A thin-layer reduced model for shallow viscoelastic flows

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Outline

Upper-Convected Maxwell (UCM) model

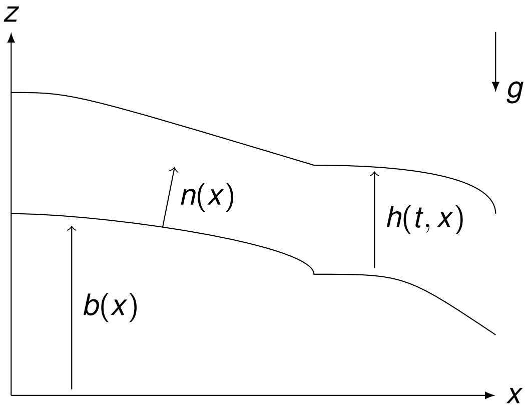
Mass and momentum equations for incompressible fluid
(velocity $\boldsymbol{u}$; pressure p ; Cauchy stress $-p\boldsymbol{I}+\boldsymbol{\tau}$)
with non-Newtonian rheology ($\boldsymbol{\tau} \not\approx \boldsymbol{D}(\boldsymbol{u}) \equiv (\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}\boldsymbol{u}^T)/2$):

$$\begin{aligned} \operatorname{div}\boldsymbol{u} &= 0 && \text{in } \mathcal{D}_t, \\ \partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} &= -\boldsymbol{\nabla}p+\operatorname{div}\boldsymbol{\tau}+\boldsymbol{f} && \text{in } \mathcal{D}_t, \\ \lambda\left(\partial_t\boldsymbol{\tau}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\tau}-(\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{\tau}-\boldsymbol{\tau}(\boldsymbol{\nabla}\boldsymbol{u})^T\right) &= \eta_p\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{\tau} && \text{in } \mathcal{D}_t, \end{aligned}$$

under gravity $\boldsymbol{f} \equiv -g\boldsymbol{e}_z$ in time-dependent domain $\mathcal{D}_t \subset \mathbb{R}^2$

$$\mathcal{D}_t = \{\boldsymbol{x}=(x,z), \quad x\in(0,L), \quad 0<z-b(x)<h(t,x)\}$$

Thin-layer geometry with non-folded interfaces



A free-surface boundary value problem

We supply the UCM model with initial and boundary conditions

$$
\begin{aligned}
\boldsymbol{u}\cdot\boldsymbol{n} &= 0, && \text{for } z = b(x), && x\in(0,L),\\
\boldsymbol{\tau}\boldsymbol{n} &= ((\boldsymbol{\tau}\boldsymbol{n})\cdot\boldsymbol{n})\,\boldsymbol{n}, && \text{for } z = b(x), && x\in(0,L),\\
\partial_t h + u_x\partial_x(b+h) &= u_z, && \text{for } z = b(x)+h(t,x), && x\in(0,L),\\
(p\boldsymbol{I}-\boldsymbol{\tau})\cdot(-\partial_x(b+h),1) &= 0, && \text{for } z = b(x)+h(t,x), && x\in(0,L),
\end{aligned}
$$

where $\boldsymbol{n}$ is the unit normal vector at the bottom inward the fluid

$$
n_x = \frac{-\partial_x b}{\sqrt{1+(\partial_x b)^2}} \qquad n_z = \frac{1}{\sqrt{1+(\partial_x b)^2}}\,.
$$

$$\partial_x u_x + \partial_z u_z = 0,$$

$$\partial_t u_x + u_x \partial_x u_x + u_z \partial_z u_x = -\partial_x p + \partial_x \tau_{xx} + \partial_z \tau_{xz},$$

$$\partial_t u_z + u_x \partial_x u_z + u_z \partial_z u_z = -\partial_z p + \partial_x \tau_{xz} + \partial_z \tau_{zz} - g,$$

$$\partial_t \tau_{xx} + u_x \partial_x \tau_{xx} + u_z \partial_z \tau_{xx} = (2\partial_x u_x)\tau_{xx} + (2\partial_z u_x)\tau_{xz} + \frac{\eta_p \partial_x u_x - \tau_{xx}}{\lambda},$$

$$\partial_t \tau_{zz} + u_x \partial_x \tau_{zz} + u_z \partial_z \tau_{zz} = (2\partial_x u_z)\tau_{xz} + (2\partial_z u_z)\tau_{zz} + \frac{\eta_p \partial_z u_z - \tau_{zz}}{\lambda},$$

$$\partial_t \tau_{xz} + u_x \partial_x \tau_{xz} + u_z \partial_z \tau_{xz} = (\partial_x u_z)\tau_{xx} + (\partial_z u_x)\tau_{zz} + \frac{\frac{\eta_p(\partial_z u_x + \partial_x u_z)}{2} - \tau_{xz}}{\lambda}$$

$$\begin{aligned}
u_z &= (\partial_x b) u_x && \text{at } z = b\,, \\
-(\partial_x b)\tau_{xx} + \tau_{xz} &= -\partial_x b\Big(-(\partial_x b)\tau_{xz} + \tau_{zz}\Big) && \text{at } z = b\,, \\
-\partial_x (b+h)(p - \tau_{xx}) - \tau_{xz} &= 0 && \text{at } z = b + h\,, \\
\partial_x (b+h)\tau_{xz} + (p - \tau_{zz}) &= 0 && \text{at } z = b + h, \\
\partial_t h + u_x \partial_x (b+h) &= u_z && \text{at } z = b + h.
\end{aligned}$$

Long-wave asymptotic regime for shallow flows

(H1) $h \sim \epsilon$ as $\epsilon \to 0$ $\quad \partial_t = O(1),\ \partial_x = O(1),\ \partial_z = O(1/\epsilon)$

$$\text{(H2) } \partial_x b = O(\epsilon) \Rightarrow u_z = (\partial_x b) u_x|_{z=b} - \int_b^z \partial_x u_x = O(\epsilon)$$

$\Rightarrow \partial_x h = O(\epsilon)$ i.e. long waves, since $\partial_t h + u_x \partial_x (b+h) = u_z|_{z=b+h}$

(H3) $\boldsymbol{\tau} = O(\epsilon)$, hence also $\eta_p \sim \epsilon\ (\lambda \sim 1), \Rightarrow \partial_z p = \partial_z \tau_{zz} - g + O(\epsilon)$

(H4) motion by slice $\partial_z u_x = O(1) \Rightarrow \partial_z \tau_{xz} = D_t u_x + O(\epsilon)$

compatible with $\tau_{xz}|_{z=b,b+h}$ if $\tau_{xz} = O(\epsilon^2), \Rightarrow \partial_z u_x = O(\epsilon)$

Momentum depth-average & $u_x(t,x,z) = u_x^0(t,x) + O(\epsilon^2)$ yields

$$0 = \int_b^{b+h} \partial_x u_x + \partial_z u_z = \partial_t h + \partial_x \int_b^{b+h} u_x = \partial_t h^0 + \partial_x (h^0 u_x^0) + O(\epsilon^2) \dots$$

Viscoelastic Saint-Venant equations

Assuming $\partial_z \tau_{xx}, \partial_z \tau_{zz} = O(1)$, we get the closed system

$$\begin{cases} \partial_t h + \partial_x (h u_x) = 0, \\ \partial_t (h u_x) + \partial_x \left(h (u_x)^2 + g \dfrac{h^2}{2} + h(\tau_{zz} - \tau_{xx}) \right) = -g(\partial_x b) h, \\ \partial_t \tau_{xx} + u_x \partial_x \tau_{xx} = 2(\partial_x u_x)\tau_{xx} + \dfrac{\eta_p}{\lambda} \partial_x u_x - \dfrac{1}{\lambda}\tau_{xx}, \\ \partial_t \tau_{zz} + u_x \partial_x \tau_{zz} = -2(\partial_x u_x)\tau_{zz} - \dfrac{\eta_p}{\lambda} \partial_x u_x - \dfrac{1}{\lambda}\tau_{zz}. \end{cases}$$

To explore the new reduced model: numerical simulations with a Finite-Volume scheme (conservativity).
Anticipating stability of the FV scheme: mathematical entropy.

UCM eqns naturally dissipate energy !

With $\boldsymbol{\sigma} = \boldsymbol{I} + \frac{2\lambda}{\eta_p}\boldsymbol{\tau}$, the UCM model rewrites

$$\lambda\left(\partial_t\boldsymbol{\sigma} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\sigma} - (\nabla\boldsymbol{u})\boldsymbol{\sigma} - \boldsymbol{\sigma}(\nabla\boldsymbol{u})^T\right) = \boldsymbol{I} - \boldsymbol{\sigma} \qquad \text{in } \mathcal{D}_t\,,$$

and thermodynamics imposes $\boldsymbol{\sigma}$ s.p.d. and a free energy

$$F(\boldsymbol{u},\boldsymbol{\sigma}) = \int_{\mathcal{D}_t}\left(\frac{1}{2}|\boldsymbol{u}|^2 + \frac{\eta_p}{4\lambda}\boldsymbol{I} : (\boldsymbol{\sigma} - \ln\boldsymbol{\sigma} - \boldsymbol{I}) - \boldsymbol{f}\cdot\boldsymbol{x}\right)d\boldsymbol{x}\,, \quad (3)$$

$$\frac{d}{dt}F(\boldsymbol{u},\boldsymbol{\sigma}) = -\frac{\eta_p}{4\lambda^2}\int_{\mathcal{D}_t}\boldsymbol{I} : (\boldsymbol{\sigma} + \boldsymbol{\sigma}^{-1} - 2\boldsymbol{I})d\boldsymbol{x}\,. \quad (4)$$

Reformulation with energy dissipation

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x \left(hu^2 + g\frac{h^2}{2} + \frac{\eta_p}{2\lambda} h(\sigma_{zz} - \sigma_{xx}) \right) = -gh\partial_x b, \\ \partial_t \sigma_{xx} + u\partial_x \sigma_{xx} - 2\sigma_{xx}\partial_x u = \frac{1 - \sigma_{xx}}{\lambda}, \\ \partial_t \sigma_{zz} + u\partial_x \sigma_{zz} + 2\sigma_{zz}\partial_x u = \frac{1 - \sigma_{zz}}{\lambda}, \end{cases}$$

$$\partial_t \left(h\frac{u^2}{2} + g\frac{h^2}{2} + gbh + \frac{\eta_p}{4\lambda} h \operatorname{tr}(\boldsymbol{\sigma} - \ln \boldsymbol{\sigma} - \boldsymbol{I}) \right)$$
$$+\partial_x \left(hu \left(\frac{u^2}{2} + g(h+b) + \frac{\eta_p}{2\lambda} \left(\frac{\operatorname{tr}(\boldsymbol{\sigma} - \ln \boldsymbol{\sigma} - \boldsymbol{I})}{2} + \sigma_{zz} - \sigma_{xx} \right) \right) \right)$$
$$= -\frac{\eta_p}{4\lambda^2} h \operatorname{tr}(\boldsymbol{\sigma} + [\boldsymbol{\sigma}]^{-1} - 2\boldsymbol{I}).$$

Outline

Flux (conservative) formulation $\partial_t U + \partial_x F(U) = S$

$$(S)\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + P(h,\mathbf{s})\right) = -gh\partial_x b, \\ \partial_t(h\mathbf{s}) + \partial_x(hu\mathbf{s}) = \dfrac{h\mathcal{S}(h,\mathbf{s})}{\lambda}, \end{cases}$$

where $\mathbf{s} = \left(\frac{\sigma_{xx}^{-1/2}}{h}, \frac{\sigma_{zz}^{1/2}}{h}\right)$, $P(h,\mathbf{s}) = g\frac{h^2}{2} + \frac{\eta_p}{2\lambda}h(\sigma_{zz} - \sigma_{xx})$,

$\mathcal{S}(h,\mathbf{s}) = \left(-\frac{\sigma_{xx}^{-3/2}}{2h}(1-\sigma_{xx}), \frac{\sigma_{zz}^{-1/2}}{2h}(1-\sigma_{zz})\right)$ is hyperbolic.

∇F: real eigenvalues $\left(\left(\frac{\partial P}{\partial h}\right)_{|\mathbf{s}} = gh + \frac{\eta_p}{2\lambda}(3\sigma_{zz} + \sigma_{xx}) > 0\right)$

$$\lambda_{1,3} = u \pm \sqrt{gh + \frac{\eta_p}{2\lambda}(3\sigma_{zz} + \sigma_{xx})} \text{ g.n.l}, \qquad \lambda_2 = u \text{ l.d.}.$$

Shall we discretize (S) by splitting: i) conservation ii) diffusion ?

Problem: how to ensure stability

There is a problem with discretizing (S) in conservative variables: the natural energy is not convex with respect to ***s*** !

$$\widetilde{E} = h\frac{u^2}{2} + g\frac{h^2}{2} + gbh + \frac{\eta_p}{4\lambda} h\left(\sigma_{xx} + \sigma_{zz} - \ln(\sigma_{xx}\sigma_{zz}) - 2\right)$$

Now, convexity is essential to entropic stability of FV schemes (Jensen) and to preserve the invariant domain $\{h, \sigma_{xx}, \sigma_{zz} \geq 0\}$.

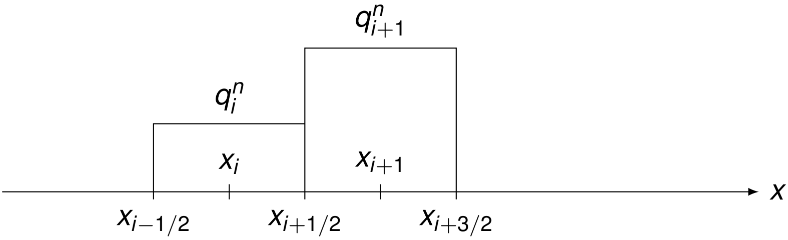
A splitted Finite-Volume approach

- Free-energy-dissipating FV scheme: piecewise constants (anticipate: approximations of *non-conservative* variables)

$$q \equiv (q_1, q_2, q_3, q_4)^T := (h, hu, h\sigma_{xx}, h\sigma_{zz})^T$$

  on a mesh of $\mathbb{R}$ with cells $(x_{i-1/2}, x_{i+1/2})$, $i \in \mathbb{Z}$ of volumes $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ at centers $x_i = \frac{x_{i-1/2}+x_{i+1/2}}{2}$
- At each discrete time t_n, variables updated by splitting:
  1. Without source: Riemann problems (Godunov approach)
  2. + topo $h\partial_x b$: preprocessing (hydrostatic reconstruction)
  3. + dissipative sources in $\boldsymbol{\sigma}$: implicit
- Main difficulties: free-energy dissipation + $h, \sigma_{xx}, \sigma_{zz} \geq 0$

Finite Volume discretization



Step 1: Godunov approach

$$q_i^n \approx \frac{1}{\Delta x_i} \int_{\Delta x_i} q(t_n, \cdot) \to q_i^{n+\frac{1}{2}} = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} q^{appr}(t_{n+1} - 0, \cdot)$$

$$q^{appr}(t, x) = R\left(\frac{x - x_{i+1/2}}{t - t_n}, q_i^n, q_{i+1}^n\right) \quad \text{for } x_i < x < x_{i+1},$$

$R(\frac{x}{t}, q_l, q_r)$ is Riemann solver of the system without source,

$$+ \text{ CFL} \begin{cases} \frac{x}{t} < -\frac{\Delta x_i}{2\Delta t} \Rightarrow R(\frac{x}{t}, q_i, q_{i+1}) = q_i, \\ \frac{x}{t} > \frac{\Delta x_{i+1}}{2\Delta t} \Rightarrow R(\frac{x}{t}, q_i, q_{i+1}) = q_{i+1}, \end{cases} \quad \text{for } \Delta t = t_{n+1} - t_n.$$

$$q_i^{n+\frac{1}{2}} = q_i^n + \frac{\Delta t}{\Delta x_i}\Bigg(\int_{-\Delta x_i/2}^{0} \left(R(\xi, q_i^n, q_{i+1}^n) - q_i^n\right) d\xi + \int_0^{\Delta x_i/2} \left(R(\xi, q_{i-1}^n, q_i^n) - q_i^n\right) d\xi\Bigg)$$

Step 1: the free-energy flux condition

$$E(q_i^{n+\frac{1}{2}}) \leq E\left(\frac{1}{\Delta x_i}\int_{-\Delta x_i/2}^{0} R(\xi, q_i^n, q_{i+1}^n)d\xi\right) + E\left(\frac{1}{\Delta x_i}\int_{0}^{\Delta x_i/2} R(\xi, q_{i-1}^n, q_i^n)d\xi\right)$$

with Jensen inequality and the definitions (whatever G)

$$G_l(q_l, q_r) = G(q_l) - \int_{-\infty}^{0}\Big(E\big(R(\xi, q_l, q_r)\big) - E\big(q_l\big)\Big)d\xi,$$
$$G_r(q_l, q_r) = G(q_r) + \int_{0}^{\infty}\Big(E\big(R(\xi, q_l, q_r)\big) - E\big(q_r\big)\Big)d\xi,$$

implies, provided $G_r(q_l, q_r) \leq G(q_l, q_r) \leq G_l(q_l, q_r)$

$$E(q_i^{n+\frac{1}{2}}) \quad \leq \quad E(q_i^n) \; + \; \frac{\Delta t}{\Delta x_i}\left(G(q_{i-1}^n, q_i^n) - G(q_i^n, q_{i+1}^n)\right)$$

Similarity with isentropic gas dynamics !

Fortunately, hyperbolic step simply advects $\boldsymbol{s}$ ($\partial_t \boldsymbol{s} + u\partial_x \boldsymbol{s} = 0$), so the situation for i) is similar to isentropic gas dynamics:

- smooth $P = g\frac{h^2}{2} + \frac{\eta_p}{2\lambda} h(\sigma_{zz} - \sigma_{xx}) = P(h, \boldsymbol{s})$ still satisfy $\partial_t(hP) + \partial_x(huP) + (h^2\partial_h P|_{\boldsymbol{s}})\partial_x u = 0$ so one can still invoke Suliciu *relaxation scheme* introducing $\pi \approx P$ as a new variable (the contact-discontinuity solution has same "structure": 3 waves with same speeds & Riemann inv.)
- thus a (discrete) entropic stability can still be established for the FV scheme under same subcharacteristic condition: choose c large enough so that it holds $h^2\partial_h P|_{\boldsymbol{s}} \leq c^2$ for all states on the left/right of the central wave with speed u.

Step 1: approximate Riemann solver

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \quad \partial_t(hu) + \partial_x \left(hu^2 + P\right) = 0, \\ \partial_t(h\sigma_{xx}) + \partial_x(hu\sigma_{xx}) - 2h\sigma_{xx}\partial_x u = 0, \\ \partial_t(h\sigma_{zz}) + \partial_x(hu\sigma_{zz}) + 2h\sigma_{zz}\partial_x u = 0, \end{cases} \tag{5}$$

$\simeq$ gas dynamics for smooth $P = g\frac{h^2}{2} + \frac{\eta_p}{2\lambda}h(\sigma_{zz} - \sigma_{xx}) = P(h, \boldsymbol{s})$

$$\partial_t \boldsymbol{s} + u\partial_x \boldsymbol{s} = 0 \qquad \partial_t(hP) + \partial_x (huP) + (h^2\partial_h P|_{\boldsymbol{s}})\partial_x u = 0$$

- *Suliciu relaxation* workable: we introduce a "pressure" variable π (such that $\partial_t(h\pi) + \partial_x(h\pi u) + c^2\partial_x u = 0$) and a variable $c > 0$ to parametrize the speeds ($c^2 \geq h^2\partial_h P|_{\boldsymbol{s}}$).
- Riemann problems of the new system for $(h, hu, h\pi, hc, h\boldsymbol{s})$ will be exactly solvable and the latter will define our approximate solutions to the initial Riemann problems.

Step 1: contact discontinuities

The initial system rewritten with $\partial_t(hu) + \partial_x(hu^2 + \pi) = 0$ and

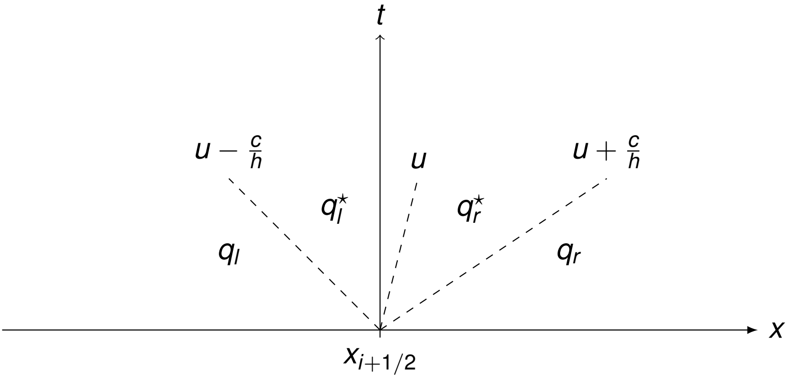
$$\partial_t c + u\partial_x c = 0 \qquad \partial_t(h\pi/c^2) + \partial_x(h\pi u/c^2 + u) = 0$$

has a quasi diagonal form with 3 waves of speeds $u, u \pm \frac{c}{h}$

$$\begin{cases} \partial_t(\pi + cu) + (u + c/h)\partial_x(\pi + cu) - \dfrac{u}{h} c\partial_x c = 0\,, \\ \partial_t(\pi - cu) + (u - c/h)\partial_x(\pi - cu) - \dfrac{u}{h} c\partial_x c = 0\,, \\ \partial_t \left(1/h + \pi/c^2\right) + u\partial_x \left(1/h + \pi/c^2\right) = 0\,, \\ \partial_t c + u\partial_x c = 0\,, \end{cases} \tag{6}$$

all linearly degenerate: Riemann problems exactly solvable !

Riemann solution with 3 contact discontinuities



Step 1: free energy dissipation

Advantage of the relaxation:

it ensures the dissipation of a *convex* energy like

$$E = h\frac{u^2}{2} + g\frac{h^2}{2} + \frac{\eta_p}{4\lambda} h \left(\sigma_{xx} + \sigma_{zz} - \ln(\sigma_{xx}\sigma_{zz}) - 2\right)$$

provided a *subcharacteristic* condition is satisfied !

Note: E without gbh, unlike $\widetilde{E}$

Step 1: energy equation

$E = hu^2/2 + he$ formally satisfies

$$\partial_t\Big(hu^2/2 + he\Big) + \partial_x\Big(\big(hu^2/2 + he + \pi\big)u\Big) = 0.$$

On introducing a new variable ê solution to the equation

$$\partial_t(\widehat{e} - \pi^2/2c^2) + u\partial_x(\widehat{e} - \pi^2/2c^2) = 0$$

solved simultaneaously with the relaxation system, we show a discrete free energy *inequality* thanks to convexity of E (under a *subcharacteristic* condition on c).

$\sigma_{xx}, \sigma_{zz} > 0$ automatically ensured by convexity (like $h \geq 0$) !

Step 1: the free-energy flux condition

$$E(q_i^{n+\frac{1}{2}}) \leq E\left(\frac{1}{\Delta x_i}\int_{-\Delta x_i/2}^{0} R(\xi, q_i^n, q_{i+1}^n)d\xi\right) + E\left(\frac{1}{\Delta x_i}\int_{0}^{\Delta x_i/2} R(\xi, q_{i-1}^n, q_i^n)d\xi\right)$$

with Jensen inequality and the definitions (whatever G)

$$G_l(q_l, q_r) = G(q_l) - \int_{-\infty}^{0}\Big(E\big(R(\xi, q_l, q_r)\big) - E(q_l)\Big)d\xi,$$
$$G_r(q_l, q_r) = G(q_r) + \int_{0}^{\infty}\Big(E\big(R(\xi, q_l, q_r)\big) - E(q_r)\Big)d\xi,$$

implies, provided $G_r(q_l, q_r) \leq G(q_l, q_r) \leq G_l(q_l, q_r)$

$$E(q_i^{n+\frac{1}{2}}) \leq E(q_i^n) + \frac{\Delta t}{\Delta x_i}\left(G(q_{i-1}^n, q_i^n) - G(q_i^n, q_{i+1}^n)\right)$$

Step 1: free-energy flux

Recall $\partial_t\Big(hu^2/2 + h\widehat{e}\Big) + \partial_x\Big(\big(hu^2/2 + h\widehat{e} + \pi\big)u\Big) = 0$ here.

Define the energy flux $G(q_l, q_r) = \Big(\big(hu^2/2 + h\widehat{e} + \pi\big)u\Big)_{x/t=0}$

$$
\begin{aligned}
G(q_l, q_r) &= G(q_l) - \int_{-\infty}^{0}\Big((hu^2/2 + \hat{e})(\xi) - E(q_l)\Big)d\xi,\\
&= G(q_r) + \int_{0}^{\infty}\Big((hu^2/2 + \hat{e})(\xi) - E(q_r)\Big)d\xi,
\end{aligned}
$$

where $G = (E + P)u$

A discrete free energy inequality finally holds provided

$$
E\big(R(\xi, q_l, q_r)\big) \le (hu^2/2 + \hat{e})(\xi)\ \forall \xi. \tag{7}
$$

Step 1: subcharacteristic condition

For the solution to Riemann problem $R(\cdot, q_l, q_r)$ initialized with

$$(\pi_l, \hat{e}_l) = (P, e)(q_l) \qquad (\pi_r, \hat{e}_r) = (P, e)(q_r)$$

the condition (7) is ensured provided

$$\begin{aligned} \forall h \in [h_l, h_l^\star] \quad & h^2 \partial_h P|_{\boldsymbol{s}}(h, \boldsymbol{s}_l) \le c_l^2, \\ \forall h \in [h_r, h_r^\star] \quad & h^2 \partial_h P|_{\boldsymbol{s}}(h, \boldsymbol{s}_r) \le c_r^2. \end{aligned} \tag{8}$$

Step 1: explicit choice of the speeds

Defining $P_l = P(h_l, \boldsymbol{s}_l)$, $P_r = P(h_r, \boldsymbol{s}_r)$, and

$$a_l = \sqrt{\partial_h P|_{\boldsymbol{s}}(h_l, \boldsymbol{s}_l)},\ a_r = \sqrt{\partial_h P|_{\boldsymbol{s}}(h_r, \boldsymbol{s}_r)},$$

the following explicit choice works

$$\frac{c_l}{h_l} = a_l + 2\left(\max\left(0, u_l - u_r\right) + \frac{\max\left(0, P_r - P_l\right)}{h_l a_l + h_r a_r}\right),$$

$$\frac{c_r}{h_r} = a_r + 2\left(\max\left(0, u_l - u_r\right) + \frac{\max\left(0, P_l - P_r\right)}{h_l a_l + h_r a_r}\right).$$

$$|P| \leq h \partial_h P|_{\boldsymbol{s}} \Rightarrow \max\left(\frac{c_l}{h_l}, \frac{c_r}{h_r}\right) \leq C\left(|u^0_{x,l}| + |u^0_{x,r}| + a_l + a_r\right),$$

Step 1: explicit Riemann solution

On each interface, by Riemann invariants conservations:

$$u_l^* = u_r^* = u^* = \frac{c_l u_l + c_r u_r + \pi_l - \pi_r}{c_l + c_r}, \qquad \pi_l^* = \pi_r^* = \frac{c_r \pi_l + c_l \pi_r - c_l c_r (u_r - u_l)}{c_l + c_r},$$

$$\frac{1}{h_l^*} = \frac{1}{h_l} + \frac{c_r(u_r - u_l) + \pi_l - \pi_r}{c_l(c_l + c_r)}, \qquad \frac{1}{h_r^*} = \frac{1}{h_r} + \frac{c_l(u_r - u_l) + \pi_r - \pi_l}{c_r(c_l + c_r)},$$

$$c_l^* = c_l, \quad c_r^* = c_r, \quad \boldsymbol{s}_l^* = \boldsymbol{s}_l, \quad \boldsymbol{s}_r^* = \boldsymbol{s}_r,$$

$$\sigma_{xx,l}^* = \sigma_{xx,l} \left(\frac{h_l}{h_l^*}\right)^2, \quad \sigma_{xx,r}^* = \sigma_{xx,r} \left(\frac{h_r}{h_r^*}\right)^2,$$

$$\sigma_{zz,l}^* = \sigma_{zz,l} \left(\frac{h_l^*}{h_l}\right)^2, \quad \sigma_{zz,r}^* = \sigma_{zz,r} \left(\frac{h_r^*}{h_r}\right)^2,$$

$$\widehat{e}_l^* = e_l - \frac{(\pi_l)^2}{2c_l^2} + \frac{(\pi_l^*)^2}{2c_l^2}, \qquad \widehat{e}_r^* = e_r - \frac{(\pi_r)^2}{2c_r^2} + \frac{(\pi_r^*)^2}{2c_r^2},$$

and we get a solution with a discrete free-energy inequality.

Step 1: CFL and flux formula

We use $\Delta t \max(|\Sigma_1|, |\Sigma_2|, |\Sigma_3|) \leq \frac{1}{2}\min(\Delta x_i, \Delta x_{i+1})$ and

$$\mathcal{F}_l = \left(\mathcal{F}^h, \mathcal{F}^{hu}, \mathcal{F}_l^{h\sigma_{xx}}, \mathcal{F}_l^{h\sigma_{zz}}\right), \qquad \mathcal{F}_r = \left(\mathcal{F}^h, \mathcal{F}^{hu}, \mathcal{F}_r^{h\sigma_{xx}}, \mathcal{F}_r^{h\sigma_{zz}}\right),$$

where the conservative part is standard

$$\mathcal{F}^h = (hu)_{x/t=0}, \qquad \mathcal{F}^{hu} = (hu^2 + \pi)_{x/t=0}$$

and denoting $\Sigma_1 = u_l - c_l/h_l$, $\Sigma_2 = u^*$, $\Sigma_3 = u_r + c_r/h_r$,

$$\begin{aligned}\mathcal{F}_{l,r}^{h\sigma_{xx,zz}} = (h\sigma_{xx,zz}u)_{l,r} &+ \min(0,\Sigma_1)\Big((h\sigma_{xx,zz})^*_{l,r} - (h\sigma_{xx,zz})_{l,r}\Big)\\ &+ \min(0,\Sigma_2)\Big((h\sigma_{xx,zz})^*_{r} - (h\sigma_{xx,zz})^*_{l,r}\Big)\\ &+ \min(0,\Sigma_3)\Big((h\sigma_{xx,zz})_{r} - (h\sigma_{xx,zz})^*_{r}\Big)\end{aligned}$$

Step 2: topographic source

We want to treat topographic source term such that for $\widetilde{E}(q,b) = E(q) + ghb$ and $\widetilde{G}(q,b) = G(q) + ghbu$:

$$\partial_t \widetilde{E}_i + \widetilde{G}_{i+1/2} - \widetilde{G}_{i-1/2} \leq 0$$

and steady states at rest are preserved.
$\Rightarrow$ [Audusse-Bouchut-Bristeau-Klein-Perthame 2004]

With $q = (h, hu, h\sigma_{xx}, h\sigma_{zz})$ and $\Delta b_{i+1/2} = b_{i+1} - b_i$

$$q_i^{n+1/2} = q_i^n - \frac{\Delta t}{\Delta x_i} \left(F_l(q_i^n, q_{i+1}^n, \Delta b_{i+1/2}) - F_r(q_{i-1}^n, q_i^n, \Delta b_{i-1/2}) \right).$$

Step 2: hydrostatic reconstruction

$$h_l^\sharp = \big(h_l - (\Delta b)_+\big)_+, \qquad h_r^\sharp = \big(h_r - (-\Delta b)_+\big)_+,$$

$$q_l^\sharp = \Big(h_l^\sharp, h_l^\sharp u_l, h_l^\sharp \sigma_{xx,l}, h_l^\sharp \sigma_{zz,l}\Big), \qquad q_r^\sharp = \Big(h_r^\sharp, h_r^\sharp u_r, h_r^\sharp \sigma_{xx,r}, h_r^\sharp \sigma_{zz,r}\Big),$$

$$F_l(q_l, q_r, \Delta b) = \mathcal{F}_l(q_l^\sharp, q_r^\sharp) + \left(0, g\frac{h_l^2}{2} - g\frac{h_l^{\sharp 2}}{2}, 0, 0\right),$$

$$F_r(q_l, q_r, \Delta b) = \mathcal{F}_r(q_l^\sharp, q_r^\sharp) + \left(0, g\frac{h_r^2}{2} - g\frac{h_r^{\sharp 2}}{2}, 0, 0\right),$$

Well-balanced scheme: preserves steady states $u = 0$, $h + b = cst$, $\sigma_{xx} = \sigma_{zz} = 1$.

Step 3: stress source terms

Compute $\sigma^{n+1}_{xx,zz}$ from $\sigma^{n+1/2}_{xx,zz}$ impicitly:

$$\left(\frac{\lambda}{\Delta t}+1\right)\sigma_i^{n+1}=\frac{\lambda}{\Delta t}\sigma_i^{n+1/2},$$

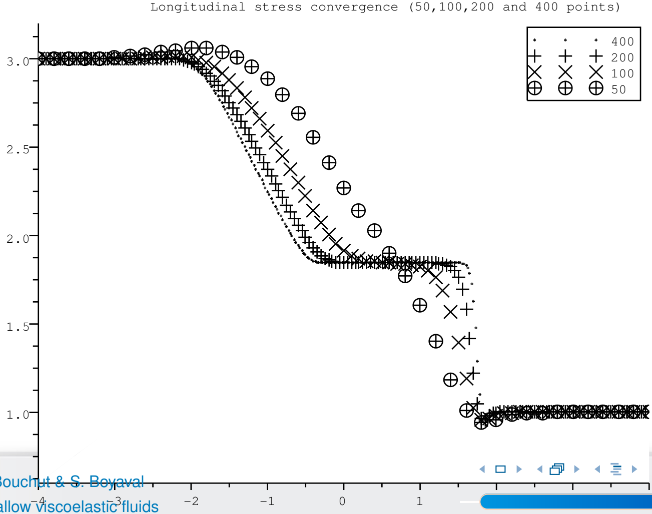
dissipative by convexity of the energy $\tilde{E}$

$$\frac{\tilde{E}_i^{n+1}-\tilde{E}_i^{n+1/2}}{\Delta t}\leq(1-\boldsymbol{\sigma}^{-1})_i^{n+1}:\left(\frac{\boldsymbol{\sigma}_i^{n+1}-\boldsymbol{\sigma}_i^{n+1/2}}{\Delta t}\right)=\frac{2-\operatorname{tr}\boldsymbol{\sigma}_i^{n+1}}{\lambda}.$$

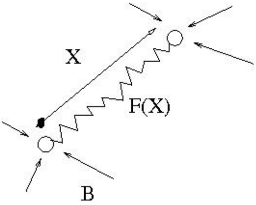
Outline

Dam break (Stoker) at $\eta_p = \lambda = 1$

$(h, hu, h\sigma_{xx}, h\sigma_{zz})_{t=0} = (3 - 2H(x))(1, 0, 1, 1)$, $b \equiv 0$.
$h\sigma_{xx}$: $50, 100, 200$ and 400 points at time $T = .2$ ($CFL = 1/2$).



Statistical physics interpretation



Assume that $\mathbf{X}(x,t) \equiv \mathbf{R}(t)$ is a 2D diffusion process solution to overdamped Langevin (Ito SDE) in every point x:

$$d\mathbf{R}(t)\,(+u\partial_x\mathbf{R}(t)dt) = \begin{pmatrix} \partial_x u & 0 \\ 0 & -\partial_x u \end{pmatrix} \mathbf{R}(t)dt - \frac{1}{2\lambda}\mathbf{R}(t)dt - \frac{1}{\sqrt{\lambda}}d\mathbf{B}(t)$$

competition between drag, extension, elasticity and Brownian collisions $\mathbf{B}(t)$ with “temperature” λ (defining a relaxation time to equilibrium) then $\sigma_{xx} = E(R_x(t)^2), \sigma_{zz} = E(R_z(t)^2)$

Literature

- Gas dynamics analogy: Suliciu Brenier Perthame Souganidis Godlewski Coquel ...
- Relaxation (stability), hydrostatic reconstruction: Bouchut
- Reduced model: Gerbeau Perthame Marche ...
  with non-Newtonian rheology: Homsy Vila Noble Chupin ... + [Bouchut Boyaval, new preprint HAL-ENPC 2013]