

where \{\tau_n^{x, u}, n \in \mathbb{N}\} are the hitting times defined in the Main Theorem.

Corollary

Assume \((H_{\text{Langevin}})\). Then, for all nonnegative \(\psi \in \mathcal{C}^\infty_c(\mathcal{D} \times \mathbb{R}^d)\) and all \(n \in \mathbb{N}^*\), \(\Gamma_n^\psi\) is a nonnegative function in \(\mathcal{C}^{1,2}_b(\bar{Q}_T) \cap \mathcal{C}(\bar{Q}_T \setminus \Sigma^0)\) and satisfies

\[
\begin{cases}
\partial_t \Gamma_n^\psi(t, x, u) - (u \cdot \nabla_x \Gamma_n^\psi(t, x, u)) - \frac{\sigma^2}{2} \Delta u \Gamma_n^\psi(t, x, u) = 0, & \text{on } Q_T, \\
\Gamma_n^\psi(0, x, u) = \psi(x, u), & \text{in } \mathcal{D} \times \mathbb{R}^d, \\
\Gamma_n^\psi(t, x, u) = \Gamma_{n-1}^\psi(t, x, u - 2(u \cdot n_D(x))n_D(x)), & \text{in } \Sigma_T^+. 
\end{cases}
\]

In addition, the set \(\{\Gamma_n^\psi, n \geq 1, \Gamma_0^\psi = \psi\}\) belongs to \(L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d))\) and admits traces that satisfy the energy equality

$$\|\Gamma_n^\psi(t)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \sigma^2 \|\nabla u \Gamma_n^\psi\|_{L^2(Q_T)}^2 + \|\Gamma_n^\psi\|_{L^2(\Sigma^-)}^2 = \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \|\Gamma_{n-1}^\psi\|_{L^2(\Sigma^-)}^2.$$
Sketch of the Phase 2: On the conditional McKean-Vlasov-Fokker-Planck equation

We construct a probability density function satisfying (in the sense of distribution):

\[
\partial_t \rho + (u \cdot \nabla_x \rho) + (B[\cdot ; \rho] \cdot \nabla_u \rho) - \frac{\sigma^2}{2} \triangle_u \rho = 0 \text{ in } (0, T) \times D \times \mathbb{R}^d,
\]
\[
\rho(0, x, u) = \rho_0(x, u) \text{ on } D \times \mathbb{R}^d,
\]
\[
\gamma(\rho)(t, x, u) = \gamma(\rho)(t, x, u - 2(u \cdot n_D(x))n_D(x)) \text{ on } (0, T) \times \partial D \times \mathbb{R}^d,
\]

Definition: Maxwellian distribution

For given \( a \in \mathbb{R}, \mu > 0, P_0 \in L^1(\mathbb{R}^d) \), such that \( P_0 \geq 0 \) on \( \mathbb{R}^d \), a Maxwellian distribution with parameters \( (a, \mu, P_0) \) is a function \( P: \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^+ \) such that

\[
P(t, u) = \exp\{at\} [m(t, u)]^\mu,
\]

where \( m: \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^+ \) is defined by \( m(t, u) = (G(\sigma^2 t) \ast P_0^\mu)(u) \), with

\[
G(t, u) = \left(\frac{1}{2\pi t}\right)^{\frac{d}{2}} \exp\{-|u|^2/2t\}.
\]
Maxwellian bounds for the linear Vlasov-FP equation

Consider the unique solution in \( V_1(\omega, Q_T) \) to the linear problem

\[
\begin{cases}
T(S) + (B \cdot \nabla u)S - \frac{\sigma^2}{2} \triangle u S = 0 \quad \text{in} \quad Q_T, \\
S(0, x, u) = \rho_0(x, u) \quad \text{on} \quad D \times \mathbb{R}^d, \\
\gamma^- (S)(t, x, u) = q(t, x, u) \quad \text{on} \quad \Sigma^-_T.
\end{cases}
\]  

(9)

Proposition

Assume (\( H_{MVFP} \)). For \( B \in L^\infty((0, T) \times D; \mathbb{R}^d) \), for \((P_0, \bar{P}_0)\) as in (\( H_{MVFP} \))-(iii), let \((p, \bar{p})\) be a couple Maxwellian distributions with parameters \((a, \mu, P_0)\) and \((\bar{a}, \bar{\mu}, \bar{P}_0)\) satisfying

(a1) \( \mu > 1 \), and \( \mu \in (\frac{1}{2}, 1) \).

(a2) \( a \leq \frac{-\mu}{2\sigma^2(\mu - 1)} \|B\|^2_{L^\infty((0,T) \times D; \mathbb{R}^d)}, \) and \( \bar{a} \geq \frac{\bar{\mu}}{2\sigma^2(1 - \bar{\mu})} \|B\|^2_{L^\infty((0,T) \times D; \mathbb{R}^d)} \).

Then

(d1) \( \sup_{t \in [0, T]} \int_{\mathbb{R}^d} (1 + |u|)\omega(u) |\bar{p}(t, u)|^2 \, du < +\infty, \) and \( \inf_{t \in [0, T]} \int_{\mathbb{R}^d} p(t, u) \, du > 0. \)

(d2) If \( \underline{p} \leq q \leq \bar{p} \), on \( \Sigma^-_T \), then \( \underline{p} \leq S \leq \bar{p} \), a.e. on \( Q_T \), and \( \underline{p} \leq \gamma^+(S) \leq \bar{p} \), on \( \Sigma^+_T \).
Add the specular condition + the nonlinear term $B[x; \rho]$

**Theorem**

Under $(H_{MVFP})$, there exists a function $\rho \in V_1(\omega, Q_T)$, and there exist $\gamma^+(\rho), \gamma^-(\rho)$ defined on $\Sigma^+_T$ and $\Sigma^-_T$ respectively, with $\gamma^\pm(\rho) \in L^2(\omega, \Sigma^\pm_T)$, s.t. $\forall t \in (0, T], \forall \psi \in C_\infty^c(Q_T)$,

$$
\int_{Q_t} \left( \rho T(\psi) + \psi (B[\cdot ; \rho] \cdot \nabla_u \rho) + \frac{\sigma^2}{2} (\nabla_u \psi \cdot \nabla_u \rho) \right) (s, x, u) \, ds \, dx \, du
$$

$$
= \int_{D \times \mathbb{R}^d} \rho(t, x, u)\psi(t, x, u) \, dx \, du - \int_{D \times \mathbb{R}^d} \rho_0(x, u)\psi(0, x, u) \, dx \, du
$$

$$
+ \int_{\Sigma^+_t} (u \cdot n_D(x)) \gamma^+(\rho)(s, x, u)\psi(s, x, u) d\lambda_{\Sigma_T}(s, x, u)
$$

$$
+ \int_{\Sigma^-_t} (u \cdot n_D(x)) \gamma^+(\rho)(s, x, u - 2(u \cdot n_D(x))n_D(x))\psi(s, x, u) d\lambda_{\Sigma_T}(s, x, u).
$$

In addition, there exist a couple of Maxwellian distributions $(\overline{P}, \underline{P})$ such that

$$\underline{P} \leq \rho \leq \overline{P}, \text{ a.e. on } Q_T,$$

$$\underline{P} \leq \gamma^\pm(\rho) \leq \overline{P}, \text{ } \lambda_{\Sigma_T}\text{-a.e. on } \Sigma^\pm_T,$$

$\overline{P}$ and $\underline{P}$ satisfy the specular boundary condition , and for all $t \in (0, T]$,

$$\sup \int (1 + |u|) \omega(u) (\overline{P}(t, u))^2 \, du < +\infty, \inf \int P(t, u) \, du > 0.$$
Sketch of the Phase 3
under \((\mathbb{P}, (\tilde{w}(t); t \in [0, T])\), the canonical process \(((x(t), u(t)); t \in [0, T])\) satisfies
\[
\begin{cases}
  x(t) = x(0) + \int_0^t u(s) \, ds, \\
  u(t) = u(0) + \sigma \tilde{w}(t) - \sum_{0 < s \leq t} 2 \left( u(s^-) \cdot n_D(x(s)) \right) n_D(x(s)) \mathbf{1}_{\{x(s) \in \partial D\}},
\end{cases}
\]
Consider \(\rho^{FP} \in V_1(\omega, Q_T)\) solution to the MVFP eq.
\[
\frac{dQ}{dP} = \exp \left\{ \frac{1}{\sigma} \int_0^T B[x(t); \rho^{FP}(t)] \, d\tilde{w}(t) - \frac{1}{2\sigma^2} \int_0^T \left| B[x(t); \rho^{FP}(t)] \right|^2 \, dt \right\}.
\]
Then, according to Girsanov Theorem, \(((x(t), u(t)); t \in [0, T])\) satisfies \(Q\)-a.s.,
\[
\begin{cases}
  x(t) = x(0) + \int_0^t u(s) \, ds, \\
  u(t) = u(0) + \int_0^t B[x(s); \rho^{FP}(s)] \, ds + \sigma w(t) - \sum_{0 < s \leq t} 2 \left( u(s^-) \cdot n_D(x(s)) \right) n_D(x(s)) \mathbf{1}_{\{x(s) \in \partial D\}},
\end{cases}
\]
where \((w(t) := \tilde{w}(t) - \int_0^t B[x(s); \rho^{FP}(s)] \, ds; t \in [0, T])\) is a \(\mathbb{R}^d\)-valued \(Q\)-Brownian motion, and \(Q(x(0) \in dx, u(0) \in du) = \rho_0(x, u) dx du\).
The mild equation

\[ \rho \in C([0, T]; L^2(D \times \mathbb{R}^d)) \] is a solution to the following linear mild equation if, for all \( t \in (0, T) \), for all \( \psi \in C_c^\infty(D \times \mathbb{R}^d) \),

\[
\langle \psi, \rho(t) \rangle = \langle \Gamma^\psi(t), \rho_0 \rangle + \int_0^t \langle \nabla_u \Gamma^\psi(t-s), B[\cdot; \rho_{FP}(s)] \rho(s) \rangle \, ds,
\] (10)

Proposition

(i) There exists at most one solution in \( C([0, T]; L^2(D \times \mathbb{R}^d)) \) to the linear mild equation (10).

(ii) The weak solution \( (\rho_{FP}(t); t \in [0, T]) \) to MVFP equation is solution to the mild equation (10).

(iii) The time marginal \( Q \circ (x(t), u(t))^{-1} \) admits a density \( \rho(t) \in L^2(\omega, D \times \mathbb{R}^d) \) which is solution to the mild equation (10).
Numerical method: stochastic particle algorithm

The PIC method
The computational space is divided in cells
We use a Particle in cell (PIC) technique to compute the Eulerian fields like 
\( \langle \mathcal{U}^{(i)} \rangle (t, x) \).
We compute the Eulerian fields (mean fields) at the center of each sub-cell only.

We introduce \( N_p \) particles \((X^{k,N_p}_t, U^{k,N_p}_t)\) in \( \mathcal{D} \).
Each cell \( C \) contains \( N_{pc} \) particles by constant mass density constraint.

\[
K(., x) = 1(., C(x)).
\]

\[
\langle F(\mathcal{U}) \rangle (t, x) \approx \frac{1}{N_p} \sum_{k=1}^{N_p} F\left(U_t^{k,N_p}\right) K(X_t^{k,N_p}, x) / \frac{1}{N_p} \sum_{k=1}^{N_p} K(X_t^{k,N_p}, x)
\]

\[
\sum_{k=1}^{N_p} K(X_t^{k,N_p}, x) = N_{pc}
\]
Connected projects

- PhD Thesis of Laurent Violeau on the numerical analysis of SLM. First rate of convergence result on the fluid particle algorithm for various cases of conditional expectation estimators.

- winpos : development of wind farm simulator (with Inria.Chile)