Numerical method for piecewise-deterministic Markov processes

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CEMRACS 2013

Outline

Plan

Processes with deterministic evolution

The process $(X_t)_{t\geq 0}$ is a jump stochastic process whose trajectories are deterministic between the jump times.
There exists a flow ϕ on $\mathbb{R}^d \times \mathbb{R}_+$, such that if X does not jump between s and t then

$$X_t = \phi(X_s, t-s).$$

The classical case supposes the flow is solution of a differential equation

$$\left\{ \begin{array}{c} \partial_t \phi(x,t) = v(\phi(x,t)), \\ \phi(x,0) = x. \end{array} \right.$$

Space

We denote $F \subset \mathbb{R}^d$ the space of the admissible stats of the process, and we define $G \subset \mathbb{R}^d$ an area where the process is forced to jump.
Given G, we can define the time to reach it from any point $x \in \mathbb{R}^d$:

$$\alpha(x) = \inf\{t \geq 0 : \phi(x, t) \in G\}.$$

The space F is $\{x \in \mathbb{R}^d : \alpha(x) > 0\}$ and the frontier

$$\Gamma = \{\phi(x, \alpha(x)) : x \in F, \alpha(x) < +\infty\}.$$

Γ represents the accessible points from F but there are not necessarily in F.
The domain G is not really useful, because many points of G are not accessible. Thus only the domain Γ will be useful.

Two kinds of jumps

When the process reaches a point x of Γ, it jumps inside F with the distribution $q(x, \mathrm{d}y)$.

But the process has stochastic jumps. We set $\lambda(x)$ the jump rate at point $x \in F$, and $Q(x, \mathrm{d}y)$ the associated distribution to the jump.

In this talk, we suppose that λ is continuous and bounded, but some integrability conditions can be sufficient.

Hypotheses of the model

The flow is assumed to be a marlovian flow until it reaches Γ:

$$\forall x \in F,\ \forall t, s \in \mathbb{R}^+, t + s < \alpha(x), \qquad \phi(\phi(x,t), s) = \phi(x, t+s),$$

and ϕ is Lipschitz with constant L_ϕ.

The function α is such that

$$\forall x \in F,\ \forall t \in [0, \alpha(x)[, \qquad \phi(x,t) \in F,$$

and

$$\alpha(\phi(x,t)) = \alpha(x) - t, \qquad \text{for all } t < \alpha(x)$$

and we assume that α is Lipschitz with constant L_α.

Remark

We can see F as union of many disjoint sets F_i. In this case, we assume that each F_i is stable by ϕ :

$$\forall i \in I,\ \forall x \in F_i,\ \forall t \in [0, \alpha(x)[, \qquad \phi(x,t) \in F_i.$$

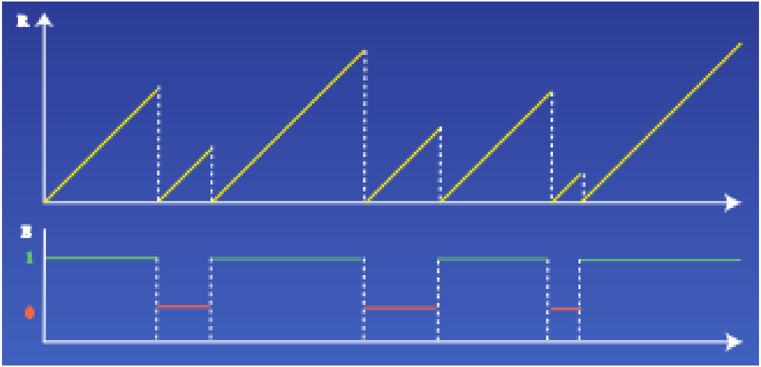
Hypotheses of the model

- Continuity of Q + control at infinity
- Continuity of q + control at infinity
- technical assumption on q : We assume that there exists $a_0 \in (0, 1)$ and $B_0 > 0$ such that

$$\sup_{x \in \Gamma} \int_F e^{-B_0 \alpha(y)} q(x, \mathrm{d}y) \leq 1 - a_0.$$

Applications

- Internet protocole TCP
- Reliability
- Server Queue
- Biology
- Economy



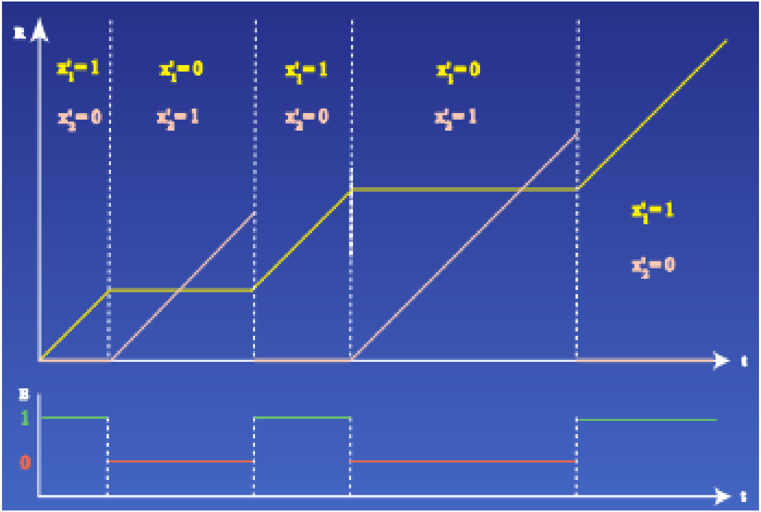


Figure: Process with switching

Figure: Application in biology

Books :

- M.H.A. Davis (1993), *Markov Models and Optimization*, Chapman & Hall, London.
- M. Jacobsen (2006), *Point process theory and applications: marked point and piecewise deterministic processes*, Series: Probability and Its Applications, Birkhäuser.
- C. Cocozza-Thivent (2013) *Renouvellement markovien et PDMP*, disponible ici : http://perso-math.univ-mlv.fr/users/cocozza.christiane/recherche-page-perso/PresentationRMetPDMP.html (in french).

Other references :

- M. Benaïm, S. Le Borgne, F. Malrieu, P.-A. Zitt (2012), *Quantitative ergodicity for some switched dynamical systems*, Electron. Commun. Probab. 17 (2012), no. 56, 1-14.
- C. Cocozza-Thivent, R. Eymard, L. G., M. Roussignol (2013), *Numerical method for piecewise deterministic Markov processes*.

Equation on the law

If we set ρ_t the law of X_t for all $t \in \mathbb{R}_+$, then we have

$$\int_F g(x,T)\rho_T(\mathrm{d}x) =$$

$$\int_F g(x,0)\rho_{\mathrm{ini}}(\mathrm{d}x) + \int_{[0,T]}\int_F \partial_{t,\phi} g(x,t)\rho_t(\mathrm{d}x)\mathrm{d}t$$

$$+ \int_{[0,T]}\int_F \lambda(x)\left(\int_F g(y,t)Q(x,\mathrm{d}y) - g(x,t)\right)\rho_t(\mathrm{d}x)\mathrm{d}t$$

$$+ \int_{[0,T]}\int_\Gamma \left(\int_F g(y,t)q(x,\mathrm{d}y) - g(x,t)\right)\sigma(\mathrm{d}x,\mathrm{d}t),$$

for all function g in $\mathcal{T}$, and where $\sigma(\mathrm{d}z,\mathrm{d}t)$ is a measure on $\Gamma \times \mathbb{R}_+$.

Test functions

We denote by $\mathcal{T}$ the space of functions $g \in C_b(F \times \mathbb{R}^+)$ with compact support in the second variable, which are continuously differentiable with respect to the operator

$$\partial_{t,\phi} g(x,t) = \lim_{\epsilon \to 0, \epsilon > 0} \frac{g(\phi(x,\epsilon), t+\epsilon) - g(x,t)}{\epsilon},$$

which is the derivative along the flow.

Moreover there exists a constant $C_g > 0$ such that

$$\sup_{\epsilon < 1} \frac{g(\phi(x,\epsilon), t+\epsilon) - g(x,t)}{\epsilon} \leq C_g.$$

Plan

Maillages

An admissible mesh $\mathcal{M}$ of F is a countable partition of F, such that

1. $\cup_{K \in \mathcal{M}} K = F$ et $\forall (K, L) \in \mathcal{M}^2, K \neq L \Rightarrow K \cap L = \emptyset$.
2. $\forall K \in \mathcal{M},\ |K| := \int_K \mathrm{d}x > 0$.
3. For all $K \in \mathcal{M}$, there exists $i \in I$ sucht that $K \subset F_i$.
4. $\sup\limits_{K \in \mathcal{M}} \mathrm{diam}(K) < +\infty$ where $\mathrm{diam}(K) = \sup\limits_{\{(x,y) \in K^2\}} |x - y|$.
   We then set $h := \sup\limits_{K \in \mathcal{M}} \mathrm{diam}(K)$.
5. $\tau > 0$ and $\delta t > 0$ are given values, and we denote by $\mathcal{D} = (\mathcal{M}, \delta t, \tau)$.

Quantities

The value $\tau > 0$ (which tends to 0) is used in the definition of the "probability mass flow" between $K \in \mathcal{M}$ and $L \in \mathcal{M}$

$$v_{KL} = \frac{1}{\tau}|\{x \in K : \alpha(x) > \tau \text{ et } \phi(x,\tau) \in L\}|.$$

Thanks to the hypotheses of the model, $v_{KL} \neq 0$ if and only if there exists F_i such that $K \subset F_i$ and $L \subset F_i$.

Quantities

We set for all $(K, L) \in \mathcal{M} \times \mathcal{M}$

$$q_{KL} = \frac{1}{\tau} \int_{\{x \in K : \alpha(x) \leq \tau\}} \int_L q(\phi(x, \alpha(x)), \mathrm{d}y)\mathrm{d}x,$$

$$q_K = \frac{1}{\tau} |\{x \in K : \alpha(x) \leq \tau\}| = \sum_{L \in \mathcal{M}} q_{KL},$$

$$\lambda_{KL} = \int_K \lambda(x) \int_L Q(x, \mathrm{d}y)\mathrm{d}x,$$

$$\lambda_K = \int_K \lambda(x)\mathrm{d}x = \sum_{L \in \mathcal{M}} \lambda_{KL}.$$

Schéma volumes finis

We define the family of real numbers $\left(p_n^{(K)}\right)_{n\in\mathbb{N},K\in\mathcal{M}}$ with the following implicit finite volume scheme :

$$|K|\frac{p_{n+1}^{(K)}-p_n^{(K)}}{\delta t}+\sum_{L\in\mathcal{M}}\left(v_{KL}p_{n+1}^{(K)}-v_{LK}p_{n+1}^{(L)}\right)$$
$$+(\lambda_K+q_K)p_{n+1}^{(K)}-\sum_{L\in\mathcal{M}}p_{n+1}^{(L)}(\lambda_{LK}+q_{LK})=0,$$
$$\forall K\in\mathcal{M},\ \forall n\in\mathbb{N},$$

with the initial condition

$$|K|\,p_0^{(K)}=\int_K\rho_{\mathrm{ini}}(\mathrm{d}x),\qquad \forall K\in\mathcal{M}.$$

We can remark that the following property is satisfied

$$\tau \left(\sum_{L \in \mathcal{M}} v_{KL} + q_K \right) = |K|, \ \forall K \in \mathcal{M}.$$

Thus the scheme can be rewritten with the alternative form

$$\left(\left(1 + \frac{\delta t}{\tau} \right) |K| + \delta t \lambda_K \right) p_{n+1}^{(K)}$$
$$- \delta t \sum_{L \in \mathcal{M}} p_{n+1}^{(L)} (v_{LK} + \lambda_{LK} + q_{LK}) = |K| \, p_n^{(K)},$$
$$\forall K \in \mathcal{M}, \ \forall n \in \mathbb{N}.$$

We define the measures $P_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t)$ and $\sigma_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t))$ on $F \times \mathbb{R}^+$ and $\Gamma \times \mathbb{R}^+$ by

$$\int_{F\times\mathbb{R}^+} f(x,t) P_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t) = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \int_K f(x, n\delta t)\mathrm{d}x,$$

for all bounded function f on $F \times \mathbb{R}^+$ with compact support in the second variable and

$$\int_{\Gamma\times\mathbb{R}^+} f(x,t) \sigma_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t) = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \frac{1}{\tau} \int_{\{x\in K:\alpha(x)\leq\tau\}} f(\phi(x,\alpha(x)), n\delta t)\mathrm{d}x,$$

for all bounded function f on $\Gamma \times \mathbb{R}^+$ with compact support in the second variable.

Lemma (Existence and uniqueness)

Under the hypotheses of the model and of the scheme, there exists a unique solution $(p_n^{(K)})_{K\in\mathcal{M},n\in\mathbb{N}}$ *of the scheme such that*

$$p_n^{(K)} \geq 0,\ \forall K \in \mathcal{M}, \qquad \forall n \in \mathbb{N}, \tag{2.1}$$

$$\sum_{K\in\mathcal{M}} |K|\, p_n^{(K)} = 1, \qquad \forall n \in \mathbb{N}. \tag{2.2}$$

Lemma (Finitness)

Let $(p_n^{(K)})_{K\in\mathcal{M},n\in\mathbb{N}}$ *be the solution of the scheme with* (2.1) *and* (2.2).

Then, for all $T > 0$, $h < \tau$ *and* $\delta t \leq T$, *there exists* $C_\sigma > 0$ *which only depends on* T, a_0, B_0, α, *such that*

$$\int_{\Gamma\times[0,T]} \sigma_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t) = \sum_{n\in\mathbb{N}, n\delta t\leq T} \delta t \sum_{K\in\mathcal{M}} q_K\, p_n^{(K)} \leq C_\sigma.$$

Lemma (Tightness of $P_{\mathcal{D}}$)

Under the hypotheses of the model, for the scheme with $T > \delta t$. Let $(p_n^{(K)})_{K \in \mathcal{M}, n \in \mathbb{N}}$ be the solution of the scheme with (2.1) and (2.2). Then, for all $\varepsilon > 0$, there exists $R > 0$ which only depends on T, ϕ, α, such that

$$\int_{(F \setminus B(0,R)) \times [t_1,t_2]} P_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t) \leq \varepsilon \, (t_2 - t_1), \; \forall 0 \leq t_1 \leq t_2 \leq T, \qquad (2.3)$$

which implies the tightness of the family of probability measures $(P_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t))_{\mathcal{D} \in \mathcal{F}}$ on $F \times [0, T]$ with respect to the family of all discretization $\mathcal{D}$ such that $\tau < 1$ and $h/\tau < 1$.

Lemma (Tightness of $\sigma_{\mathcal{D}}$)

Under the hypotheses of the model, for the scheme with $T > \delta t$. Let $(p_n^{(K)})_{K\in\mathcal{M},n\in\mathbb{N}}$ be the solution of the scheme with (2.1) and (2.2). Then, for all $\varepsilon > 0$, there exists $R > 0$ which only depends on T, α, such that

$$\int_{(\Gamma\setminus B(0,R))\times[0,T]} \sigma_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t) \leq \varepsilon, \tag{2.4}$$

which implies the tightness of the family of probability measures $(\sigma_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t))_{\mathcal{D}\in\mathcal{F}}$ on $\Gamma \times [0, T]$ with respect to the family of all discretization $\mathcal{D}$ such that $\tau < 1$ and $h/\tau < 1$.

Plan

Lemma (Uniqueness)

Under the hypotheses of the model, there exists a unique couple (μ, σ) which is solution of

$$0 = \int_F g(x,0)\rho_{\mathrm{ini}}(\mathrm{d}x) + \int_{\mathbb{R}_+}\int_F \partial_{t,\phi} g(x,t)\mu(\mathrm{d}x,\mathrm{d}t)$$

$$+ \int_{\mathbb{R}_+}\int_F \lambda(x)\left(\int_F g(y,t)Q(x,\mathrm{d}y) - g(x,t)\right)\mu(\mathrm{d}x,\mathrm{d}t)$$

$$+ \int_{\mathbb{R}_+}\int_\Gamma \left(\int_F g(y,t)q(z,\mathrm{d}y) - g(z,t)\right)\sigma(\mathrm{d}z,\mathrm{d}t)$$

for all function $g \in \mathcal{T}$.

In order to prove the uniqueness, we inverse the operator acting on g. Actually, given $\bar{I} \in C_b(F \times \mathbb{R}_+, \mathbb{R})$ and $\bar{J} \in C_b(\Gamma \times \mathbb{R}_+, \mathbb{R})$ with compact support in the second variable, we are looking for g such that

$$\bar{I}(x,t) = \partial_{t,\phi} g(x,t) + \lambda(x) \left(\int_F g(y,t) Q(x, \mathrm{d}y) - g(x,t) \right),$$

for all $(x,t) \in F \times \mathbb{R}_+$, and

$$\bar{J}(z,t) = \int_F g(x,t) q(z, \mathrm{d}x) - g(z,t),$$

for all $(z,t) \in \Gamma \times \mathbb{R}_+$.

Lemma (Decomposition)

Under the hypotheses of the model, there exists a family of measures $(\rho_t)_{t\in\mathbb{R}_+}$ on $F\cup\Gamma$ such that for all function $g\in\mathcal{C}_b(F\cup\Gamma\times\mathbb{R}_+,\mathbb{R})$ we have

$$\int_{\mathbb{R}_+}\int_{F\cup\Gamma} g(x,t)\mu(\mathrm{d}x,\mathrm{d}t) = \int_{\mathbb{R}_+}\int_{F\cup\Gamma} g(x,t)\rho_t(\mathrm{d}x)\mathrm{d}t.$$

Lemma (Consistency)

Under the hypotheses of the model, for all $t\in\mathbb{R}_+$, $\rho_t(\Gamma)=0$.

Plan

Simple case

We study a single server queue with Poisson arrivals (parameter λ) and general service times distribution χ. Random jumps occur with constant rate λ and

$$Q(x, \mathrm{d}y) := \chi(\mathrm{d}y - x).$$

It is possible to modelize the workload of this queue as a PDMP with frontier
Let $F = F_0 \cup F_1$ be a domain of $\mathbb{R}^2$ with $F_0 = (0, 0)$ and $F_1 = \mathbb{R}_+^* \times \{1\}$. The first component of the process represents the workload and the second component marks the fact that the workload is null.

Simple case

When the process is in F_0, it stays at point $(0,0)$ until an arrival occurs at a random exponential time. At this time the process jumps on F_1 with the distribution $Q((0,0),\mathrm{d}(y_1,y_2)) = \chi(\mathrm{d}y_1) \times \delta_1(\mathrm{d}y_2)$.
When the process is in F_1, it follows the deterministic flow

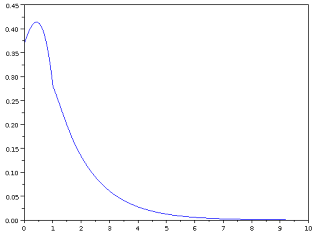
$$\phi((x,1),t) = (x-t,1)$$

until a jump occurs. So there are two kinds of jumps:

- Random ones occur with constant rate λ and the jump distribution is $Q((x,1),\mathrm{d}(y_1,y_2)) = \chi(\mathrm{d}y_1 - x) \times \delta_1(\mathrm{d}y_2)$
- Deterministic jumps occur when the process reaches the frontier $\Gamma = \{(0,1)\}$. At this time the process jumps on F_0 with distribution

$$q((0,1),\mathrm{d}(y_1,y_2)) = \delta_{(0,0)}(\mathrm{d}(y_1,y_2)).$$

We consider the case where χ is the uniform distribution on $(0,1)$. We use the scheme with $\tau = h$, and large values for δt, in order to approximate the asymptotic stationary state.



(a) $\lambda = 3/2$ and $\nu(F_0) = 1/4$
~ 0.2426 for $h = 1/100$
~ 0.2464 for $h = 1/200$

(b) $\lambda = 9/5$ and $\nu(F_0) = 1/10$
~ 0.0911 for $h = 1/100$
~ 0.0956 for $h = 1/200$

Figure: Asymptotic distribution on F_1 in long time

Thanks for your attention.