Properties of a stochastic variational inequality and random mechanics

Presentation at CEMRACS 2013, Marseille.

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Motivation: risk analysis of failure for mechanical structures under random forcing

- **Engineering Mechanics**:
  - $\longrightarrow$ study of elastic-plastic oscillators with noise
  - 2 References in Engineering literature:
    $\longrightarrow$ an elasto-plastic model with noise (Karnopp & Scharton, 1966)
    $\longrightarrow$ estimation of the variance of the deformation (Feau, 2007)

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  - 2 References in Engineering literature:
    $\longrightarrow$ an elasto-plastic model with noise (Karnopp & Scharton, 1966)
    $\longrightarrow$ estimation of the variance of the deformation (Feau, 2007)
- **Mathematical Framework:**
  - $\longrightarrow$ study of stochastic variational inequalities (SVI)
  - A mathematical reference:
    $\longrightarrow$ connection between this 1D elasto-plastic model and a SVI (Bensoussan & Turi 2007)
  - mathematical properties of SVI? Can we improve engineering methods with this new mathematical framework?

Outline of the presentation

1. Example of a mechanical structure modeled by an elasto-plastic oscillator
2. Framework of the stochastic variational inequality for the elasto-plastic problem and ergodic property
3. Short cycles : a new characterization of the ergodic measure
4. Long cycles : characterization of the growth rate of the deformation
5. Open problems

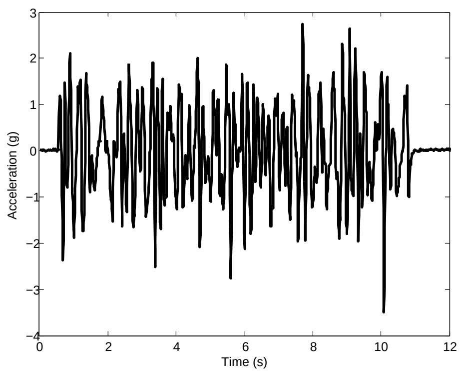
Example of a mechanical structure modeled by an elasto-plastic oscillator

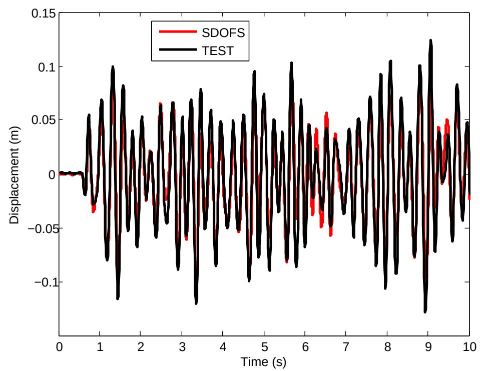
Illustrative example: piping system

For a class of structures: A one dimensional (1d) model

- global behavior of the structure

Seismic excitation (left) / Mass displacement (right): Test *vs* 1d model results





Elastic behavior: start with a linear oscillator ...

$$c_0 > 0 : \text{damping coefficient}$$
$$k > 0 : \text{stiffness}$$
$$\text{``}\frac{dw(t)}{dt}\text{''} : \text{external force white noise}$$
$$x(t) : \text{response of the oscillator}$$

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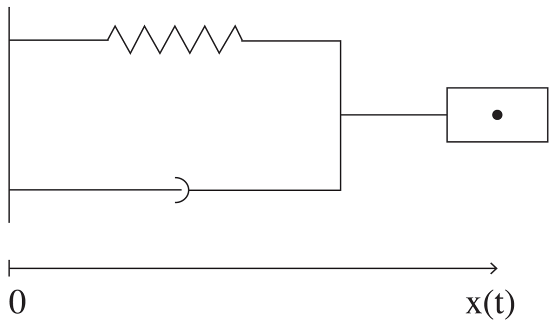
linear case: $x(t)$ solves

$$\ddot{x}(t) + c_0 \dot{x}(t) + F(t) = \text{``}\frac{dw(t)}{dt}\text{''}, \qquad F(t) = kx(t)$$

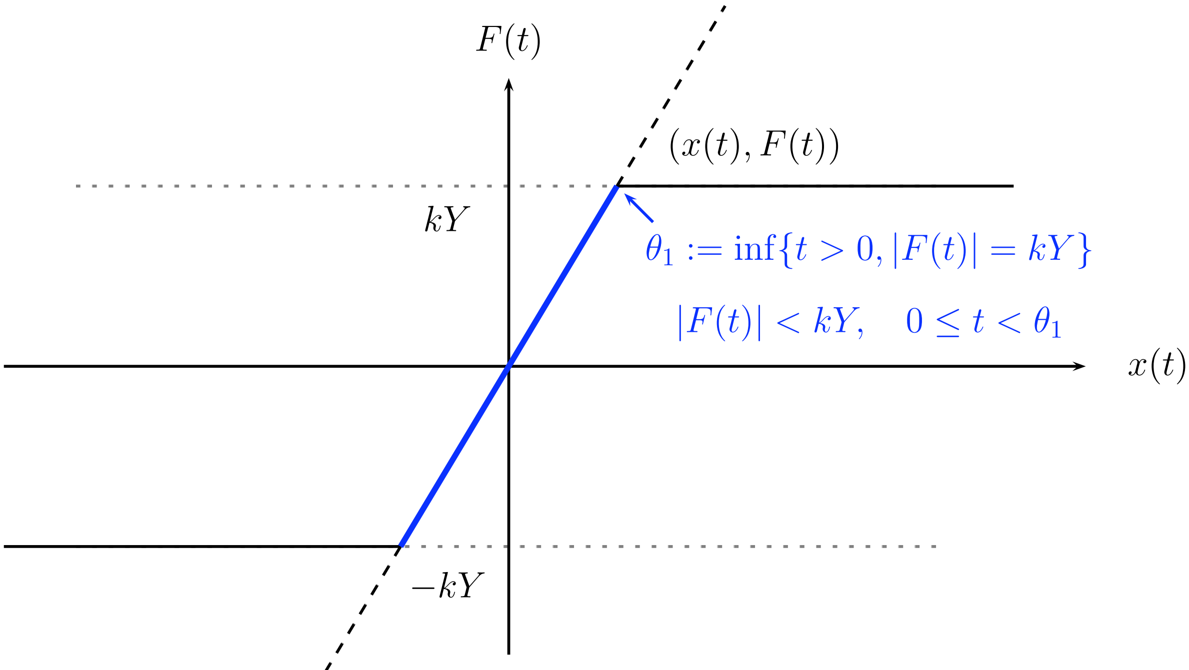


Elastic-perfectly-plastic behavior

elasto-perfectly-plastic case: $x(t)$ solves

$$\ddot{x}(t) + c_0\dot{x}(t) + F(t) = \text{“}\frac{dw(t)}{dt}\text{”}$$

- $|F(t)| \leq kY$, Y : elasto-plastic bound
  θ_1 : first time going into plastic phase



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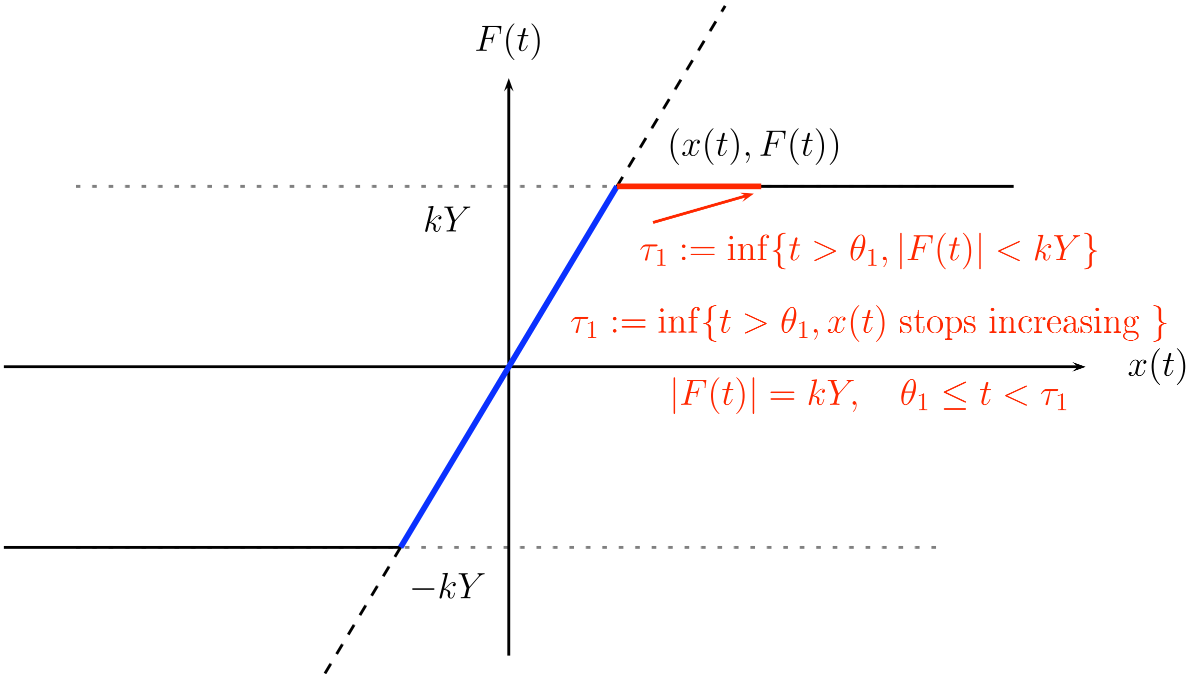
$$\ddot{x}(t) + c_0\dot{x}(t) + F(t) = \text{“}\frac{dw(t)}{dt}\text{”}, \quad F(t) = k(x(t) - \Delta(t))$$

- $\rightarrow$ a plastic deformation $\Delta(t)$ occurs in $x(t)$ when $|F(t)| = kY$.
  τ_1: first time going out of plastic phase



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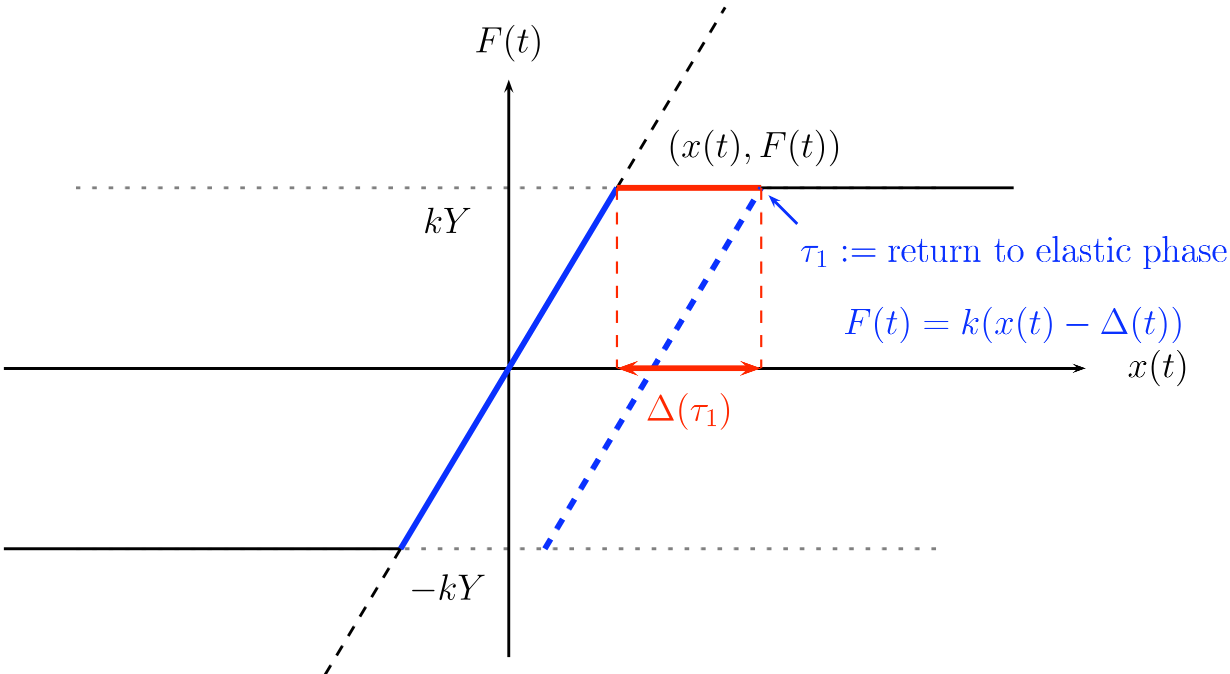
$\Delta(t)$ stops increasing
when $x(t)$ stops increasing.



Balance between elastic and plastic

Denote

$$y(t) = \dot{x}(t) \quad , \quad z(t) = x(t) - \Delta(t)$$

then

$$\ddot{x}(t) + c_0 \dot{x}(t) + kz(t) = \text{``}\frac{dw(t)}{dt}\text{''}$$

becomes

- **elastic** $|z(t)| < Y$**:**

$$\begin{cases} \dot{y}(t) &= -(c_0 y(t) + kz(t)) + \text{``}\frac{dw(t)}{dt}\text{''}, \\ \dot{z}(t) &= y(t) \end{cases}$$

- **plastic** $z(t) = Y, y(t) > 0$ **or** $z(t) = -Y, y(t) < 0$**:**

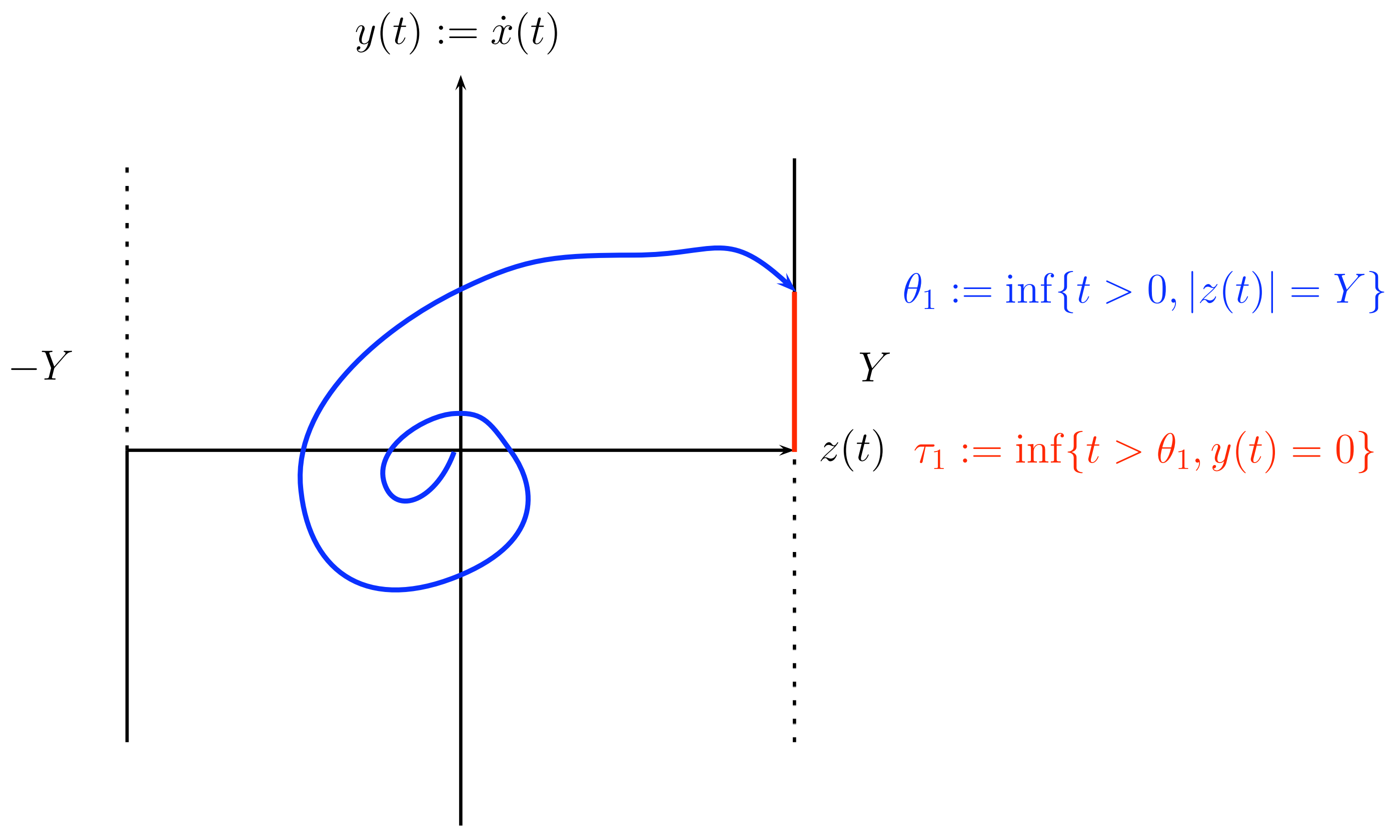
$$\begin{cases} \dot{y}(t) &= -(c_0 y(t) \pm kY) + \text{``}\frac{dw(t)}{dt}\text{''}, \\ \dot{z}(t) &= 0 \end{cases}$$

Key idea: Switching between elastic and plastic phases

An example of phase transition

$$\begin{cases} 1^{st} \text{ elastic phase: } [0, \theta_1) \\ \dot{y}(t) = -(c_0 y(t) + kz(t)) + \text{“}\frac{dw(t)}{dt}\text{”}, \\ \dot{z}(t) = y(t) \end{cases}$$

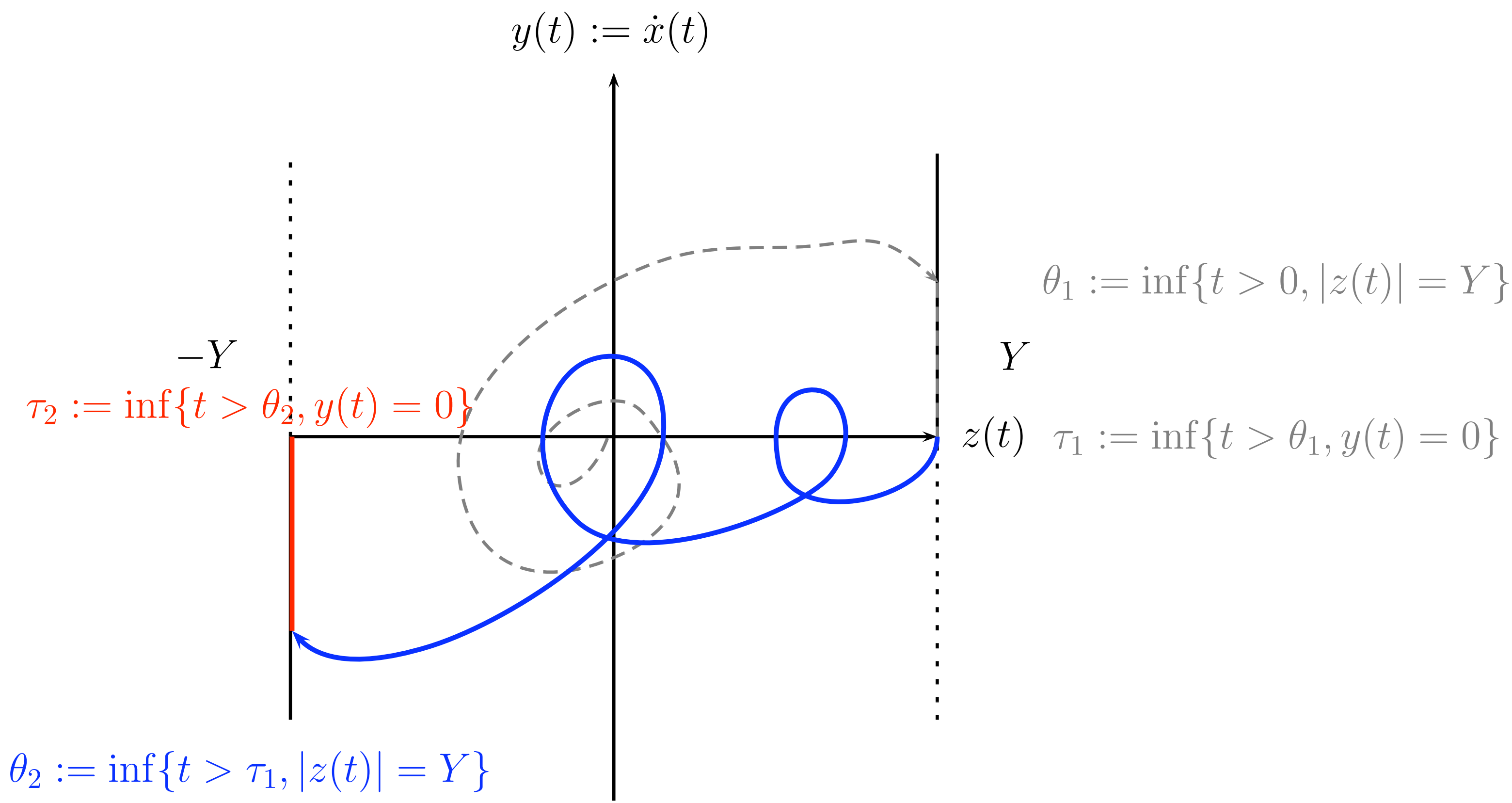
$$\begin{cases} 1^{st} \text{ plastic phase: } [\theta_1, \tau_1) \\ \dot{y}(t) = -(c_0 y(t) + kY) + \text{“}\frac{dw(t)}{dt}\text{”}, \\ z(t) = Y \end{cases}$$



An example of phase transition

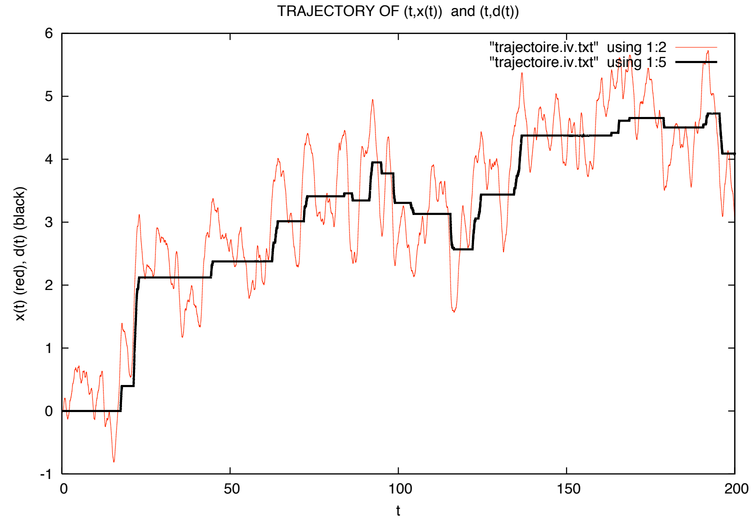
$$\begin{cases} 2^{st} \text{ elastic phase: } [\tau_1, \theta_2) \\ \dot{y}(t) = -(c_0 y(t) + kz(t)) + \text{“}\frac{dw(t)}{dt}\text{”}, \\ \dot{z}(t) = y(t) \end{cases}$$

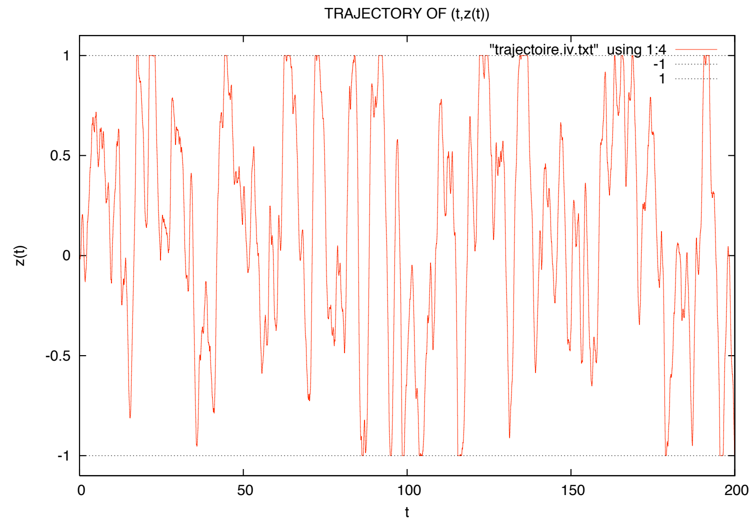
$$\begin{cases} 2^{st} \text{ plastic phase: } [\theta_2, \tau_2) \\ \dot{y}(t) = -(c_0 y(t) - kY) + \text{“}\frac{dw(t)}{dt}\text{”}, \\ z(t) = -Y \end{cases}$$



A numerical example of the deformation

Figure: on the top $t, x(t)$ (red) $t, \Delta(t)$ (black : plastic deformation) and at the bottom $t, z(t)$ for $c_0 = 1, k = 1, Y = 1$





Part 2: SVI for the elasto-plastic problem

Stochastic variational inequality modeling an elasto-plastic problem with noise

The problem can be seen as a

- $(y(t), z(t))$ reflected diffusion, $\Delta(t)$: reflection process
  References:
  SVIs: [Bensoussan-Lions1982].

Theorem (Bensoussan-Turi 2007)

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$$dy(t) = -(c_0 y(t) + kz(t))dt + dw(t),$$
$$(dz(t) - y(t)dt)(\phi - z(t)) \geq 0, \quad \forall |\phi| \leq Y, \quad |z(t)| \leq Y$$

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- without the plastic deformation $\Delta(t)$,

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The problem can be formulated

- without the plastic deformation $\Delta(t)$,
- without the instants of phase transition.
  The variational inequality: nicely adapted to plastic/elastic transition.
  $\rightarrow$ noise effect at the transition from plastic to elastic

Characterization of the invariant probability (balance between elastic and plastic state)

Theorem (Bensoussan-Turi 2007)

$(y(t), z(t))$ *ergodic Markov process*

- *unique invariant probability distribution* ν *for* $(y(t), z(t))$ *and* $(y(t), z(t)) \xrightarrow[t\to\infty]{\mathcal{L}} \nu$ *(independently of the initial condition).*

- elastic domain: $D := (-\infty, +\infty) \times (-Y, Y)$

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- elastic domain: $D := (-\infty, +\infty) \times (-Y, Y)$
- plastic domains: $D^- := (-\infty, 0) \times \{-Y\}$ and $D^+ := (0, +\infty) \times \{Y\}$
- ν has a density denoted by m is characterized by: $\forall \varphi$ smooth,

$$\int_D m(y,z)\{\overbrace{-y\varphi_z + (c_0 y + kz)\varphi_y - \frac{1}{2}\varphi_{yy}}^{A\varphi :=}\}dydz$$
$$+ \int_{D^+} m(y,Y)\{\underbrace{(c_0 y + kY)\varphi_y(y,Y) - \frac{1}{2}\varphi_{yy}(y,Y)}_{B_+\varphi :=}\}dy$$
$$+ \int_{D^-} m(y,-Y)\{\underbrace{(c_0 y - kY)\varphi_y(y,-Y) - \frac{1}{2}\varphi_{yy}(y,-Y)}_{B_-\varphi :=}\}dy = 0.$$

Alternative method to the Monte-Carlo simulation (1):

From ergodic theory, we know the limiting behavior of $(y(t), z(t))$.

- For all bounded function f and $\forall (y_0, z_0) \in \bar{D}$,

$$\lim_{t\to\infty} \mathbb{E} f(y^{y_0}(t), z^{z_0}(t)) = \int_D f(y,z)m(y,z)dydz + \int_{D^+} f(Y,y)m(Y,y)dy + \int_{D^-} f(-Y,y)m(-Y,y)dy.$$

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- But, it is also well known that

$$\lim_{t\to\infty} \mathbb{E} f(y^{y_0}(t), z^{z_0}(t)) = \lim_{\lambda\to 0} \lambda \int_0^\infty e^{-\lambda t} \mathbb{E} f(y^{y_0}(t), z^{z_0}(t))dt.$$

Alternative method to the Monte-Carlo simulation (2)

- Denote $u_\lambda(y_0, z_0; f) = \mathbb{E}\left[\int_0^\infty \exp(-\lambda t) f(y^{y_0}(t), z^{z_0}(t)) dt\right]$.

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- Denote $u_\lambda(y_0, z_0; f) = \mathbb{E}\left[\int_0^\infty \exp(-\lambda t) f(y^{y_0}(t), z^{z_0}(t))dt\right]$.
- Equivalent characterization of the asymptotic limit:

$$\begin{aligned}
\lambda u_\lambda + A u_\lambda &= f(y,z) && \text{in } D \\
\lambda u_\lambda + B_+ u_\lambda &= f(y,Y) && \text{in } D^+ \\
\lambda u_\lambda + B_- u_\lambda &= f(y,-Y) && \text{in } D^-
\end{aligned}$$

Nonlocal problem : $y \to u_\lambda(y, \pm Y; f)$ are continuous.

$\forall (y_0, z_0) \in \bar{D}$

$$\lim_{\lambda \to 0} \lambda u_\lambda(y_0, z_0; f) = \int_D f(y,z) m(y,z) dydz + \int_{D^+} f(y,Y) m(y,Y) dy + \int_{D^-} f(y,-Y) m(y,-Y) dy$$

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$\boxed{\text{Nonlocal problem}: y \to u_\lambda(y, \pm Y; f) \text{ are continuous.}}$

$\forall (y_0, z_0) \in \bar{D}$

$$\lim_{\lambda \to 0} \lambda u_\lambda(y_0, z_0; f) = \int_D f(y,z) m(y,z) dydz + \int_{D^+} f(y,Y) m(y,Y) dy + \int_{D^-} f(y,-Y) m(y,-Y) dy$$

- This result is fundamental for the numerical resolution of m : alternative method to Monte-Carlo, that requires simulations for long durations.
Publication: [Bensoussan, Mertz, Pironneau,Turi 2009], SIAM Journal on Numerical Analysis , Volume 47 Issue 5

Alternative method to the Monte-Carlo simulation (3)

Superposition of three local problems:

- $$\begin{cases} \lambda v_\lambda + A v_\lambda = f & \text{in } D, \\ \lambda v_\lambda + B_+ v_\lambda = f_+ & \text{in } D^+, \\ \lambda v_\lambda + B_- v_\lambda = f_- & \text{in } D^-, \end{cases}$$

  with $v_\lambda(0^+, Y) = 0, v_\lambda(0^-, -Y) = 0$,

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- $$\begin{cases} \lambda \pi_\lambda^+ + A \pi_\lambda^+ &= 0 \quad \text{in} \ \ D, \\ \lambda \pi_\lambda^+ + B_+ \pi_\lambda^+ &= 0 \quad \text{in} \ \ D^+, \\ \lambda \pi_\lambda^+ + B_- \pi_\lambda^+ &= 0 \quad \text{in} \ \ D^-, \end{cases}$$

  with $\pi^+(0^+, Y) = 1, \pi^+(0^-, -Y) = 0,$

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- $$\begin{cases} \lambda \pi_\lambda^- + A \pi_\lambda^- &= 0 \quad \text{in } D, \\ \lambda \pi_\lambda^- + B_+ \pi_\lambda^- &= 0 \quad \text{in } D^+, \\ \lambda \pi_\lambda^- + B_- \pi_\lambda^- &= 0 \quad \text{in } D^-, \end{cases}$$

  with $\pi^+(0^+, Y) = 0, \pi^-(0^-, -Y) = 1$.

Alternative method to the Monte-Carlo simulation (4)

- We look for p_+ and p_-:

$$v_\lambda + p_+ \pi_\lambda^+ + p_- \pi_\lambda^- \qquad \text{continuous in } (0, \pm Y)$$

Alternative method to the Monte-Carlo simulation (4)

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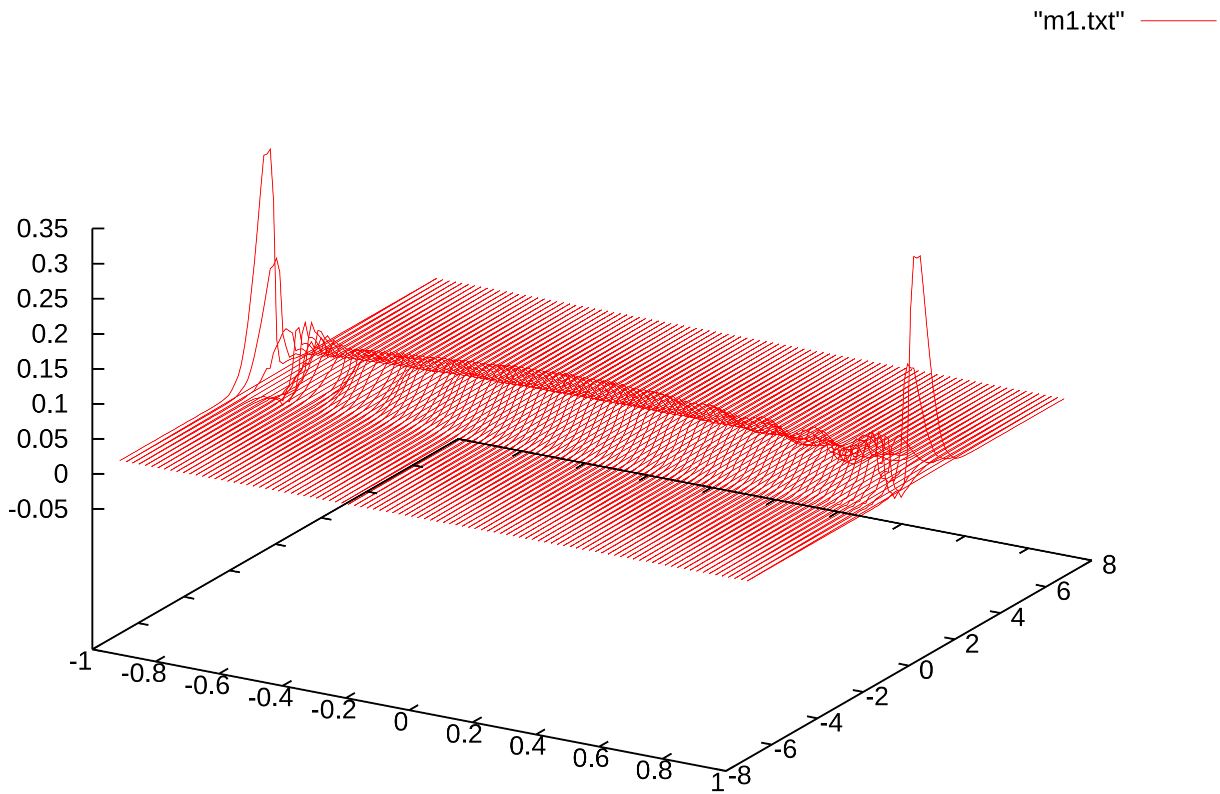
- Finally, we solve the following linear system:
  with

$$\Pi := \begin{pmatrix} \pi^+(0^+, Y) - \pi^+(0^-, Y) & \pi^-(0^+, Y) - \pi^-(0^-, Y) \\ \pi^+(-0^+, -Y) - \pi^+(0^-, -Y) & \pi^-(0^+, -Y) - \pi^-(0^-, -Y) \end{pmatrix}$$

then

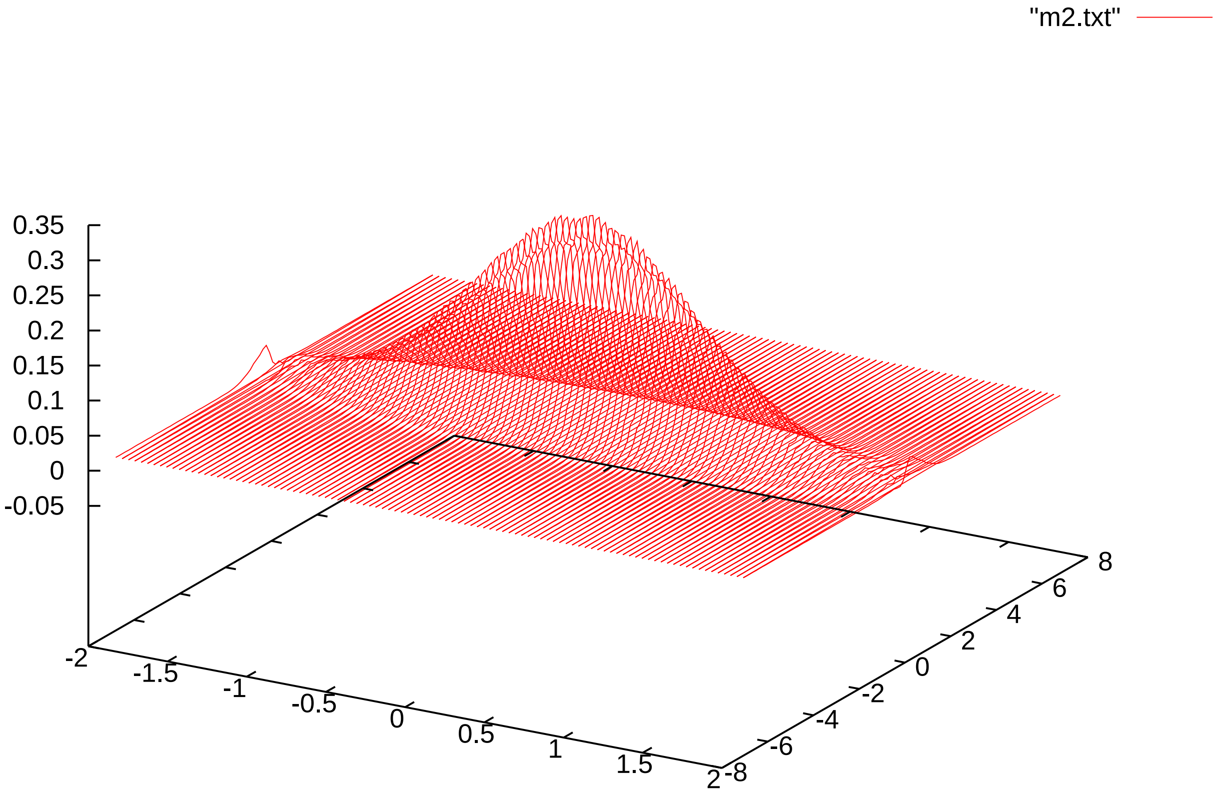
$$\Pi \begin{pmatrix} p_+ \\ p_- \end{pmatrix} = \begin{pmatrix} v_\lambda(0^-, Y) - v_\lambda(0^+, Y) \\ v_\lambda(0^-, -Y) - v_\lambda(0^+, -Y) \end{pmatrix}$$

Numerical result: $c_0 = 1$, $k = 1$ and $Y = 1$



- plot of m with the deterministic method, $-1 \leq z \leq 1, -7 \leq y \leq 7$.

Numerical result: $c_0 = 1, k = 1$ and $Y = 2$



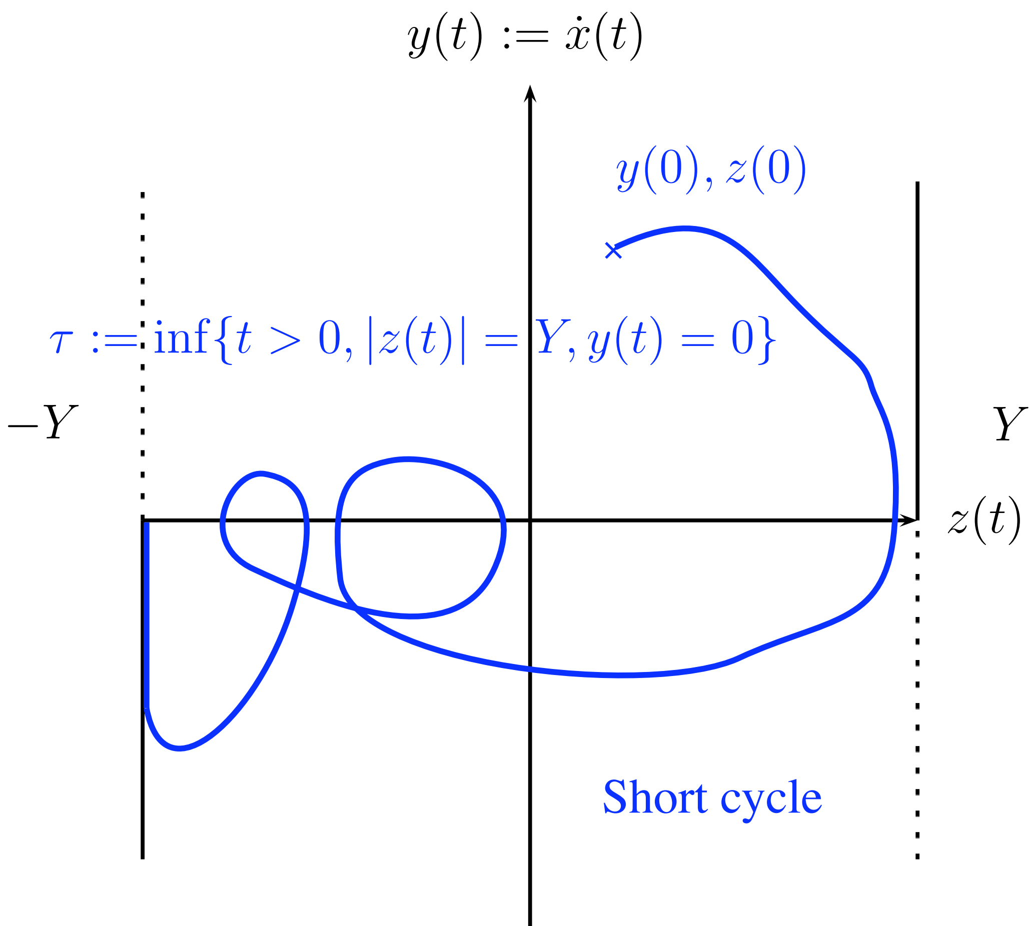
- plot of m with the deterministic method, $-2 \leq z \leq 2, -7 \leq y \leq 7$.

Part 3: Short cycles for a new characterization of the ergodic measure

Short cycle (trajectory)

Short cycle: path, solution of the SVI, starting from a point $(y, z) \in D$ and which contains

- only one phase evolving in D (elastic domain)
- and only one phase evolving in D^+ or D^- (plastic domains).



Short cycle (PDE)

- Goal: characterization of a short cycle by a PDE approach of

$$\mathbb{E}\int_0^{\tau} f(y(s), z(s))\mathrm{d}s, \quad \forall f \quad \text{bounded function on } D.$$

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- Goal: characterization of a short cycle by a PDE approach of

$$\mathbb{E}\int_0^\tau f(y(s),z(s))ds, \quad \forall f \quad \text{bounded function on } D.$$

- Define $v(y,z;f)$ the solution of

$$\begin{cases} \text{(elastic phase)} & \\ -yv_z + (c_0 y + kz)v_y - \frac{1}{2}v_{yy} & = f \text{ in } D, \\ \text{(plastic phase)} & \\ (c_0 y + kY)v_y - \frac{1}{2}v_{yy} & = f \text{ in } D^+, \\ (c_0 y - kY)v_y - \frac{1}{2}v_{yy} & = f \text{ in } D^- \end{cases} \qquad (P_v)$$

with the local boundary conditions

$$v(0^+, Y; f) = 0 \text{ and } v(0^-, -Y; f) = 0.$$

We call $v(y,z;f)$ a short cycle.

Analysis of short cycles/ new ergodic theorem

Theorem (Analysis of short cycles, A. Bensoussan, L.M.)

There exists a unique solution to (P_v) *of the form*

$$v(y,z;f) = \varphi^+(y;f)\mathbf{1}_{\{y>0\}} + \varphi^-(y;f)\mathbf{1}_{\{y<0\}} + w(y,z;f)$$

where

- *w is a bounded function*
- $-(c_0 y + kY)\varphi^+_y + \frac{1}{2}\varphi^+_{yy} = f(y,Y), \quad y>0, \quad \varphi^+(0^+;f)=0$ *and*
  $-(c_0 y - kY)\varphi^-_y + \frac{1}{2}\varphi^-_{yy} = f(y,-Y), \quad y<0, \quad \varphi^-(0^-;f)=0.$

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  $-(c_0 y - kY)\varphi_y^- + \frac{1}{2}\varphi_{yy}^- = f(y,-Y), \quad y<0, \quad \varphi^-(0^-;f) = 0.$

Theorem (New ergodic theorem, A. Bensoussan, L.M.)

A new characterization of the ergodic measure:

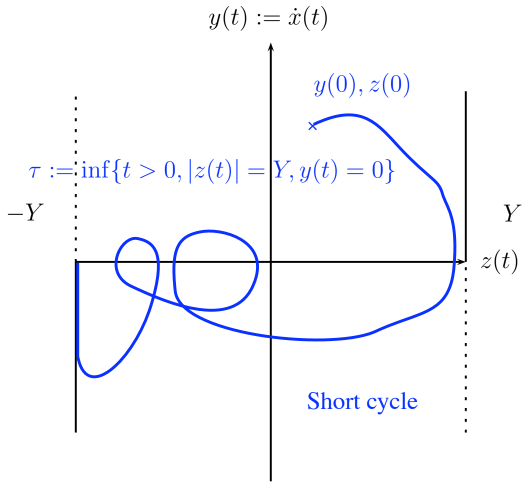
$$\nu(f) = \frac{v(0^-,Y;f) + v(0^+,-Y;f)}{2v(0^-,Y;1)}$$

[Bensoussan, Mertz 2012], CRAS An analytical approach to the ergodic theory of a stochastic variational inequality

[Khasminskii,1980] → ergodic measures written as cycle ratio

How can we apply the short cycles in engineering?

the frequency of occurence of short cycles
$=$ the frequency of occurence of plastic phases



Therefore

$$\text{The frequency of plastic phases} = \frac{1}{2v(0^-, Y; 1)}.$$

This quantity is used in engineering methods for estimating the variance of the plastic deformation.

Part 3 : Long cycles for the characterization of the growth rate related to the deformation

Long cycle behavior

- Find a repeating pattern (independent) in the trajectory.

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- Long cycle: path, solution of the SVI, starting and ending in one of the two points of $\{(0, Y), (0, -Y)\}$, knowing that the trajectory had a stop by the other point.

Long cycle behavior

- Find a repeating pattern (independent) in the trajectory.

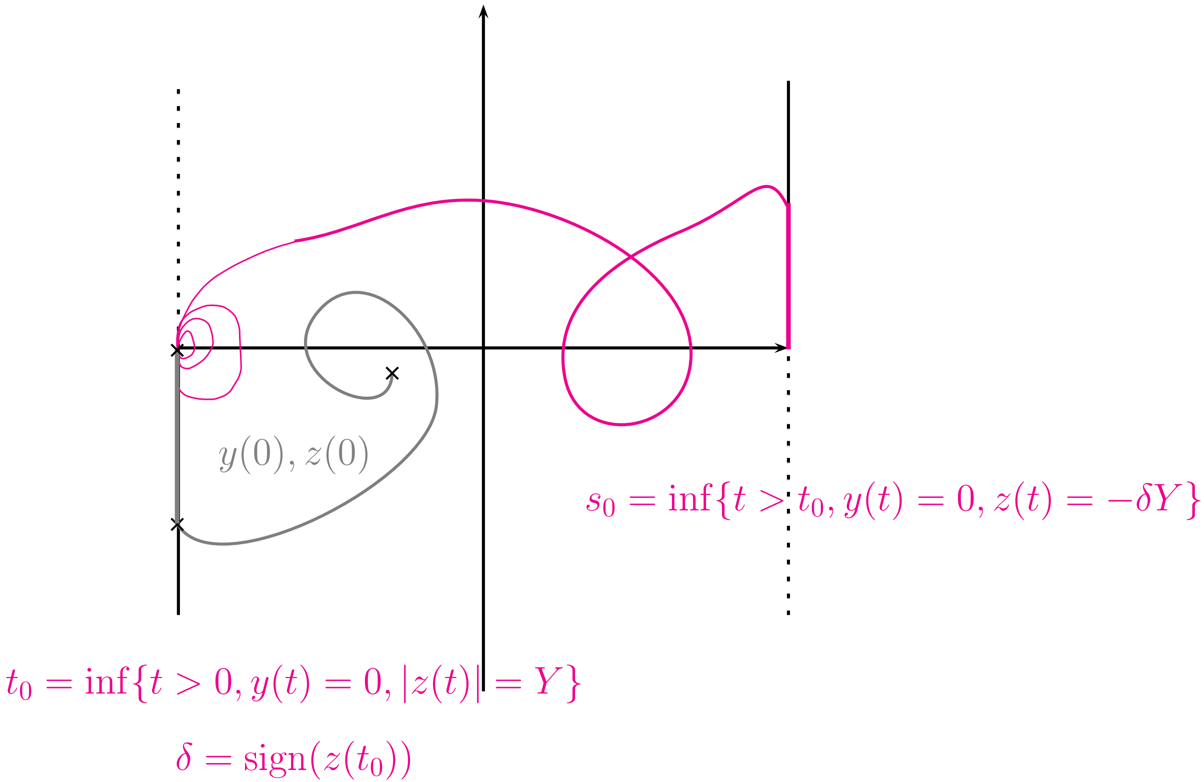
- Long cycle: path, solution of the SVI, starting and ending in one of the two points of $\{(0, Y), (0, -Y)\}$, knowing that the trajectory had a stop by the other point.

- Long cycles help to characterize the plastic behavior.

Definition and analysis of long cycles

- Define

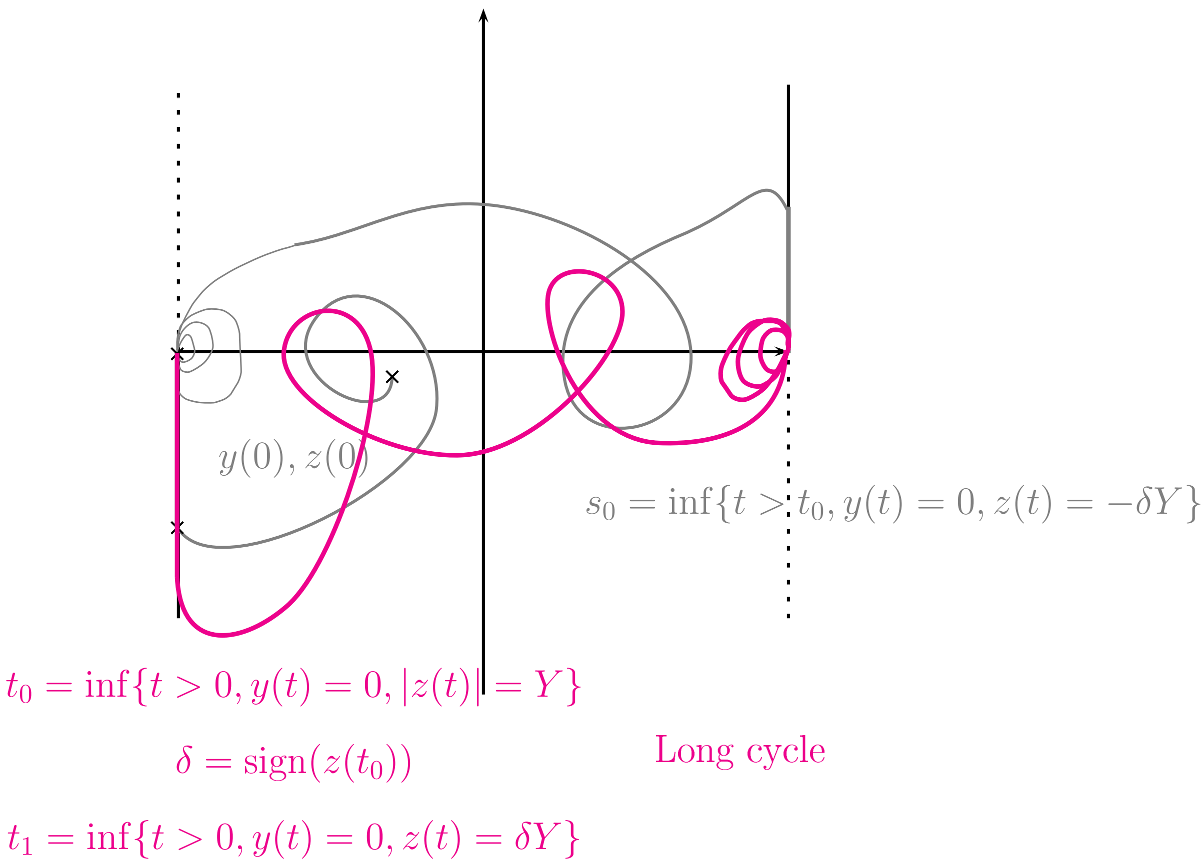
$$\begin{cases} t_0 = \inf\{t > 0, \quad y(t) = 0, |z(t)| = Y\}, \\ \delta = \text{sign}(z(t_0)), \\ s_0 = \inf\{t > t_0, \quad y(t) = 0, z(t) = -\delta Y\}. \end{cases}$$



Definition and analysis of long cycles

- Define

$$t_1 = \inf\{t > s_0, \quad y(t) = 0, z(t) = \delta Y\},$$



Long cycle behavior of the variance of the deformation

Theorem (Long cycle behavior, A. Bensoussan, L.M.)

In this context, we have proven

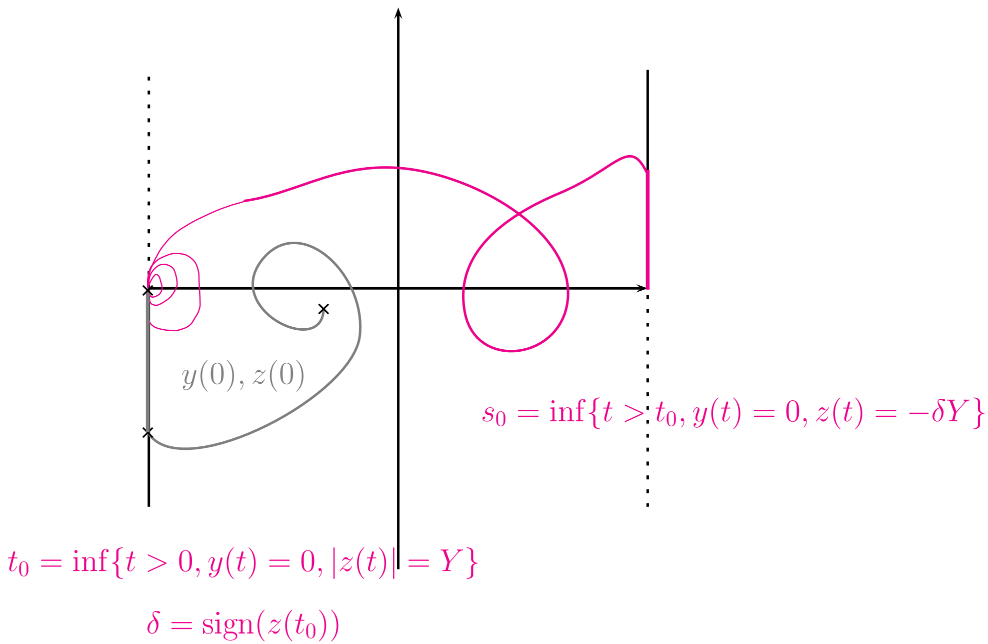
$$\lim_{t \to +\infty} \frac{\sigma^2(\Delta(t))}{t} = \frac{\mathbb{E}(\Delta(t_1) - \Delta(t_0))^2}{\mathbb{E}(t_1 - t_0)}$$

Key idea: We use a PDE framework to show $\mathbb{E}(t_1 - t_0)$ is finite

- Formulation of relevant quantity for the risk analysis of failure of a simple mechanical structure which is
  - Exact
  - Simple
  - and Easy to implement.

CRAS [Bensoussan, Mertz 2012] Behavior of the plastic deformation of an elasto-perfectly-plastic oscillator with noise

PDEs related to Long cycles (type one way)

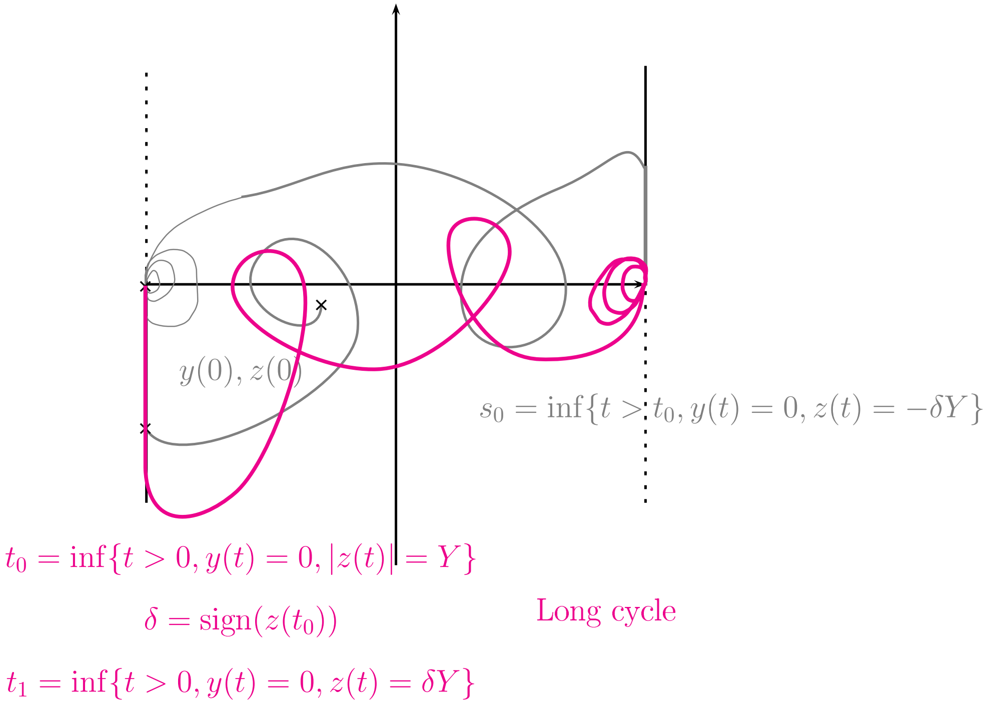


$$A\bar{v} = f, \quad B_+\bar{v} = f, \quad B_-\bar{v} = f \text{ and } \bar{v}(0^+, Y) = 0$$

where $\boxed{\text{nonlocal problem} : y \longrightarrow \bar{v}(y, -Y) \text{ is continuous}}$

$$\bar{v}(0, -Y) = \text{“}\mathbb{E}_{(0,-Y)}\left(\int_{t_0}^{s_0} f(y(s), z(s))ds\right)\text{”}$$

PDEs related to Long cycles (type return)



$$A\underline{v} = f, \quad B_+\underline{v} = f, \quad B_-\underline{v} = f \text{ and } \underline{v}(0^-, -Y) = 0$$

where $\boxed{\text{nonlocal problem : } y \longrightarrow \underline{v}(y, Y) \text{ is continuous}}$

$$\underline{v}(0, Y) = \text{“}\mathbb{E}_{(0,Y)}\left(\int_{s_0}^{t_1} f(y(s), z(s))ds\right)\text{”}$$

Numerical results in support of our prediction

In this section, we provide computational results which confirm our theoretical results.

$c_0 = 1, k = 1$				
Y	$\frac{\sigma^2(\Delta(t))}{t}, t = 500$	$\frac{\mathbb{E}(\Delta(t_1)-\Delta(t_0))^2}{\mathbb{E}(t_1-t_0)}$	$\mathbb{E}(t_1 - t_0)$	Relative error %
0.1	$0.807^{\pm 0.031}$	$0.834^{\pm 0.069}$	$6.61^{\pm 0.11}$	3.2
0.2	$0.649^{\pm 0.026}$	$0.624^{\pm 0.047}$	$8.74^{\pm 0.13}$	3.8
0.3	$0.493^{\pm 0.020}$	$0.464^{\pm 0.034}$	$10.45^{\pm 0.16}$	5.8
0.4	$0.361^{\pm 0.014}$	$0.355^{\pm 0.026}$	$12.12^{\pm 0.18}$	1.7
0.5	$0.266^{\pm 0.011}$	$0.257^{\pm 0.019}$	$13.80^{\pm 0.21}$	3.3
0.6	$0.195^{\pm 0.008}$	$0.198^{\pm 0.014}$	$16.15^{\pm 0.26}$	1.5
0.7	$0.137^{\pm 0.005}$	$0.149^{\pm 0.011}$	$18.84^{\pm 0.31}$	8
0.8	$0.103^{\pm 0.004}$	$0.112^{\pm 0.008}$	$22.80^{\pm 0.39}$	8
0.9	$0.071^{\pm 0.003}$	$0.086^{\pm 0.006}$	$26.79^{\pm 0.47}$	15

Table: Monte-Carlo simulations $t = 500$, $\delta t = 10^{-4}$ and $MC = 10000$. 95% of confidence

How can we apply the long cycles for the risk of failure analysis?1/2

with Cyril Feau, CEA (French Atomic Commission)

Define

- $f_{\text{lc}} := \frac{1}{\mathbb{E}(t_1 - t_0)}$ frequency of occurrence of long cycles, so that

$$[f_{\text{lc}} t] = \text{ number of long cycles up to the time } t,$$

- $\sigma_{\text{lc}} := \sqrt{\mathbb{E}\left[(\Delta(t_1) - \Delta(t_0))^2\right]}$ the standard deviation of the plastic deformation on a long cycle,
- and define a surrogate variable for $\Delta(t)$:

$$\Delta_{app}(t) := \sigma_{\text{lc}} \sum_{k=1}^{[f_{lc} t]} \beta_k,$$

where $\{\beta_k, k \geq 1\}$ are i.i.d. and $\mathbb{P}(\beta = -1) = \mathbb{P}(\beta = 1) = \frac{1}{2}$.

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How can we apply the long cycles for the risk of failure analysis?2/2

The risk of failure :

$$\begin{cases} b \longrightarrow \mathbb{P}(\max_{t\in[0,T]} |\Delta(t)| \geq b) \\ b \longrightarrow \mathbb{P}(\max_{t\in[0,T]} |\Delta_{app}(t)| \geq b) \end{cases} \quad (1)$$

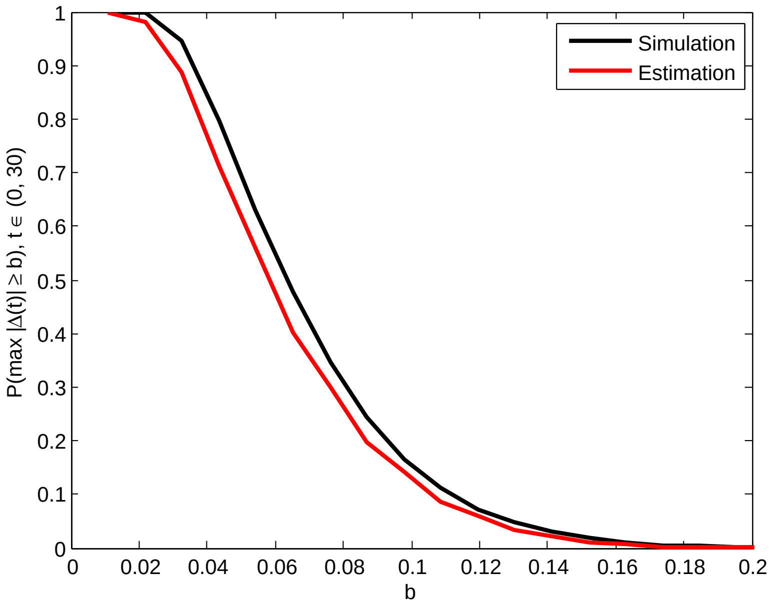

Figure: $b \rightarrow \mathbb{P}(\max_{t\in[0,T]} |\Delta_{app}(t)| \geq b) \leftarrow$ there exists an explicit formula.

Open problems

Cycle properties for a SVI with a filtered noise

What are the right cycles when the dimension increases?

- A more realistic case to take $d\xi_\alpha(t)$ for the noise where

$$\mathrm{d}\xi_\alpha(t) = -\alpha\xi_\alpha(t)\mathrm{d}t + \mathrm{d}w(t), \alpha > 0,$$

then

$$\begin{cases} \mathrm{d}y(t) = -(c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}\xi_\alpha(t), \\ (\mathrm{d}z(t) - y(t)\mathrm{d}t)(\phi - z(t)) \geq 0, \quad \forall |\phi| \leq Y, \quad |z(t)| \leq Y \end{cases} \tag{2}$$

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- We can consider an unbounded behavior for the restoring force: therefore the SVI involves the total deformation in the drift

$$\begin{cases} \mathrm{d}x(t) = y(t)\mathrm{d}t \\ \mathrm{d}y(t) = -(c_0 y(t) + k(1-\alpha)z(t) + k\alpha x(t))\mathrm{d}t + \mathrm{d}w(t), \\ (\mathrm{d}z(t) - y(t)\mathrm{d}t)(\phi - z(t)) \geq 0, \quad \forall |\phi| \leq Y, \quad |z(t)| \leq Y. \end{cases} \tag{3}$$

Fragility (optimal stopping problem) (1/2):

- Let
  - $T > 0$ be a maturity
  - and $b > 0$ be certain threshold of plastic deformation.

- Then, it could be relevant for engineering purposes to know

$$\min_{\tau \in \mathcal{T}_{0,T}} \{\mathbb{E}(|\Delta(\tau)| - b)^2\}, \text{ where } \mathcal{T}_{0,T} = \text{set of stopping times.}$$

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-
  - For engineering purposes, we can provide a rigorous framework to define a notion of fragility for a mechanical structure.
  - Mathematically: new type of free boundary problem for an optimal stopping problem.

Fragility time (optimal stopping problem) (2/2):

- Recall the elastic and plastic operators:
  - $A\varphi = -y\varphi_z + (c_0 y + kz)\varphi_y - \frac{1}{2}\varphi_{yy}$,
  - $B_{\pm}\varphi = (c_0 y \pm kY)\varphi_y - \frac{1}{2}\varphi_{yy}$,

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- then the problem of finding $\min_{\tau \in \mathcal{T}_{0,T}} \{\mathbb{E}(\Delta(\tau) - b)^2\}$ corresponds to find $V(\Delta, y, z)$ such that

$$\max\left(\frac{\partial V}{\partial t} + AV, -((|\Delta| - b)^2 + V)\right) = 0 \text{ in } D$$

$$\max\left(\frac{\partial V}{\partial t} + B_+ V, -((|\Delta| - b)^2 + V)\right) = 0 \text{ in } D^+$$

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with a non local boundary condition

$$(\Delta, y) \to V(\Delta, y, \pm Y) \text{ is continuous.}$$

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with a non local boundary condition

$$(\Delta, y) \to V(\Delta, y, \pm Y) \text{ is continuous.}$$

Critical stochastic excitation

- We would like to calculate a function $\sigma(t)$ such that

$$\dot{y} = -(c_0 y + kz) + \sigma(t)\dot{w},$$
$$(\dot{z} - y)(\phi - z) \geq 0, \forall |\phi| \leq Y, |z(t)| \leq Y.$$

  and $\mathbb{E}(\Delta(T) - b)^2$ is minimal.

-
  - For engineering purposes, we can characterize the type of excitation which leads to a certain level of plastic deformation.
  - Mathematically: a new problem of stochastic optimisation.

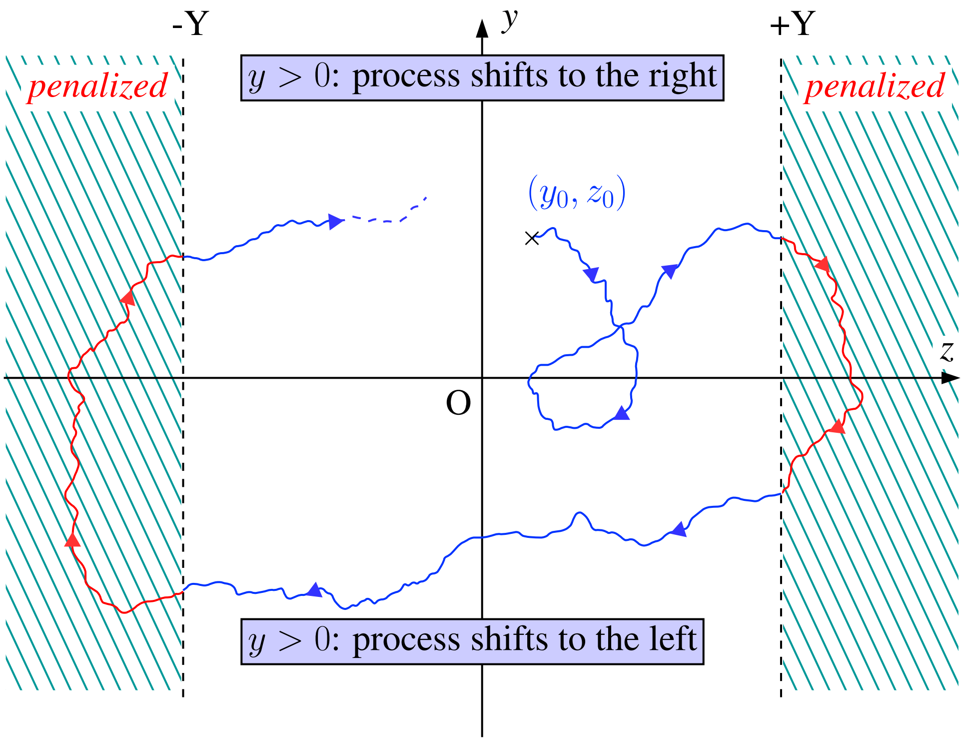
Our project in CEMRACS : MECALEA

With Mathieu Laurière,

- we study a penalized equation $(y_n(t), z_n(t)) \in \mathbb{R}^2$,

$$\begin{cases} dy_n(t) = -(c_0 y_n(t) + k z_n(t))dt + dw(t), \\ dz_n(t) = y_n(t)dt - n(z_n(t) - \pi(z_n(t)))dt \end{cases} \tag{4}$$

where $\pi(z)$ is the projection of z on $[-Y, Y]$.



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The end.

Thanks for your attention,

any questions?