CEMRACS
Quantitative theory in stochastic homogenization
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1 Introduction, setting, and main results

This course relies on a work in preparation with A. Gloria [2], which is a continuum version of [1]. It slightly differs from [2] because the present analysis does not rely on Green's functions and treats the periodic case. For related work on an emerging quantitative theory of stochastic homogenization, including many references, we refer to three preprints, which are available on my web page: [5] requires the least machinery, [3] gives an extensive introduction next to a couple of quantitative results, and [4] uses both to give a full error estimate.

We start by introducing the relevant deterministic notions: The corrector $\phi(a;\cdot)$ and the homogenized coefficient $a_{hom}(a)$ for an arbitrary coefficient field a on the torus of side length L, which we sometimes denote as $[-\frac{L}{2},\frac{L}{2})^d$ to single out the point 0.

Definition 1.
Space of coefficient fields. *For a given side-length L let Ω be the space of all $[-\frac{L}{2},\frac{L}{2})^d$-periodic fields of symmetric $d\times d$ matrices a that are uniformly elliptic in the sense*

$$\forall\, x\in[-\frac{L}{2},\frac{L}{2})^d,\ \xi\in\mathbb{R}^d\quad \lambda|\xi|^2\le\xi\cdot a(x)\xi\le|\xi|^2,$$

where $\lambda>0$ is a number fixed throughout the course.
Corrector. *For given $a\in\Omega$, the corrector $\phi(a;\cdot)$ is an $[-\frac{L}{2},\frac{L}{2})^d$-periodic function defined through the elliptic equation*

$$-\nabla\cdot a(\nabla\phi(a;\cdot)+\xi)=0\quad \textit{and}\quad \int_{[-\frac{L}{2},\frac{L}{2})^d}\phi(a;\cdot)=0, \tag{1}$$

where $\xi \in \mathbb{R}^d$ with $|\xi| = 1$ is a direction which is fixed throughout the course. For further reference we note that ϕ is "stationary" in the sense of

$$\phi(a(\cdot + z), x) = \phi(a, x + z) \tag{2}$$

for all points $x \in \mathbb{R}^d$, coefficient fields $a \in \Omega$, and shift vectors $z \in \mathbb{R}^d$.

Homogenized coefficient. *The homogenized coefficient in direction ξ is defined via*

$$\xi \cdot a_{hom}(a)\xi := L^{-d} \int_{[-\frac{L}{2}, \frac{L}{2})^d} (\nabla\phi(a; \cdot) + \xi) \cdot a(\nabla\phi(a; \cdot) + \xi). \tag{3}$$

We now introduce our example of an ensemble on the space of coefficient fields on the torus.

Definition 2. *By the "Poisson ensemble" we understand the following probability measure on Ω:*

Let the configuration of points $X := \{X_n\}_{n=1,\cdots,N}$ on the torus be distributed according to the Poisson point process with density one. This means the following

- *For any two disjoint (Lebesgue measurable) subsets D and D' of the torus we have that the configuration of points in D and the configuration of points in D' are independent. In other words, if ζ is a function of X that depends on X only through $X_{|D}$ and ζ' is a function of X that depends on X only through $X_{|D'}$ we have*

$$\langle \zeta\zeta' \rangle_0 = \langle \zeta \rangle_0 \langle \zeta' \rangle_0, \tag{4}$$

  where $\langle \cdot \rangle_0$ denotes the expectation w. r. t. the Poisson point process.

- *For any (Lebesgue measurable) subset D of the torus, the number of points in D is Poisson distributed; the expected number is given by the Lebesgue measure of D.*

Note that N is random, too.

With any realization $X = \{X_n\}_{n=1,\cdots,N}$ of the Poisson point process, we associate the coefficient field $a \in \Omega$ via

$$a(x) = \left\{ \begin{array}{ccc} \lambda & \mathit{if} & x \in \bigcup_{n=1}^N B_1(X_n) \\ 1 & \mathit{else} & \end{array} \right\} \mathrm{id}. \tag{5}$$

Here and throughout the course, balls like $B_1(X_n)$ refer to the distance function of the torus. This defines a probability measure on Ω by "push-forward" of $\langle\cdot\rangle_0$. We denote the expectation w. r. t. this ensemble with $\langle\cdot\rangle$.

For our result, we only need the following two properties of the Poisson ensemble.

Lemma 1.
STATIONARITY. *The Poisson ensemble is stationary which means that for any shift vector $z \in \mathbb{Z}^d$ the random field a and its shifted version $a(\cdot+z)\colon x \mapsto a(x+z)$ have the same distribution. In other words, for any (integrable) function $\zeta\colon \Omega \to \mathbb{R}$ (which we think of as a random variable) we have that $a \mapsto \zeta(a(\cdot+z))$ and ζ have the same expectation:*

$$\langle \zeta(a(\cdot + z))\rangle = \langle \zeta\rangle. \tag{6}$$

SPECTRAL GAP ESTIMATE. *The Poisson ensemble satisfies a Spectral Gap Estimate by which we understand the following: There exists a radius R only depending on d such that for any function $\zeta\colon \Omega \to \mathbb{R}$, we have*

$$\langle (\zeta - \langle\zeta\rangle)^2\rangle \le \langle \sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}\zeta)^2\rangle. \tag{7}$$

Here, for a (Lebesgue measurable) subset D of the torus, the (essential) oscillation $\mathrm{osc}_D\zeta$ of ζ with respect to D is a random variable defined through

$$\begin{aligned}(\mathrm{osc}_D\zeta)(a) &= \sup\{\zeta(\tilde a)|\tilde a \in \Omega \textit{ with } \tilde a = a \textit{ outside } D\}\\ &\quad - \inf\{\zeta(\tilde a)|\tilde a \in \Omega \textit{ with } \tilde a = a \textit{ outside } D\}.\end{aligned} \tag{8}$$

It measures how sensitively $\zeta(a)$ depends on $a_{|D}$. Note that $(\mathrm{osc}_D\zeta)(a)$ does not depend on $a_{|D}$.

The main result of the course is a Central Limit Theorem-type scaling of the variance of the homogenized coefficient in terms of the system volume L^d.

Theorem 1. *Suppose $\langle\cdot\rangle$ is stationary and satisfies the Spectral Gap Estimate. Then we have the following estimate on the variance of the homogenized coefficient*

$$\langle (\xi\cdot a_{hom}\xi - \langle \xi\cdot a_{hom}\xi\rangle)^2\rangle \le C(d,\lambda)L^{-d}.$$

In this course, we prove Theorem 1 only for $d > 2$. We shall derive it from the following result of independent interest, which is only true for $d > 2$.

Proposition 1. *Let $d > 2$ and suppose $\langle\cdot\rangle$ is stationary and satisfies the Spectral Gap Estimate. Then all moments of the corrector are bounded independently of L, that is, for any $1 \le p < \infty$ we have*

$$\langle \phi^{2p} \rangle \le C(d, \lambda, p).$$

Here and in the entire text, we write ϕ^{2p} for $(\phi^2)^p$, so that expressions like above make sense also for a non-integer exponent p.

2 Auxiliary results

We need the following $L^p(\Omega)$-version of the Spectral Gap Estimate.

Lemma 2. *Let $\langle\cdot\rangle$ satisfy the Spectral Gap Estimate. Then it satisfies an $L^p(\Omega)$-version of a Spectral Gap Estimate in the following sense: Let R be the radius from (7). Then we have for any ($2p$-integrable) function $\zeta\colon \Omega \to \mathbb{R}$ and any $1 \le p < \infty$*

$$\langle (\zeta - \langle \zeta \rangle)^{2p} \rangle \lesssim \langle (\sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2})^d} (\mathrm{osc}_{B_R(z)} \zeta)^2)^p \rangle. \tag{9}$$

Here $\lesssim$ means up to a generic constant that only depends on p.

Together with the previous lemma, the following lemma gives an estimate of ϕ in terms of $\nabla\phi + \xi$.

Lemma 3. *Suppose $d > 2$ and that $\langle\cdot\rangle$ is stationary. Then we have for any $\frac{d}{d-2} < p < \infty$ and any $R \lesssim 1$*

$$\langle (\sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2})^d} (\mathrm{osc}_{B_R(z)} \phi(\cdot;0))^2)^p \rangle \lesssim \langle (\int_{B_1} |\nabla\phi + \xi|^2)^p \rangle, \tag{10}$$

where $\lesssim$ means up to a generic constant only depending on d, λ, p, and R.

The last lemma in turn gives an estimate of $\nabla\phi + \xi$ in terms of ϕ.

Lemma 4. *Suppose that $\langle\cdot\rangle$ is stationary. Then we have for any $2 \le p < \infty$*

$$\langle (\int_{B_1} |\nabla\phi + \xi|^2)^p \rangle \lesssim \langle \phi^{2(p-1)} \rangle + 1,$$

where $\lesssim$ means up to a generic constant only depending on d, λ, and p.

3 Proofs

Proof of Lemma 1.

Step 1. Generalization and reduction. The most natural form of the result of the lemma is the following: For any measurable partition $D_1, \cdots, D_N$ of the torus we have

$$\langle(\zeta - \langle\zeta\rangle)^2\rangle \le \langle\sum_{n=1}^{N}(\mathrm{osc}_{B_1(D_n)}\zeta)^2\rangle, \tag{11}$$

where $B_1(D)$ is the set of all points on the torus that have distance less than one to D. In this step, we will derive this from the following similar estimate on the Poisson point process itself:

$$\langle(\zeta_0 - \langle\zeta_0\rangle_0)^2\rangle_0 \le \langle\sum_{n=1}^{N}(\mathrm{osc}_{0,D_n}\zeta_0)^2\rangle_0. \tag{12}$$

Here $\langle\cdot\rangle_0$ denotes the expectation w. r. t. to the Poisson point process $X := \{X_n\}_{n=1,\cdots,N}$, ζ is a (square integrable) function of the point configuration X, and the oscillation $\mathrm{osc}_{0,D}$ is defined in a similar way to (8):

$$\begin{aligned}(\mathrm{osc}_{0,D}\zeta_0)(X) &= \sup\{\zeta_0(\tilde{X})|\ \tilde{X} = X \text{outside of } D\} \\ &\quad - \inf\{\zeta_0(\tilde{X})|\ \tilde{X} = X \text{outside of } D\}. \end{aligned} \tag{13}$$

Indeed, (11) is an immediate consequence of (12) because of the following two facts.

- We recall that (5) defines a mapping $X \mapsto a$ from point configurations to coefficient fields. As such, it pulls back functions according to $\zeta_0(X) = \zeta(a(X))$ and pushes forward the ensemble according to

  $$\langle\zeta\rangle = \langle\zeta_0\rangle_0. \tag{14}$$

  In particular we have for the variance

  $$\langle(\zeta - \langle\zeta\rangle)^2\rangle = \langle(\zeta_0 - \langle\zeta_0\rangle_0)^2\rangle_0.$$

- By definition (5), if the point configurations X and $\tilde{X}$ coincide outside of D, then the corresponding coefficient fields $a(X;\cdot)$ and $a(\tilde{X};\cdot)$

coincide outside of $B_1(D)$. Hence for a given configuration X, the set $\{a(\tilde{X})|\tilde{X}=X$ outside of $D\}$ is contained in the set $\{\tilde{a}|\tilde{a}=a(X)$ outside of $B_1(D)\}$ so that

$$\begin{aligned}&\sup\{\zeta(a(\tilde{X}))|\tilde{X}=X \text{ outside of } D\}\\ \leq\ &\sup\{\zeta(\tilde{a})|\tilde{a}=a(X) \text{ outside of } B_1(D)\},\end{aligned}$$

and the opposite inequality if we replace the supremum by the infimum. From the definitions (8) and (13) of the oscillation we thus see

$$(\operatorname{osc}_{0,D}\zeta_0)(X)\leq(\operatorname{osc}_{B_1(D)}\zeta)(a(X)).$$

By (14), this implies as desired

$$\langle(\operatorname{osc}_{0,D}\zeta_0)^2\rangle_0\leq\langle(\operatorname{osc}_{B_1(D)}\zeta)^2\rangle.$$

Step 2. Conditional expectations and independence. From now on, we prove statement (12) for the Poisson point process. For brevity, we drop the subscript 0.

For a given (Lebesgue measurable) subset D of the torus, we denote by $\langle\cdot|D\rangle$ the expectation conditioned on the restriction $X_{|D}$ of the (random) point configuration X on D. We note that for a function $\zeta\colon\Omega\to\mathbb{R}$ which is square integrable, $\langle\zeta|D\rangle$ is the $L^2(\Omega)$-orthogonal projection of ζ onto the space of square integrable functions $\tilde{\zeta}\colon\Omega\to\mathbb{R}$ that only depend on X via $X_{|D}$.

With help of these conditional expectations the independence assumption (4) can be rephrased as follows: For any two (Lebesgue measurable) subsets D, D' that are *disjoint* and any (square integrable) function $\zeta\colon\Omega\to\mathbb{R}$ that does *not* depend on $X_{|D}$ we have

$$\langle\zeta|D\cup D'\rangle=\langle\zeta|D'\rangle. \tag{15}$$

Here comes the argument: By definition of conditional expectation, (15) follows if for any pair of (bounded and measurable) test functions u and u' which only depend on $X_{|D}$ and $X_{|D'}$, respectively, we have

$$\langle\zeta uu'\rangle=\langle\langle\zeta|D'\rangle uu'\rangle.$$

Indeed, on the one hand, since ζ only depends on $X_{|D^c}$ (where D^c denotes the complement of D) and u' does only depend on $X_{|D'}$ (and thus a fortiori only on $X_{|D^c}$) while u only depends on $X_{|D}$, we have from (4):

$$\langle \zeta u u' \rangle = \langle \zeta u' \rangle \langle u \rangle.$$

On the other hand, since $\langle \zeta | D' \rangle u'$ only depends on $X_{|D'}$ (and in particular only on $X_{|D^c}$) while u only depends on $X_{|D}$, we have from (4):

$$\langle \langle \zeta | D' \rangle u u' \rangle = \langle \langle \zeta | D' \rangle u' \rangle \langle u \rangle = \langle \langle \zeta u' | D' \rangle \rangle \langle u \rangle = \langle \zeta u' \rangle \langle u \rangle,$$

where the middle identity holds since u' only depends on $X_{|D'}$.

Step 3. Conditional expectation and oscillation. For any (Lebesgue measurable) disjoint subsets D and D' of the torus and any (square integrable) function, we have

$$|\langle \zeta | D \cup D' \rangle - \langle \zeta | D' \rangle| \le \langle \mathrm{osc}_D \zeta | D' \rangle. \tag{16}$$

By exchanging ζ with $-\zeta$, we see that it is enough to show

$$\langle \zeta | D \cup D' \rangle \le \langle \zeta | D' \rangle + \langle \mathrm{osc}_D \zeta | D' \rangle, \tag{17}$$

We note that $\sup_D \zeta \le \zeta + \mathrm{osc}_D \zeta$, where we've set for abbreviation

$$(\sup_D \zeta)(X) := \sup\{\zeta(\tilde{X}) | \tilde{X} = X \text{ outside of } D\}.$$

Hence (17) follows from

$$\langle \zeta | D \cup D' \rangle \le \langle \sup_D \zeta | D' \rangle.$$

The latter inequality can be seen as follows

$$\begin{aligned} &\langle \zeta | D \cup D' \rangle \\ &\quad \le \quad \langle \sup_D \zeta | D \cup D' \rangle \quad \text{since } \zeta \le \sup_D \zeta \\ &\quad \overset{(15)}{=} \quad \langle \sup_D \zeta | D' \rangle \quad \text{since } \sup_D \zeta \text{ does not depend on } X_{|D}. \end{aligned}$$

Step 4. Martingale decomposition. For conciseness, we only prove (12) for $N = 3$. So let $\{D_1, D_2, D_3\}$ be a partition of the torus, we claim

$$\begin{aligned} &\langle (\zeta - \langle \zeta \rangle)^2 \rangle \\ &\quad = \langle (\zeta - \langle \zeta | D_1 \cup D_2 \rangle)^2 \rangle + \langle (\langle \zeta | D_1 \cup D_2 \rangle - \langle \zeta | D_1 \rangle)^2 \rangle + \langle (\langle \zeta | D_1 \rangle - \langle \zeta \rangle)^2 \rangle. \end{aligned} \tag{18}$$

Indeed, this follows from the fact that

$$\zeta-\langle\zeta|D_1\cup D_2\rangle,\quad \langle\zeta|D_1\cup D_2\rangle-\langle\zeta|D_1\rangle,\quad \langle\zeta|D_1\rangle-\langle\zeta\rangle \quad \text{are } L^2(\Omega)-\text{orthogonal.}$$

The latter can be seen as follows: By definition of $\langle\cdot|D\rangle$ as $L^2(\Omega)$-orthogonal projection, the two last functions $\langle\zeta|D_1 \cup D_2\rangle - \langle\zeta|D_1\rangle$ and $\langle\zeta|D_1\rangle - \langle\zeta\rangle$ do only depend on $X_{|D_1\cup D_2}$, so that they are orthogonal to the first function $\zeta - \langle\zeta|D_1 \cup D_2\rangle$. It remains to argue that the two last functions $\langle\zeta|D_1 \cup D_2\rangle - \langle\zeta|D_1\rangle$ and $\langle\zeta|D_1\rangle - \langle\zeta\rangle$ are orthogonal. To that purpose, we rewrite the middle function as

$$\langle\zeta|D_1 \cup D_2\rangle - \langle\zeta|D_1\rangle = \zeta' - \langle\zeta'|D_1\rangle \quad \text{where } \zeta' := \langle\zeta|D_1 \cup D_2\rangle.$$

Since the last function only depends on $X_{|D_1}$, they are orthogonal.

Step 5. Conclusion, i. e. (11) for $N = 3$. By Step 4, it remains to estimate the three r. h. s. terms of (18). For the first term, we use (16) with $D = D_1 \cup D_2$ and $D' = D_3$ and obtain because of $\zeta = \langle\zeta|D_1 \cup D_2 \cup D_3\rangle$

$$\begin{aligned}\langle(\zeta - \langle\zeta|D_1 \cup D_2\rangle)^2\rangle &\leq \langle\langle \mathrm{osc}_{D_3}\zeta|D_1 \cup D_2\rangle^2\rangle\\ &\overset{\text{Jensen}}{\leq} \langle\langle(\mathrm{osc}_{D_3}\zeta)^2|D_1 \cup D_2\rangle\rangle = \langle(\mathrm{osc}_{D_3}\zeta)^2\rangle.\end{aligned}$$

The other two terms follow the same way.

Proof of Lemma 2.
W. l. o. g. we may assume that $\langle\zeta\rangle = 0$.
Step 1. Application of the original Spectral Gap Estimate to ζ^p. We claim that this yields

$$\langle\zeta^{2p}\rangle \lesssim \langle\zeta^p\rangle^2 + \langle(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}\zeta)^2)^p\rangle. \tag{19}$$

Indeed, (7) applied to ζ^p at first gives

$$\langle(\zeta^p - \langle\zeta^p\rangle)^2\rangle \lesssim \langle \sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}(\zeta^p))^2\rangle. \tag{20}$$

Using the triangle inequality in $L^2(\Omega)$ on the l. h. s. of (20) in form of

$$\langle(\zeta^p)^2\rangle^{\frac{1}{2}} \leq \langle(\zeta^p - \langle\zeta^p\rangle)^2\rangle^{\frac{1}{2}} + |\langle\zeta^p\rangle|,$$

we see that (19) follows from (20) by Young's inequality provided we can show

$$
\begin{aligned}
&\langle \sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}(\zeta^p))^2\rangle \\
&\lesssim \ \langle\zeta^{2p}\rangle^{1-\frac{1}{p}}\langle(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}\zeta)^2)^p\rangle^{\frac{1}{p}} + \langle(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}\zeta)^2)^p\rangle.
\end{aligned} \tag{21}
$$

The latter can be seen as follows: From the elementary real-variable estimate

$$
|\tilde\zeta^p-\zeta^p| \lesssim |\zeta|^{p-1}|\tilde\zeta-\zeta|+|\tilde\zeta-\zeta|^p,
$$

we obtain by definition of osc that

$$
\mathrm{osc}_{B_R(z)}(\zeta^p) \lesssim |\zeta|^{p-1}\mathrm{osc}_{B_R(z)}\zeta+(\mathrm{osc}_{B_R(z)}\zeta)^p.
$$

Using that the discrete $\ell^{2p}(\mathbb{Z}^d)$-norm is estimated by the discrete $\ell^2(\mathbb{Z}^d)$-norm, this implies

$$
\begin{aligned}
&\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}(\zeta^p))^2 \\
&\lesssim \ \zeta^{2(p-1)} \sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}\zeta)^2 + (\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}\zeta)^2)^p.
\end{aligned}
$$

Hölder's inequality w. r. t. to $\langle\cdot\rangle$ applied to the first r. h. s. term with exponents $(\frac{p}{p-1},p)$ yields (21).

Step 2. Conclusion in case of $p\geq 2$ (the other case is easier and not needed later). It remains to treat the first r. h. s. term of (19). By Hölder's inequality w. r. t. $\langle\cdot\rangle$ we have

$$
\langle\zeta^p\rangle \leq \langle\zeta^{2p}\rangle^{\frac{p-2}{2p-2}}\langle\zeta^2\rangle^{\frac{p}{2p-2}}. \tag{22}
$$

Using $\langle\zeta\rangle=0$ we obtain from the original Spectral Gap Estimate applied to ζ itself

$$
\langle\zeta^2\rangle \lesssim \langle \sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}\zeta)^2\rangle \overset{\text{Jensen}}{\leq} \langle(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}\zeta)^2)^p\rangle^{\frac{1}{p}}. \tag{23}
$$

Inserting (23) into (22) we obtain

$$
\langle\zeta^p\rangle^2 \lesssim \langle\zeta^{2p}\rangle^{\frac{p-2}{p-1}}\langle(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} (\mathrm{osc}_{B_R(z)}\zeta)^2)^p\rangle^{\frac{1}{p-1}}.
$$

Inserting this into (19) and using Young's inequality yields the claim of the lemma.

PROOF OF LEMMA 3.

Step 1. Regularity theory for a-harmonic functions. Without proof we will use the following two ingredients from De Giorgi's theory for uniformly elliptic equations: For any $a \in \Omega$ and any a-harmonic function u in B_2 we have

$$\sup_{B_1} |u| \lesssim (\int_{B_2} u^2)^{\frac{1}{2}}. \tag{24}$$

Moreover, there exists a Hölder exponent $\alpha > 0$ only depending on d and λ such that

$$\sup_{x_1, x_2 \in B_1} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} \lesssim (\int_{B_2} |\nabla u|^2)^{\frac{1}{2}}. \tag{25}$$

Here and in the sequel, we write $B_R = B_R(0)$ for brevity. The crucial element of these estimates is that the constants depend on the coefficient field a only through the ellipticity ratio λ (as indicated by the use of $\lesssim$).

Step 2. In this step, we derive an auxiliary a priori estimate involving dyadic annuli. Let u be a function and g a vector field on the torus related by the elliptic equation $-\nabla \cdot a \nabla u = \nabla \cdot g$ and normalized by $\int_{[-\frac{L}{2}, \frac{L}{2})^d} u = 0$. We claim that if g vanishes in B_1 we have

$$|u(0)| \lesssim \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} (\int_{B_{2^n} - B_{2^{n-1}}} |g|^2)^{\frac{1}{2}}. \tag{26}$$

We note that this sum is actually finite since for $2^n \gg L$, the ball $B_{2^{n-1}}$ invades the entire torus so that the "annulus" $B_{2^n} - B_{2^{n-1}}$ is actually void. Estimate (26) will be derived from (24) and an elementary scaling argument. Indeed, for $n \in \mathbb{N}$, we introduce

$$g_n := \left\{ \begin{array}{ll} g & \text{on} \quad B_{2^n} - B_{2^{n-1}} \\ 0 & \text{else} \end{array} \right\},$$

so that $g = \sum_{n=1}^{\infty} g_n$. Let u_n denote the solution of $-\nabla \cdot a \nabla u_n = \nabla \cdot g_n$ on the torus normalized by $\int_{[-\frac{L}{2}, \frac{L}{2})^d} u_n = 0$, so that $u = \sum_{n=1}^{\infty} u_n$. Hence by the triangle inequality, for (26) it is enough to show

$$|u_n(0)| \lesssim (2^n)^{1-\frac{d}{2}} (\int_{[-\frac{L}{2}, \frac{L}{2})^d} |g_n|^2)^{\frac{1}{2}}. \tag{27}$$

We now give the argument for (27). Testing $-\nabla \cdot a\nabla u_n = \nabla \cdot g_n$ with u_n, using the uniform ellipticity, and Cauchy-Schwarz' inequality, we obtain

$$(\int_{[-\frac{L}{2},\frac{L}{2})^d} |\nabla u_n|^2)^{\frac{1}{2}} \lesssim (\int_{[-\frac{L}{2},\frac{L}{2})^d} |g_n|^2)^{\frac{1}{2}}.$$

Since $d > 2$, Sobolev's embedding together with $\int_{[-\frac{L}{2},\frac{L}{2})^d} u_n = 0$ yields

$$(\int_{[-\frac{L}{2},\frac{L}{2})^d} |u_n|^{\frac{2d}{d-2}})^{\frac{d-2}{2d}} \lesssim (\int_{[-\frac{L}{2},\frac{L}{2})^d} |g_n|^2)^{\frac{1}{2}}.$$

By Hölder's inequality on $B_{2^{n-1}}$ with exponents $(\frac{d}{d-2}, \frac{d}{2})$ we obtain

$$(\int_{B_{2^{n-1}}} |u_n|^2)^{\frac{1}{2}} \lesssim 2^n(\int_{[-\frac{L}{2},\frac{L}{2})^d} |g_n|^2)^{\frac{1}{2}}.$$

We note that u_n is a-harmonic on $B_{2^{n-1}}$ (since g_n vanishes there). Hence by (24), which we rescale from B_2 to $B_{2^{n-1}}$, we have

$$|u_n(0)| \le \sup_{B_{2^{n-2}}} |u_n| \lesssim ((2^n)^{-d} \int_{B_{2^{n-1}}} |u_n|^2)^{\frac{1}{2}}.$$

The combination of the two last estimates yields (27).

Step 3. As a preliminary, we study the local dependence of $\nabla\phi + \xi$ on a: Let the two coefficient fields a and $\tilde{a}$ agree outside of B_R. Then we have

$$\int_{B_R} |\nabla\phi(\tilde{a};\cdot) + \xi|^2 \lesssim \int_{B_R} |\nabla\phi(a;\cdot) + \xi|^2. \tag{28}$$

Indeed, we note that the function $\phi(\tilde{a};\cdot) - \phi(a;\cdot)$ satisfies

$$-\nabla \cdot \tilde{a}\nabla(\phi(\tilde{a};\cdot) - \phi(a;\cdot)) = \nabla \cdot (\tilde{a} - a)(\nabla\phi(a;\cdot) + \xi).$$

We test this equation with $\phi(\tilde{a};\cdot)-\phi(a;\cdot)$ and obtain from uniform ellipticity and Cauchy-Schwarz' inequality

$$\int_{[-\frac{L}{2},\frac{L}{2})^d} |\nabla(\phi(\tilde{a};\cdot) - \phi(a;\cdot))|^2 \lesssim \int_{[-\frac{L}{2},\frac{L}{2})^d} |(\tilde{a} - a)(\nabla\phi(a;\cdot) + \xi)|^2.$$

Since by assumption, $\tilde{a}-a$ vanishes outside B_R, the above yields

$$\begin{aligned}
&\int_{B_R} |\nabla(\phi(\tilde{a};\cdot)-\phi(a;\cdot))|^2 \\
&\quad\le \int_{[-\frac{L}{2},\frac{L}{2})^d} |\nabla(\phi(\tilde{a};\cdot)-\phi(a;\cdot))|^2 \lesssim \int_{B_R} |\nabla\phi(a;\cdot)+\xi|^2. \qquad (29)
\end{aligned}$$

This implies (28) by the triangle inequality in $L^2(B_R)$.

Step 4. In this step, we derive the central deterministic estimate

$$\begin{aligned}
&\Big(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^2-B_{R+1}} (\mathrm{osc}_{B_R(z)}\phi(\cdot;0))^2\Big)^{\frac{1}{2}} \\
&\quad\lesssim \sum_{n=1}^{\infty}(2^n)^{1-\frac{d}{2}}\Big(\sum_{z\in\mathbb{Z}^d\cap B_{2^n+R}} \Big(\int_{B_R(z)}|\nabla\phi+\xi|^2\Big)^p\Big)^{\frac{1}{2p}}. \qquad (30)
\end{aligned}$$

Given a coefficient field a on the torus and a point on the integer lattice $z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d$, we denote by a_z an arbitrary coefficient field on the torus that agrees with a outside of $B_R(z)$. We note that the function $\phi(a_z;\cdot)-\phi(a;\cdot)$ satisfies

$$-\nabla\cdot a\nabla(\phi(a_z;\cdot)-\phi(a;\cdot)) = \nabla\cdot(a_z-a)(\nabla\phi(a_z;\cdot)+\xi). \qquad (31)$$

Given a discrete field $\{\omega_z\}_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d}$ we consider the function u and the vector field g on the torus defined through

$$\begin{aligned}
u(x) &:= \sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} \omega_z(\phi(a_z;x)-\phi(a;x)), \\
g(x) &:= \sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} \omega_z(a_z(x)-a(x))(\phi(a_z;x)+\xi)
\end{aligned}$$

and note that (31) translates into $-\nabla\cdot a\nabla u=\nabla\cdot g$. Provided $\omega_z=0$ for $z\in B_{R+1}$, we have $g(x)=0$ for $x\in B_1$. Under this assumption, we may apply (26) from Step 2 and obtain

$$\begin{aligned}
&\Big|\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} \omega_z(\phi(a_z;0)-\phi(a;0))\Big| \\
&\quad\lesssim \sum_{n=1}^{\infty}(2^n)^{1-\frac{d}{2}}\Big(\int_{B_{2^n}-B_{2^{n-1}}}\Big|\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} \omega_z(a_z-a)(\nabla\phi(a_z;\cdot)+\xi)\Big|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Since $|a_z - a| \leq 1$ is supported in $B_R(z)$ and since $\{B_R(z)\}_{z\in\mathbb{Z}^d}$ locally have a finite overlap, this turns into

$$
\begin{aligned}
&|\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} \omega_z(\phi(a_z;0)-\phi(a;0))| \\
&\lesssim \sum_{n=1}^{\infty}(2^n)^{1-\frac{d}{2}}(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} \omega_z^2 \int_{B_R(z)\cap B_{2^n}} |\nabla\phi(a_z,\cdot)+\xi|^2)^{\frac{1}{2}} \\
&\overset{(28)}{\lesssim} \sum_{n=1}^{\infty}(2^n)^{1-\frac{d}{2}}(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} \omega_z^2 \int_{B_R(z)\cap B_{2^n}} |\nabla\phi(a,\cdot)+\xi|^2)^{\frac{1}{2}} \\
&\overset{\text{Hölder in } z}{\lesssim} (\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d} \omega_z^{2q})^{\frac{1}{2q}} \\
&\qquad \times \sum_{n=1}^{\infty}(2^n)^{1-\frac{d}{2}}(\sum_{z\in\mathbb{Z}^d\cap B_{2^n+R}} (\int_{B_R(z)} |\nabla\phi(a,\cdot)+\xi|^2)^p)^{\frac{1}{2p}},
\end{aligned}
$$

where p and q are dual exponents, that is, $\frac{1}{p}+\frac{1}{q}=1$. Since $\{\omega_z\}_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d}$ was arbitrary under the constraint that $\omega_z = 0$ for $z \in B_{R+1}$ this implies by the duality of $\ell^{2q}(\mathbb{Z}^d)$ and $\ell^{\frac{2q}{2q-1}}(\mathbb{Z}^d)$

$$
\begin{aligned}
&(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d-B_{R+1}} |\phi(a_z;0)-\phi(a;0)|^{\frac{2q}{2q-1}})^{\frac{2q-1}{2q}} \\
&\quad\lesssim \sum_{n=1}^{\infty}(2^n)^{1-\frac{d}{2}}(\sum_{z\in\mathbb{Z}^d\cap B_{2^n+R}} (\int_{B_R(z)} |\nabla\phi(a,\cdot)+\xi|^2)^p)^{\frac{1}{2p}}.
\end{aligned}
$$

Since for any $z\in\mathbb{Z}^d$, a_z was an arbitrary coefficient field that agrees with a outside of $B_R(z)$, this implies by the definition of $\mathrm{osc}_{B_R(z)}$

$$
\begin{aligned}
&(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d-B_{R+1}} (\mathrm{osc}_{B_R(z)}\phi(\cdot;0))^{\frac{2q}{2q-1}})^{\frac{2q-1}{2q}} \\
&\quad\lesssim \sum_{n=1}^{\infty}(2^n)^{1-\frac{d}{2}}(\sum_{z\in\mathbb{Z}^d\cap B_{2^n+R}} (\int_{B_R(z)} |\nabla\phi+\xi|^2)^p)^{\frac{1}{2p}}.
\end{aligned}
$$

On the l. h. s. we use that since $\frac{2q}{2q-1}\leq 2$, the discrete $\ell^{\frac{2q}{2q-1}}(\mathbb{Z}^d)$-norm dominates the discrete $\ell^2(\mathbb{Z}^d)$-norm to obtain (30).

Step 5. Using stationarity, we upgrade Step 4 to the stochastic estimate

$$\langle(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d-B_{R+1}}(\mathrm{osc}_{B_R(z)}\phi(\cdot;0))^2)^p\rangle^{\frac{1}{2p}}\lesssim\langle(\int_{B_R}|\nabla\phi+\xi|^2)^p\rangle^{\frac{1}{2p}}. \qquad (32)$$

Indeed, we start from (30) in Step 4 and apply the triangle inequality to the sum over n w. r. t. the norm $L^{2p}(\Omega)$:

$$\begin{aligned}&\langle(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d-B_{R+1}}\mathrm{osc}_{B_R(z)}(\phi(\cdot;0))^2)^p\rangle^{\frac{1}{2p}}\\ &\lesssim\sum_{n=1}^{\infty}(2^n)^{1-\frac{d}{2}}(\sum_{z\in\mathbb{Z}^d\cap B_{2^n+R}}\langle(\int_{B_R(z)}|\nabla\phi(a,\cdot)+\xi|^2)^p\rangle)^{\frac{1}{2p}}. \qquad (33)\end{aligned}$$

We now note that the stationarity (2) of ϕ also yields

$$\nabla\phi(a;x+z)=\nabla\phi(a(\cdot+z);x)$$

and thus

$$\int_{B_R(z)}|\nabla\phi(a,x')+\xi|^2dx'=\int_{B_R}|\nabla\phi(a(\cdot+z),x)+\xi|^2dx.$$

By stationarity of $\langle\cdot\rangle$, cf. (6) applied to $\zeta(a)=\int_{B_R(z)}|\nabla\phi(a;x)+\xi|^2dx$, this implies

$$\langle(\int_{B_R(z)}|\nabla\phi(\cdot,x')+\xi|^2dx')^p\rangle=\langle(\int_{B_R}|\nabla\phi(\cdot,x)+\xi|^2dx)^p\rangle. \qquad (34)$$

Inserting this into (33) yields (because of $\sum_{z\in\mathbb{Z}^d\cap B_{2^n+R}}1\lesssim(2^n+R)^d$)

$$\begin{aligned}&\langle(\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d-B_{R+1}}\mathrm{osc}_{B_R(z)}(\phi(\cdot;0))^2)^p\rangle^{\frac{1}{2p}}\\ &\lesssim\sum_{n=1}^{\infty}(2^n)^{1-\frac{d}{2}}(2^n+R)^{\frac{d}{2p}}\langle(\int_{B_R}|\nabla\phi(a,\cdot)+\xi|^2)^p\rangle^{\frac{1}{2p}}. \qquad (35)\end{aligned}$$

Since for $p>\frac{d}{d-2}$ the exponent $1-\frac{d}{2}+\frac{d}{2p}<0$ is negative we have $\sum_{n=1}^{\infty}(2^n)^{1-\frac{d}{2}}(2^n+R)^{\frac{d}{2p}}\lesssim 1$. Hence (35) turns into the desired (32).

Step 6. It remains to treat $z \in \mathbb{Z}^d \cap B_{R+1}$ in (10). By stationarity, it will be enough to consider $z = 0$, cf. Step 7. In this step, we will derive from the Hölder continuity a priori estimate (25) the deterministic estimate

$$\mathrm{osc}_{B_R} \phi(a;0) \lesssim (\int_{B_{2R}} |\nabla\phi(a;\cdot) + \xi|^2)^{\frac{1}{2}}. \tag{36}$$

Let $a \in \Omega$ be given and $\tilde{a} \in \Omega$ agree with a outside of B_R and otherwise be arbitrary. On the one hand, since $d > 2$ and $\int_{[-\frac{L}{2},\frac{L}{2})^d} (\phi(\tilde{a};\cdot) - \phi(a;\cdot)) \overset{(1)}{=} 0$, we have by Sobolev's embedding

$$\begin{aligned}
(\int_{B_R} |\phi(\tilde{a};\cdot) - \phi(a;\cdot)|^{\frac{2d}{d-2}})^{\frac{d-2}{2d}} \quad &\leq \quad (\int_{[-\frac{L}{2},\frac{L}{2})^d} |\phi(\tilde{a};\cdot) - \phi(a;\cdot)|^{\frac{2d}{d-2}})^{\frac{d-2}{2d}} \\
&\lesssim \quad (\int_{[-\frac{L}{2},\frac{L}{2})^d} |\nabla(\phi(\tilde{a};\cdot) - \phi(a;\cdot))|^2)^{\frac{1}{2}} \\
&\overset{(29)}{\lesssim} \quad (\int_{B_R} |\nabla\phi(a;\cdot) + \xi|^2)^{\frac{1}{2}}.
\end{aligned} \tag{37}$$

On the other hand, we obtain from (25) applied to the a-harmonic function $u(x) = \phi(a;x) + \xi \cdot x$ (rescaled from B_1 to B_R):

$$\sup_{x_1,x_2 \in B_R} \frac{|\phi(a;x_1) - \phi(a;x_2) + \xi \cdot (x_1 - x_2)|}{|x_1 - x_2|^\alpha} \lesssim (\int_{B_{2R}} |\nabla\phi(a;\cdot) + \xi|^2)^{\frac{1}{2}}. \tag{38}$$

Replacing a by $\tilde{a}$ in the above and using (28) from Step 3 (with B_R replaced by B_{2R}) we likewise have

$$\sup_{x_1,x_2 \in B_R} \frac{|\phi(\tilde{a};x_1) - \phi(\tilde{a};x_2) + \xi \cdot (x_1 - x_2)|}{|x_1 - x_2|^\alpha} \lesssim (\int_{B_{2R}} |\nabla\phi(a;\cdot) + \xi|^2)^{\frac{1}{2}}. \tag{39}$$

Combining (38) and (39), we obtain

$$\sup_{x_1,x_2 \in B_R} \frac{|(\phi(\tilde{a};x_1) - \phi(a;x_1)) - (\phi(\tilde{a};x_2) - \phi(a;x_2))|}{|x_1 - x_2|^\alpha} \lesssim (\int_{B_{2R}} |\nabla\phi(a;\cdot)+\xi|^2)^{\frac{1}{2}}. \tag{40}$$

By the following elementary interpolation estimate, valid for an arbitrary function u,

$$\sup_{x \in B_R} |u(x)| \lesssim (\int_{B_R} |u|^{\frac{2d}{d-2}})^{\frac{d-2}{2d}} + \sup_{x_1,x_2 \in B_R} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha},$$

we see that (37) and (40) combine to

$$|\phi(\tilde{a};0)-\phi(a;0)|\lesssim(\int_{B_{2R}}|\nabla\phi(a;\cdot)+\xi|^2)^{\frac{1}{2}}.$$

Since $\tilde{a}$ was arbitrary besides agreeing with a outside of B_R, we obtain (36) by definition of osc.

Step 7. We upgrade Step 6 to the stochastic estimate

$$\langle(\sum_{z\in\mathbb{Z}^d\cap B_{R+1}}(\mathrm{osc}_{B_R(z)}\phi(\cdot;0))^2)^p\rangle\lesssim\langle(\int_{B_{3R+1}}|\nabla\phi+\xi|^2)^p\rangle. \tag{41}$$

Indeed, (36) from Step 6, with the origin replaced by z, implies after summation

$$\sum_{z\in\mathbb{Z}^d\cap B_{R+1}}(\mathrm{osc}_{B_R(z)}\phi(\cdot;0))^2\lesssim\int_{B_{3R+1}}|\nabla\phi+\xi|^2.$$

Taking the p-th power and the expectation yields (41).

Step 8. From Steps 5 and 7 we learn that (10) is satisfied with B_1 replaced by B_{3R+1} on the r. h. s. . We appeal once more to stationarity to get for a generic $R\lesssim 1$

$$\langle(\int_{B_R}|\nabla\phi+\xi|^2)^p\rangle\lesssim\langle(\int_{B_1}|\nabla\phi+\xi|^2)^p\rangle. \tag{42}$$

Indeed, there exist points $z_1,\cdots,z_N$ on the torus such that $B_R\subset\bigcup_{n=1}^N B_1(z_n)$ and we can arrange for $N\lesssim 1$ because of $R\lesssim 1$. Thus we have

$$\int_{B_R}|\nabla\phi+\xi|^2\le\sum_{n=1}^N\int_{B_1(z_n)}|\nabla\phi+\xi|^2.$$

Taking the p-th power gives

$$(\int_{B_R}|\nabla\phi+\xi|^2)^p\lesssim\sum_{n=1}^N(\int_{B_1(z_n)}|\nabla\phi+\xi|^2)^p;$$

taking the expectation yields

$$\langle(\int_{B_R}|\nabla\phi+\xi|^2)^p\rangle\le\max_{n=1,\cdots,N}\langle(\int_{B_1(z_n)}|\nabla\phi+\xi|^2)^p\rangle.$$

By stationarity, cf. (34), this yields (42).

PROOF OF LEMMA 4

Step 1. We start by establishing the deterministic estimate

$$(\int_{B_1} |\nabla\phi+\xi|^2)^p \lesssim \int_{B_2} (\phi+\xi\cdot x)^{2(p-1)}|\nabla\phi+\xi|^2, \tag{43}$$

which we will use in form of

$$(\int_{B_1} |\nabla\phi+\xi|^2)^p \lesssim \int_{B_2} (\phi^{2(p-1)}+1)(|\nabla\phi|^2+1). \tag{44}$$

Estimate (43) relies on the fact that $u(x) := \phi(x)+\xi\cdot x$ is a-harmonic, that is,

$$-\nabla\cdot a\nabla u = 0.$$

We test this equation with $\eta^2 u$, where η is a cut-off function for B_1 in B_2. By uniform ellipticity we obtain

$$\lambda\int(\eta|\nabla u|)^2 \le 2\int|\nabla\eta||u|\,\eta|\nabla u|.$$

We now use Young's inequality (and $p\ge 2$) on the r. h. s. integrand in form of

$$\frac{2}{\lambda}|\nabla\eta||u|\,\eta|\nabla u| \le \frac{1}{2}(\eta|\nabla u|)^2 + C(|\nabla\eta|u)^{2\frac{p-1}{p}}(\eta|\nabla u|)^{\frac{2}{p}},$$

which yields

$$\int(\eta|\nabla u|)^2 \lesssim \int(|\nabla\eta|u)^{2\frac{p-1}{p}}(\eta|\nabla u|)^{\frac{2}{p}}.$$

By the choice of η, this implies

$$\int_{B_1}|\nabla u|^2 \lesssim \int_{B_2} u^{2\frac{p-1}{p}}|\nabla u|^{\frac{2}{p}}.$$

It remains to apply Jensen's inequality on the r. h. s. to obtain as desired

$$\int_{B_1}|\nabla u|^2 \lesssim (\int_{B_2} u^{2(p-1)}|\nabla u|^2)^{\frac{1}{p}}.$$

Step 2. We continue with the deterministic estimate

$$\int_{[-\frac{L}{2},\frac{L}{2})^d} \phi^{2(p-1)}|\nabla\phi|^2 \lesssim \int_{[-\frac{L}{2},\frac{L}{2})^d} \phi^{2(p-1)}, \tag{45}$$

which we will use in form of

$$\int_{[-\frac{L}{2},\frac{L}{2})^d}(\phi^{2(p-1)}+1)|\nabla\phi|^2 \lesssim \int_{[-\frac{L}{2},\frac{L}{2})^d}(\phi^{2(p-1)}+1). \tag{46}$$

Indeed, we test $-\nabla\cdot a(\nabla\phi+\xi)=0$ with the monotone-in-ϕ expression $\frac{1}{2p-1}\phi|\phi|^{2(p-1)}$ over the entire torus. Because of $\nabla\frac{1}{2p-1}\phi|\phi|^{2(p-1)}=\phi^{2(p-1)}\nabla\phi$ and by uniform ellipticity, we obtain

$$\lambda\int_{[-\frac{L}{2},\frac{L}{2})^d}\phi^{2(p-1)}|\nabla\phi|^2 \le \int_{[-\frac{L}{2},\frac{L}{2})^d}\phi^{2(p-1)}|\nabla\phi|.$$

Using Cauchy-Schwarz' inequality on the r. h. s. of that inequality yields (45).

Step 3. Conclusion using stationarity. We take the expectation of (44):

$$\langle(\int_{B_1}|\nabla\phi+\xi|^2)^p\rangle \lesssim \langle\int_{B_2}(\phi^{2(p-1)}+1)(|\nabla\phi|^2+1)\rangle.$$

By stationarity, we have

$$\langle\int_{B_2}(\phi^{2(p-1)}+1)(|\nabla\phi|^2+1)\rangle = |B_2|L^{-d}\langle\int_{[-\frac{L}{2},\frac{L}{2})^d}(\phi^{2(p-1)}+1)(|\nabla\phi|^2+1)\rangle.$$

We now use the expectation of (46):

$$|B_2|L^{-d}\langle\int_{[-\frac{L}{2},\frac{L}{2})^d}(\phi^{2(p-1)}+1)|\nabla\phi|^2\rangle \lesssim L^{-d}\langle\int_{[-\frac{L}{2},\frac{L}{2})^d}(\phi^{2(p-1)}+1)\rangle.$$

We use once more stationarity in form of

$$L^{-d}\langle\int_{[-\frac{L}{2},\frac{L}{2})^d}(\phi^{2(p-1)}+1)\rangle = \langle\phi^{2(p-1)}\rangle+1.$$

Proof of Proposition 1.
By Jensen's inequality, it is enough to prove the statement for $p>\frac{d}{d-2}$. We apply Lemma 2 to $\zeta(a)=\phi(a;0)$. We note that by stationarity of ϕ and $\langle\cdot\rangle$ we have

$$\langle\phi\rangle = \langle L^{-d}\int_{[-\frac{L}{2},\frac{L}{2})^d}\phi\rangle \overset{(1)}{=} 0.$$

Hence the statement of Lemma 2 assumes the form

$$\langle \phi^{2p} \rangle \lesssim \langle (\sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2})^d} (\mathrm{osc}_{B_R(z)} \phi(\cdot; 0))^2)^p \rangle.$$

Estimating the r. h. s. by Lemmas 3 and 4, this turns into

$$\langle \phi^{2p} \rangle \leq C(\langle \phi^{2(p-1)} \rangle + 1).$$

We conclude by using Jensen's and Young's inequalities in form of $C\langle \phi^{2(p-1)} \rangle \leq C\langle \phi^{2p} \rangle^{\frac{p-1}{p}} \leq \frac{1}{2}\langle \phi^{2p} \rangle + \tilde{C}$.

Proof of Theorem 1.
Step 1. Application of Lemma 1 to $\zeta = \xi \cdot a_{hom}\xi$ yields

$$\langle (\xi \cdot a_{hom}\xi - \langle \xi \cdot a_{hom}\xi \rangle)^2 \rangle \lesssim \langle \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2})^d} (\mathrm{osc}_{B_R(z)} \xi \cdot a_{hom}\xi)^2 \rangle. \qquad (47)$$

Step 2. Deterministic estimate of the oscillation. We claim

$$\mathrm{osc}_{B_R(z)} \xi \cdot a_{hom}\xi \lesssim L^{-d} \int_{B_R(z)} |\nabla \phi + \xi|^2. \qquad (48)$$

Indeed, consider two arbitrary coefficient fields $a_0, a_1 \in \Omega$ that agree outside of $B_R(z)$. We write for abbreviation $\phi_i(x) = \phi(a_i; x)$ and $a_{hom,i} = a_{hom}(a_i)$ for $i = 0, 1$. By definition of osc it is enough to show

$$L^d |\xi \cdot a_{hom,1}\xi - \xi \cdot a_{hom,0}\xi| \lesssim \int_{B_R(z)} |\nabla \phi_0 + \xi|^2. \qquad (49)$$

Indeed, we have by definition of a_{hom} and of ϕ

$$\begin{aligned}
&L^d(\xi \cdot a_{hom,1}\xi - \xi \cdot a_{hom,0}\xi) \\
&\overset{(3)}{=} \int_{[-\frac{L}{2}, \frac{L}{2})^d} (\nabla \phi_1 + \xi) \cdot a_1 (\nabla \phi_1 + \xi) - \int_{[-\frac{L}{2}, \frac{L}{2})^d} (\nabla \phi_0 + \xi) \cdot a_0 (\nabla \phi_0 + \xi) \\
&= \int_{[-\frac{L}{2}, \frac{L}{2})^d} \nabla(\phi_1 - \phi_0) \cdot a_1 (\nabla \phi_1 + \xi) + \int_{[-\frac{L}{2}, \frac{L}{2})^d} \nabla(\phi_1 - \phi_0) \cdot a_0 (\nabla \phi_0 + \xi) \\
&\quad + \int_{[-\frac{L}{2}, \frac{L}{2})^d} (\nabla \phi_1 + \xi) \cdot (a_1 - a_0)(\nabla \phi_0 + \xi) \\
&\overset{(1)}{=} \int_{[-\frac{L}{2}, \frac{L}{2})^d} (\nabla \phi_1 + \xi) \cdot (a_1 - a_0)(\nabla \phi_0 + \xi),
\end{aligned}$$

so that we obtain

$$\begin{aligned} & L^d|\xi\cdot a_{hom,1}\xi-\xi\cdot a_{hom,0}\xi| \\ \le\ & (\int_{B_R(z)}|\nabla\phi_1+\xi|^2\int_{B_R(z)}|\nabla\phi_0+\xi|^2)^{\frac{1}{2}}. \end{aligned}$$

Now (49) follows from this and Step 3 in the proof of Lemma 3 in form of $\int_{B_R(z)}|\nabla\phi_1+\xi|^2\lesssim\int_{B_R(z)}|\nabla\phi_0+\xi|^2$.

Step 3. Stochastic estimate based on Proposition 1. We claim

$$\langle(\int_{B_R(z)}|\nabla\phi+\xi|^2)^2\rangle\lesssim 1. \tag{50}$$

Indeed, by Step 8 from the proof of Lemma 3 (and stationarity to replace z by 0) we have

$$\langle(\int_{B_R(z)}|\nabla\phi+\xi|^2)^2\rangle\lesssim\langle(\int_{B_1}|\nabla\phi+\xi|^2)^2\rangle.$$

An application of Lemma 4 with $p=2$ and of Proposition 1 with $p=1$ yields (50).

Step 4. Conclusion:

$$\begin{aligned} & \langle(\xi\cdot a_{hom}\xi-\langle\xi\cdot a_{hom}\xi\rangle)^2\rangle \\ \overset{(47)}{\lesssim}\ & \langle\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d}(\mathrm{osc}_{B_R(z)}\xi\cdot a_{hom}\xi)^2\rangle \\ \overset{(48)}{\lesssim}\ & L^{-2d}\langle\sum_{z\in\mathbb{Z}^d\cap[-\frac{L}{2},\frac{L}{2})^d}(\int_{B_R(z)}|\nabla\phi+\xi|^2)^2\rangle \\ \overset{(50)}{\lesssim}\ & L^{-d}. \end{aligned}$$

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