On a Monte-Carlo Semi-Lagrangian scheme: variance estimate and simulations

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The Monte-Carlo method

Problem: approximation of the solution of PDEs like

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + c(x).\nabla u(t,x), \quad 0 < t \leq T, x \in \mathbb{R}^d,$$
$$u(0,x) = u_0(x), \text{ for any } x \in \mathbb{R}^d.$$

Probabilistic representation:

$$u(t,x) = \mathbb{E}u_0(X_t^x)$$
$$dX_t^x = c(X_t^x)dt + dB_t, \quad X_0^x = x.$$

Monte-Carlo method:

$$\mathbb{E}\Big|\frac{1}{N}\sum_{m=1}^{N} u_0(X_t^{x,(m)}) - u(t,x)\Big|^2 \leq \frac{C(t,x)}{N}.$$

Problems:

- Realizations of X_t^x? Need of a numerical scheme.
- Computations at points in a grid: $u(n\delta t, j\delta x)$. Dependence of $C(t,x)$ with respect to δt and δx?

Semi-lagrangian scheme

Recursive construction of u_j^n approximating $u(n\delta t, j\delta x)$.

1. Representation formula between $n\delta t$ and $(n+1)\delta t$:

$$u((n+1)\delta t, j\delta x) = \mathbb{E} u(n\delta t, X_{\delta t}^{j\delta x});$$

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3. Use of an interpolation operator $\mathcal{J}$:

$$v_j^{n+1} := \mathbb{E}\big(\mathcal{J}(u^n)(j\delta x + \delta t c(j\delta x) + \sqrt{\delta t}\mathcal{N})\big);$$

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4. Monte-Carlo approximation:

$$u_j^{n+1} := \frac{1}{N}\sum_{m=1}^{N}\big(\mathcal{J}(u^n)(j\delta x + \delta t c(j\delta x) + \sqrt{\delta t}\mathcal{N}^{n,m,j})\big).$$

Structure of the noise

Independence of the $\mathcal{N}^{n,m,j} \sim \mathcal{N}(0, I)$, with respect to:

- m: Monte-Carlo approximation.
  Consequence: variance of size $\mathcal{C}(\delta t, \delta x)/N$;
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- n: independent increments/Markov property;
- j: spatial independence.

Consequences:

Improved estimate for the l^2-norm:

$$\|u\|_{l^2}^2 := \delta x \sum_j u_j^2;$$

but irregular functions in the h^1 semi-norm:

$$|u|_{h^1}^2 := \delta x \sum_j \frac{(u_{j+1}-u_j)^2}{\delta x^2}.$$

Simplified setting for a convergence estimate

In dimension $d = 1$.

No drift: $c(x) = 0$.

Boundary conditions: periodic.

Interpolation: linear.

Initial condition: u_0 is of class $\mathcal{C}^2$.

First convergence result

Theorem

For any final time $T > 0$, there exists a constant $C > 0$, such that for any $\delta t > 0$, $\delta x > 0$ and $N \in \mathbb{N}^*$ we have

$$\sup_{j \in \mathbb{N}; 0 \leq j\delta x < 1} |u(n\delta t, x_j) - v_j^n| \leq C \frac{\delta x^2}{\delta t} \sup_{x \in [0,1]} |u_0^{''}(x)|$$

and for any $n \in \mathbb{N}$ with $n\delta t \leq T$

$$\mathbb{E}\|u^n - v^n\|_{l^2}^2 = \delta x \sum_{j; 0 \leq j\delta x < 1} \mathbb{E}|u_j^n - v_j^n|^2$$

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and for any $n \in \mathbb{N}$ with $n\delta t \leq T$

$$\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 \leq 2C|u^0|_{h^1}^2 (1 + \frac{\delta x^2}{\delta t}) \frac{\delta t}{N}$$

whenever N is sufficiently large:

$$\frac{C}{N} \mathcal{A}(\delta t, \delta x) \leq \frac{1}{2}.$$

with $\mathcal{A}(\delta t, \delta x) = 1 + \frac{\delta x}{\delta t} + \frac{\delta x^2}{\delta t^2}(1 + |\log(\delta t)|)$.

Second convergence result

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Explicit method, with no CFL.

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$$\mathbb{E}\|u^n - v^n\|_{l^2}^2 \le C_p |u^0|_{h^1}^2 \Big(1 + \frac{\delta x^2}{\delta t}\Big) \mathcal{A}(\delta t, \delta x)^p \left(\frac{\delta t}{N} + \frac{1}{N^{p+1}}\right)$$

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Interpretation: anti-CFL condition and variance of size $\frac{\delta t}{N}$.

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Numerical illustration

Periodic case:

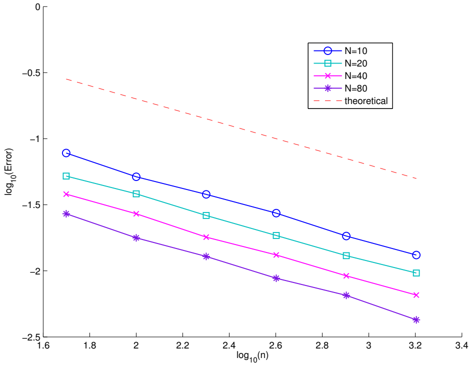

Figure: Error for periodic boundary conditions when $\delta t = \delta x = 1/n$, in logarithmic scales.

Dirichlet case:

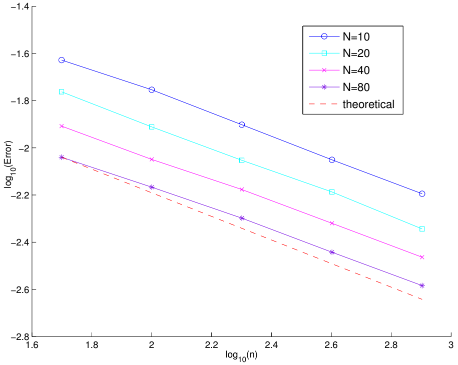

Figure: Error for Dirichlet boundary conditions when $\delta t = \delta x = 1/n$, in logarithmic scales.

Possible extensions

In higher dimension $d \geq 2$: no change in principle for the implementation; but for the analysis?

Addition of a drift $c(x)$: discretization of a SDE via the Euler scheme. Analysis: need to change the norm?

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Dirichlet (resp. Neumann) boundary conditions: killed (resp. reflected) diffusion processes.

Non-linear equations: more involved schemes.

Dirichlet boundary conditions

Still in the simplified setting ($d = 1$, $c(x) = 0$).

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Equation:

$$\partial_t u = \frac{1}{2}\partial_{xx}^2 u;$$
$$u(t, b) = u(t, a) = 0 \text{ for any } t \geq 0,$$
$$u(0, x) = u_0(x) \text{ for any } x \in [a, b].$$

The probabilistic representation formula:

$$u(t, x) = \mathbb{E}[u_0(X_t^x)\mathbb{1}_{t<\tau^x}],$$

with

$$\tau^x = \inf\left\{s > 0; x + B_s \notin (a, b)\right\}.$$

Algorithm for the Dirichlet boundary conditions

Difficulty: evaluation of (the law of) the exit time τ^x.

Solution (cf Gobet): Law $(B_{n\delta t+t})_{0\leq t\leq \delta t}$ conditionally to $B_{n\delta t}=x$ and $B_{(n+1)\delta t}=y$ is the law of the process

$$(\tilde{B}_t^{x,y,\delta t}:=\frac{t}{\delta t}y+\frac{\delta t-t}{\delta t}x+W_t-\frac{t}{\delta t}W_{\delta t})_{0\leq t\leq \delta t},$$

a Brownian Bridge.

Application: a more performant test for the exit.
Requires a refined time-step (and a larger Monte-Carlo parameter).
Solution: decomposition of the domain into two zones.

An non-linear example: 2D-Burgers equations

$$\frac{\partial u}{\partial t} + (u.\nabla)u = \nu\Delta u + f.$$

Domain $(-1,1)^2$, with homogeneous Dirichlet B.C.

A semi-implicit scheme: for $n\delta t \leq t \leq (n+1)\delta t$ and $x \in D$

$$\frac{\partial v^{n+1}}{\partial t} + (u^n.\nabla)v^{n+1} = \nu\Delta v^{n+1} + f^n,$$

with initial condition is $v^{n+1}(n\delta t,.) = u^n = v^n(n\delta t,.)$.

Approximation $u^n = v^n(n\delta t,.)$.

Forcing: $f^n(t,x) = f(n\delta t, x, u^n(x))$.

On $[n\delta t,(n+1)\delta t]$:

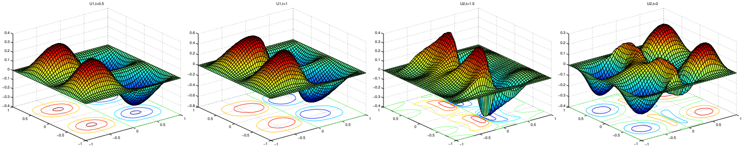
$$v^{n+1}(t,x)=\mathbb{E}[v^{n+1}(n\delta t,X_t^x)\mathbb{1}_{t<\tau^x}+\int_{n\delta t}^{t\wedge\tau^x}f^n(X_s^x)ds],$$

with

$$dX_t^x=-u^n(X_t^x)dt+\sqrt{2\nu}dB_t, X_{n\delta t}^x=x.$$

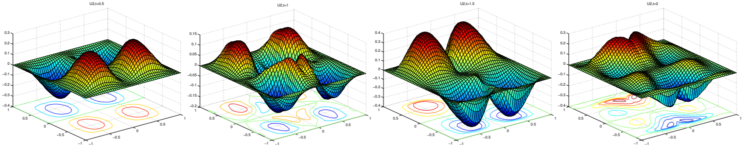
and the associated stopping time τ^x.

Simulations: $u_0(x)=0$,
$f(t,x)=(-\sin(\pi t)\sin(\pi x)\sin(\pi y)^2,-\sin(\pi t)\sin(\pi x)^2\sin(\pi y))$,
$\nu=0.001$.
$\delta t=0.02,\ \delta x=0.04,\ N=10.$



(a) $u_1, t = 0.5$ (b) $u_1, t = 1$ (c) $u_1, t = 1.5$ (d) $u_1, t = 2$

Figure: Solution of the 2D Burgers equation at different times



(a) $u_2, t = 0.5$ (b) $u_2, t = 1$ (c) $u_2, t = 1.5$ (d) $u_2, t = 2$

Idea of the proof (1)

$$P_{j,k}^{(n,m)} := \phi_k(j\delta x + \delta t c(j\delta x) + \sqrt{\delta t}\mathcal{N}^{n,m,j}),$$

$$Q_{j,k} = \mathbb{E}P_{j,k}^{(n,m)}.$$

Matrix formulation:

$$u^{n+1} = \Big(\frac{1}{N}\sum_{m=1}^{N} P^{(n,m)}\Big)u^n = P^{(n)}u^n,$$

$$v^{n+1} = Qv^n.$$

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Property

$P^{(n,m)}$ is a stochastic matrix, with independent rows.
Q is symmetric.
$\mathbb{E}\|P^{(n,m)}u\|_{l^2}^2 \leq \|u\|^2$. *But* $\mathbb{E}|P^{(n,m)}u|_{h^1}^2$ *?*

Idea of the proof (2)

One-step variance:

$$\mathbb{E}\|(P^{(n)} - Q)u\|_{l^2}^2 \leq C\frac{\delta t + \delta x^2}{N}|u|_{h^1}^2.$$

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$$+ \sum_{k=0}^{n-1}\left(Q^{2(n-1-k)}\right)_{1,1}\mathbb{E}\|(P^{(k)} - Q)\left(P^{(k-1)}\dots P^{(0)} - Q^k\right)u^0\|_{l^2}^2.$$

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Blue part: improved bound.
Red part: recursion, with p.

Conclusion

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Conclusion

- A numerical scheme combining Semi-Lagrangian and Monte-Carlo techniques.
- Simple and can be widely generalized (dimension, domains, boundary conditions).
- Example of applications: fluids, kinetic equations.
- The variance estimate: interesting but only for a simple case...
- A lot to be done!

An additional example

Incompressible Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + (u.\nabla)u - \nu\Delta u + \nabla p = f$$
$$\mathrm{div}(u) = 0,$$
$$u(0,.) = u_0,$$
$$+\text{boundary conditions.}$$

Use of a projection method:

$$\frac{\partial v^{n+1}}{\partial t} + (u^n.\nabla)v^{n+1} - \nu\Delta v^{n+1} = f(n\delta t,.), \tag{1}$$

and $u^{n+1} = v^{n+1} - \nabla p^{n+1}$ with

$$\Delta p^{n+1} = \mathrm{div}(v^{n+1}). \tag{2}$$

Example: driven cavity, with different Reynolds numbers.

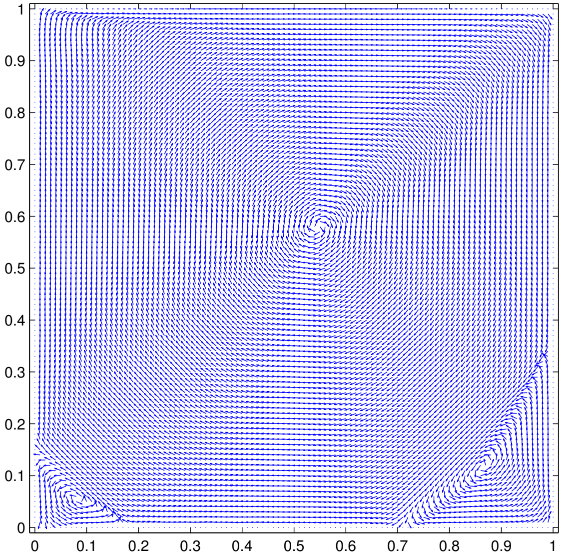

Figure: The driven cavity flow for $Re = 1000$.

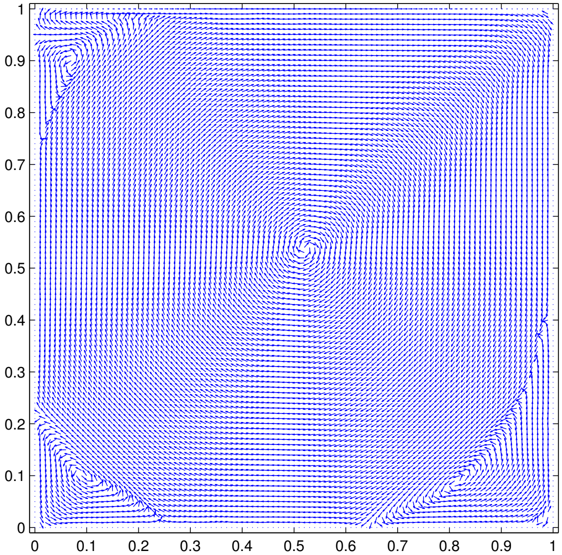

Figure: The driven cavity flow for $Re = 5000$.

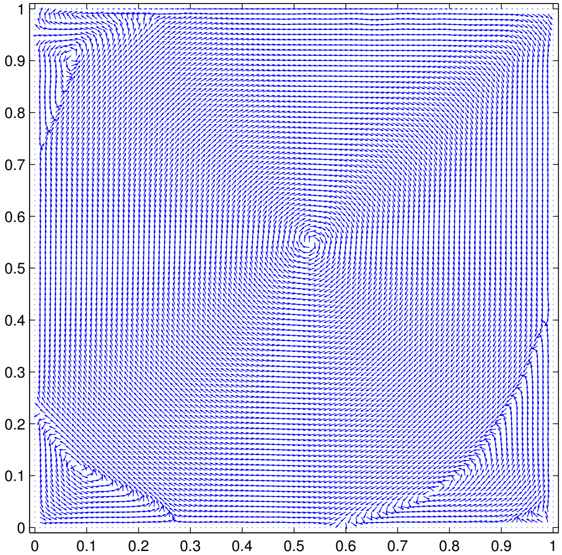

Figure: The driven cavity flow for $Re = 10000$.

Thanks for your attention.