Reduced basis method for viscous flows in complex parametrized systems: applications to inverse problems and optimal control

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SISSA Mathlab

EPFL - MATHICSE - CMCS

CEMRACS 2013

Modelling and simulation of complex systems: stochastic and deterministic approaches

CIRM, Marseille, France. July 22 - 26, 2013.

Sponsors: Swiss National Science Foundation, SISSA excellence grant NOFYSAS and ERC-AdG MATHCARD.

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4. **Applications and results**
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    - Application to a surface reconstruction problem in haemodynamics (L1)
    - Stokes constraint: a numerical test (L2) and a data assimilation problem for blood flows (L3)
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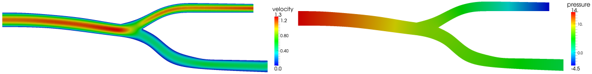
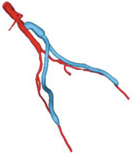
    - Brezzi-Rappaz-Raviart theory to obtain error bounds
    - Benchmark test: vorticity minimization (NL1)
    - Bypass graft design via boundary optimal control (NL2)

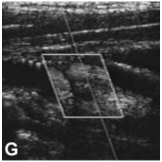
Complexity in haemodynamics



- The main obstacle to make mathematical models extensively useful and reliable in the clinical context is that they have to be personalized

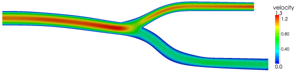
- Many quantities required by the numerical simulations cannot be always obtained through direct measurements and thus need to be estimated using the available clinical measurements





- The ultimate goal would be to optimize the therapeutic intervention depending on the patient attributes

Complexity in haemodynamics



- **Parametrized Simulation Problems**

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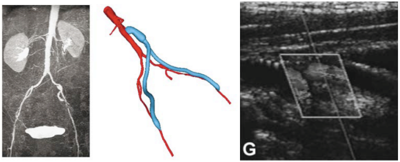

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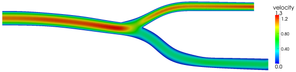
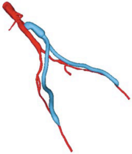
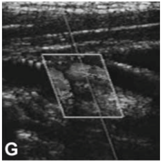


- **Parametrized Simulation Problems**

- **Parametrized Data assimilation and Inverse Problems**





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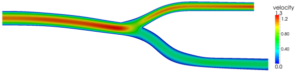

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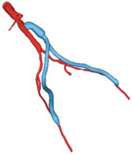
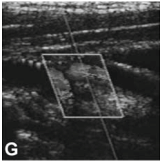
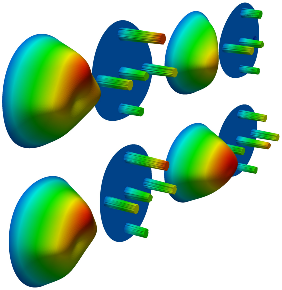
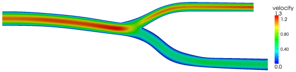




- **Parametrized Optimization Problems**

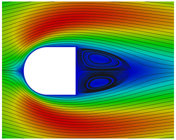
Models and problems

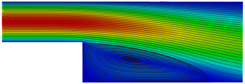




- surface reconstruction of blood flow profiles
- inverse problems: reconstruction of boundary conditions by experimental measures/observations

- flow control: vorticity reduction by suction/injection of fluid through the boundary





Steady state system: advection-diffusion, Stokes or Navier-Stokes equations

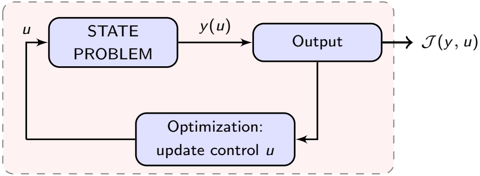
Control variables: distributed in the domain or along the boundary

Parameters: they can be physical/geometrical quantities describing the state system or related to observation measurements in the cost functional

Optimal control problems [Lions, 1971]

In general, an optimal control problem (OCP) consists of:

- a control function u, which can be seen as an input for the system,
- a controlled system, i.e. an input-output process: $\mathcal{E}(y, u) = 0$, being y the state variable
- an objective functional to be minimized: $\mathcal{J}(y, u)$



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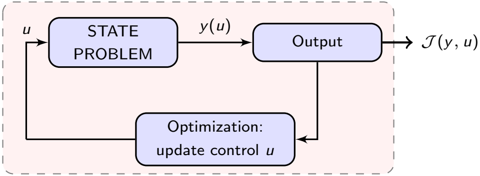

find the optimal control u^* and the state $y(u^*)$ such that the cost functional $\mathcal{J}(y, u)$ is minimized subject to $\mathcal{E}(y, u) = 0$ $\quad (OCP)$

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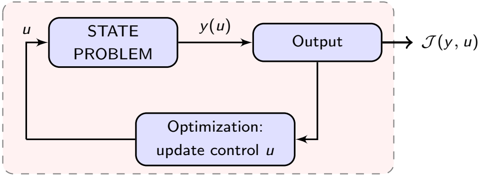


find the optimal control u^* *and the state* $y(u^*)$ *such that the cost functional* $\mathcal{J}(y, u)$ *is minimized subject to* $\mathcal{E}(y, u) = 0$ (OCP)

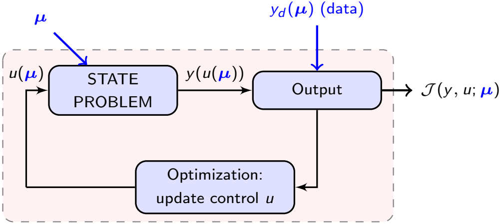
We restrict attention to:

- quadratic cost functionals, e.g. $\mathcal{J}(y, u) = \frac{1}{2}\|y - y_d\|^2 + \frac{\alpha}{2}\|u\|^2$

Parametrized optimal control problems

A parametrized optimal control problem ($OCP_{\boldsymbol{\mu}}$) consists of:

- a control function $u(\boldsymbol{\mu})$, which can be seen as an input for the system,
- a controlled system, i.e. an input-output process: $\mathcal{E}(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0$,
- an objective functional to be minimized: $\mathcal{J}(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}); \boldsymbol{\mu})$



given $\boldsymbol{\mu} \in \mathcal{D}$, find the optimal control $u^*(\boldsymbol{\mu})$ and the state $y^*(\boldsymbol{\mu})$ such that the cost functional $\mathcal{J}(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}); \boldsymbol{\mu})$ is minimized subject to $\mathcal{E}(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0$ $\quad (OCP_{\boldsymbol{\mu}})$

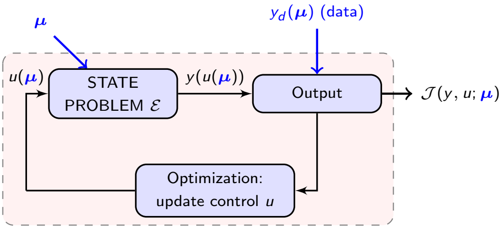
where $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^p$ denotes a p-vector whose components can represent:

- coefficients in boundary conditions
- geometrical configurations
- physical parametrization
- data (observation)

Reduction strategies for Parametrized Optimal Control Problems

PROBLEM: given $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^p$,

$$\min_{y,u} \ \mathcal{J}(y,u;\boldsymbol{\mu})$$
$$\text{s.t.} \quad \mathcal{E}(y,u;\boldsymbol{\mu}) = 0$$



The computational effort may be unacceptably high and, often, unaffordable when

- performing the optimization process for many different parameter values (**many-query** context)
- for a given new configuration, we want to compute the solution in a rapid way (**real-time** context)

Goal: to achieve the **accuracy** and **reliability** of a high fidelity approximation but at greatly **reduced cost** of a **low order model**

Main ingredients: **linear** state equation case

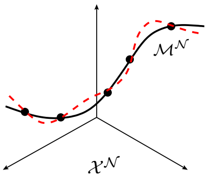
We build the Reduced Basis (RB) approximation directly on the optimality (KKT) system:

- we firstly recast the problem in the framework of **saddle-point** problem [Gunzburger & Bochev, 2004]
- we then apply the well-known **Brezzi-Babuška theory** [Brezzi & Fortin, 1991]

This way we can exploit the analogies with the already developed theory of RB method for Stokes-type problems [Rozza & Veroy, 2007] [Rozza *et al.*, n.d.] [Gerner & Veroy, 2012]

The usual ingredients of the RB methodology are provided:

- Galerkin projection onto a **low-dimensional space** of basis functions properly selected by a greedy algorithm for optimal parameters sampling;
- affine parametric dependence $\rightarrow$ **Offline-Online** computational procedure [EIM];
- an **efficient and rigorous a posteriori error estimation** on the solution variables as well as on the cost functional.



$$\mathcal{M}^{\mathcal{N}} = \{\mathrm{U}^{\mathcal{N}}(\boldsymbol{\mu}) \in \mathcal{X}^{\mathcal{N}} : \boldsymbol{\mu} \in \mathcal{D}\}$$
$$\mathcal{X}^{N} = \text{span}\{\mathrm{U}^{\mathcal{N}}(\boldsymbol{\mu}^i), i = 1, \dots, N\}$$

Main ingredients: **nonlinear** state equation (Navier-Stokes) case

Again, we work directly on the optimality system, in this case a **nonlinear** system of PDEs

- **Newton-SQP** method: **sequence of saddle-point problems** featuring the same structure of the optimality system in the linear case [Ito & Kunisch, 2008]
- we then apply the **Brezzi-Rappaz-Raviart theory** [Brezzi, Rappaz, Raviart, 1980]

This way we can exploit the analogies with the already developed theory of RB method for nonlinear equations (in particular Navier-Stokes) [Patera, Veroy, R., Deparis, Manzoni]

The usual ingredients of the RB methodology are provided:

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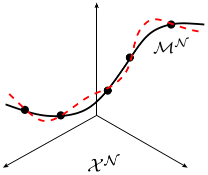

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Optimality system

Let $x = (y, u)$ be the optimization variable (state and control variables),

$$\text{given } \boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^p, \qquad \min_{x \in X} \mathcal{J}(x; \boldsymbol{\mu}) \qquad \text{s.t.} \qquad \mathcal{E}(x; \boldsymbol{\mu}) = 0 \text{ in } Q'$$

Lagrangian functional: $$\mathcal{L}(x, p; \boldsymbol{\mu}) = \mathcal{J}(x, \boldsymbol{\mu}) + \langle \mathcal{E}(x, \boldsymbol{\mu}), p \rangle,$$

By requiring the first derivatives to vanish we obtain the optimality (KKT) system

Optimality system

$$\begin{cases} \mathcal{J}_x(x; \boldsymbol{\mu}) + \mathcal{E}_x(x; \boldsymbol{\mu})^* p & = 0 \\ \mathcal{E}(x; \boldsymbol{\mu}) & = 0 \end{cases}$$

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Linear state equation: $\mathcal{E}(\cdot; \boldsymbol{\mu}) \colon X \to Q'$ is linear,

$$\text{let} \quad \mathcal{E}(x; \boldsymbol{\mu}) = B(\boldsymbol{\mu})x - g(\boldsymbol{\mu}) \quad \Longrightarrow \quad \mathcal{E}_x(x; \boldsymbol{\mu})^* = B^*(\boldsymbol{\mu}) \text{ independent of } x$$

$$\mathcal{J}(x; \boldsymbol{\mu}) = \frac{1}{2}\langle A(\boldsymbol{\mu})x, x \rangle - \langle f(\boldsymbol{\mu}), x \rangle \quad \Longrightarrow \quad \mathcal{J}_x(x; \boldsymbol{\mu}) = A(\boldsymbol{\mu})x - f(\boldsymbol{\mu})$$

Algebraic formulation:

$$\begin{pmatrix} A(\boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ B(\boldsymbol{\mu}) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}(\boldsymbol{\mu}) \\ \mathbf{p}(\boldsymbol{\mu}) \end{pmatrix} = \begin{pmatrix} \mathbf{F}(\boldsymbol{\mu}) \\ \mathbf{G}(\boldsymbol{\mu}) \end{pmatrix}$$

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$$\begin{cases} \mathcal{J}_x(x; \boldsymbol{\mu}) + \mathcal{E}_x(x; \boldsymbol{\mu})^* p & = 0 \\ \mathcal{E}(x; \boldsymbol{\mu}) & = 0 \end{cases}$$

Nonlinear state equation: $\mathcal{E}(\cdot; \boldsymbol{\mu}) : X \to Q'$ is nonlinear. Newton's method on the optimality system: for $k = 1, 2, \ldots$

solve for (s_x^k, s_p^k)
$$\begin{cases} \mathcal{L}_{xx}(x^k, p^k; \boldsymbol{\mu})\, s_x^k + \mathcal{E}_x(x^k; \boldsymbol{\mu})^* s_p^k & = -\mathcal{L}_x(x^k, p^k; \boldsymbol{\mu}) \\ \mathcal{E}_x(x^k; \boldsymbol{\mu})\, s_x^k & = -\mathcal{E}(x^k, \boldsymbol{\mu}) \end{cases}$$

update
$$x^{k+1} = x^k + s_x^k, \qquad p^{k+1} = p^k + s_p^k$$

Optimality system

Let $x = (y, u)$ be the optimization variable (state and control variables),

$$\text{given } \boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^p, \qquad \min_{x \in X} \mathcal{J}(x; \boldsymbol{\mu}) \qquad \text{s.t.} \qquad \mathcal{E}(x; \boldsymbol{\mu}) = 0 \text{ in } Q'$$

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solve for $(\mathbf{s}_x^k, \mathbf{s}_p^k)$

$$\begin{pmatrix} A^k(\boldsymbol{\mu}) & B^k(\boldsymbol{\mu})^T \\ B^k(\boldsymbol{\mu}) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{s}_x^k(\boldsymbol{\mu}) \\ \mathbf{s}_p^k(\boldsymbol{\mu}) \end{pmatrix} = \begin{pmatrix} \mathbf{F}^k(\boldsymbol{\mu}) \\ \mathbf{G}^k(\boldsymbol{\mu}) \end{pmatrix}$$

update

$$x^{k+1} = x^k + s_x^k, \qquad p^{k+1} = p^k + s_p^k$$

The abstract optimization problem

Notation: $y, z \in Y$ state space $\quad u, v \in U$ control space

$p, q \in Q\,(\equiv Y)$ adjoint space $\quad \mathcal{Z}$ observation space s.t. $Y \subset \mathcal{Z}$

Parametrized optimal control problem: given $\boldsymbol{\mu} \in \mathcal{D}$

$$\text{minimize} \quad J(y, u; \boldsymbol{\mu}) = \frac{1}{2} m(y - y_d(\boldsymbol{\mu}), y - y_d(\boldsymbol{\mu}); \boldsymbol{\mu}) + \frac{\alpha}{2} n(u, u; \boldsymbol{\mu})$$

$$\text{s.t.} \quad a(y, q; \boldsymbol{\mu}) = c(u, q; \boldsymbol{\mu}) + \langle G(\boldsymbol{\mu}), q \rangle \qquad \forall q \in Q.$$

The abstract optimization problem: saddle-point formulation

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$p, q \in Q \, (\equiv Y)$ adjoint space $\quad \mathcal{Z}$ observation space s.t. $Y \subset \mathcal{Z}$

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$$\text{s.t.} \quad a(y, q; \boldsymbol{\mu}) = c(u, q; \boldsymbol{\mu}) + \langle G(\boldsymbol{\mu}), q \rangle \qquad \forall q \in Q.$$

Let $X \equiv Y \times U$ be the state and control space, the constrained optimization problem can be recast in the form:

Saddle-point formulation: given $\boldsymbol{\mu} \in \mathcal{D}$

$$\begin{cases} \min \, \mathcal{J}(x; \boldsymbol{\mu}) = \frac{1}{2} \mathcal{A}(x, x; \boldsymbol{\mu}) - \langle F(\boldsymbol{\mu}), x \rangle, & \text{s.t.} \\ \mathcal{B}(x, q; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q \rangle \qquad \forall q \in Q. & \end{cases}$$

notation:

$x = (y, u) \in X$

$w = (z, v) \in X$

where

$$\mathcal{A}(x, w; \boldsymbol{\mu}) = m(y, z; \boldsymbol{\mu}) + \alpha n(u, v; \boldsymbol{\mu}), \qquad \langle F(\boldsymbol{\mu}), w \rangle = m(y_d(\boldsymbol{\mu}), z; \boldsymbol{\mu})$$

$$\mathcal{B}(w, q; \boldsymbol{\mu}) = a(z, q; \boldsymbol{\mu}) - c(v, q; \boldsymbol{\mu})$$

Saddle-point formulation: applying Brezzi theory

- the optimal control problem

$$\min_{x \in X} \mathcal{J}(x; \boldsymbol{\mu}) \quad \text{subject to} \quad \mathcal{B}(x, q; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q \rangle \quad \forall q \in Q.$$

  has a unique solution $x = (y, u) \in X$ for any $\boldsymbol{\mu} \in \mathcal{D}$

- that solution can be determined by solving the **optimality system**

$$\begin{cases} \mathcal{A}(x(\boldsymbol{\mu}), w; \boldsymbol{\mu}) + \mathcal{B}(w, p(\boldsymbol{\mu}); \boldsymbol{\mu}) & = \langle F(\boldsymbol{\mu}), w \rangle \qquad \forall w \in X, \\ \mathcal{B}(x(\boldsymbol{\mu}), q; \boldsymbol{\mu}) & = \langle G(\boldsymbol{\mu}), q \rangle \qquad \forall q \in Q, \end{cases}$$

Compact form

given $\boldsymbol{\mu} \in \mathcal{D}$, find $\mathrm{U}(\boldsymbol{\mu}) \in \mathcal{X}$ s.t:

$$\mathrm{B}(\mathrm{U}(\boldsymbol{\mu}), \mathrm{W}; \boldsymbol{\mu}) = \mathrm{F}(\mathrm{W}; \boldsymbol{\mu}) \quad \forall \mathrm{W} \in \mathcal{X}.$$

$$\mathcal{X} = X \times Q, \quad \mathrm{U} = (x, p), \quad \mathrm{W} = (w, q)$$

$$\mathrm{B}(\mathrm{U}, \mathrm{W}; \boldsymbol{\mu}) = \mathcal{A}(x, w; \boldsymbol{\mu}) + \mathcal{B}(w, p; \boldsymbol{\mu}) + \mathcal{B}(x, q; \boldsymbol{\mu})$$

$$\mathrm{F}(\mathrm{W}; \boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}), w \rangle + \langle G(\boldsymbol{\mu}), q \rangle$$

- at this point we may apply the Galerkin-FE approximation

Optimize - then - discretize

μ-OCP, optimality system

$\mathsf{Pb}(\boldsymbol{\mu}; \mathsf{U}(\boldsymbol{\mu}))$

$$\mathsf{U}(\boldsymbol{\mu}) \in \mathcal{X} : \quad \mathsf{B}(\mathsf{U}(\boldsymbol{\mu}), \mathsf{W}; \boldsymbol{\mu}) = \mathsf{F}(\mathsf{W}) \qquad \forall \mathsf{W} \in \mathcal{X}$$

Truth approximation (FEM)

$\mathsf{Pb}_{\mathcal{N}}(\boldsymbol{\mu}; \mathsf{U}^{\mathcal{N}}(\boldsymbol{\mu}))$

$$\mathsf{U}^{\mathcal{N}}(\boldsymbol{\mu}) \in X^{\mathcal{N}} : \quad \mathsf{B}(\mathsf{U}^{\mathcal{N}}(\boldsymbol{\mu}), \mathsf{W}; \boldsymbol{\mu}) = \mathsf{F}(\mathsf{W}) \quad \forall \mathsf{W} \in \mathcal{X}^{\mathcal{N}}$$

Optimize - then - discretize - then - **reduce** approach

μ-OCP, optimality system

$\mathrm{Pb}(\boldsymbol{\mu}; \mathrm{U}(\boldsymbol{\mu}))$

$$\mathrm{U}(\boldsymbol{\mu}) \in \mathcal{X} : \quad \mathrm{B}(\mathrm{U}(\boldsymbol{\mu}), \mathrm{W}; \boldsymbol{\mu}) = \mathrm{F}(\mathrm{W}) \qquad \forall \mathrm{W} \in \mathcal{X}$$

Truth approximation (FEM)

$\mathrm{Pb}_{\mathcal{N}}(\boldsymbol{\mu}; \mathrm{U}^{\mathcal{N}}(\boldsymbol{\mu}))$

$$\mathrm{U}^{\mathcal{N}}(\boldsymbol{\mu}) \in X^{\mathcal{N}} : \quad \mathrm{B}(\mathrm{U}^{\mathcal{N}}(\boldsymbol{\mu}), \mathrm{W}; \boldsymbol{\mu}) = \mathrm{F}(\mathrm{W}) \quad \forall \mathrm{W} \in \mathcal{X}^{\mathcal{N}}$$

Sampling (Greedy)

Space Construction

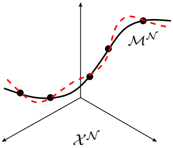
(Hierarchical Lagrange basis)

OFFLINE

$$S_N = \{\boldsymbol{\mu}^i, \quad i = 1, \dots, N\}$$

$$\mathcal{X}_N = \mathrm{span}\{U^{\mathcal{N}}(\boldsymbol{\mu}^i), \ \ i = 1, \dots, N\}$$

$$\dim(\mathcal{X}_N) = N \ll \mathcal{N} = \dim(\mathcal{X}^{\mathcal{N}})$$



Reduced Basis (RB) approximation

$\mathrm{Pb}_N(\boldsymbol{\mu}; \mathrm{U}_N(\boldsymbol{\mu}))$

Galerkin projection

ONLINE

$$\mathrm{U}_N(\boldsymbol{\mu}) \in \mathcal{X}_N : \quad \mathrm{B}(\mathrm{U}_N(\boldsymbol{\mu}), \mathrm{W}; \boldsymbol{\mu}) = \mathrm{F}(\mathrm{W}) \qquad \forall \mathrm{W} \in \mathcal{X}_N$$

[Patera, Rozza 2006] [Rozza *et al.*, 2008] (review)

Reduced Basis Method: approximation stability

Reduced Basis (RB) approximation: given $\boldsymbol{\mu} \in \mathcal{D}$, find $(x_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \times Q_N$:

$$\begin{cases} \mathcal{A}(x_N(\boldsymbol{\mu}), w; \boldsymbol{\mu}) + \mathcal{B}(w, p_N(\boldsymbol{\mu}); \boldsymbol{\mu}) & = \langle F(\boldsymbol{\mu}), w \rangle \qquad \forall w \in X_N \\ \mathcal{B}(x_N(\boldsymbol{\mu}), q; \boldsymbol{\mu}) & = \langle G(\boldsymbol{\mu}), q \rangle \qquad \forall q \in Q_N \end{cases} \qquad (*)$$

How to define the reduced basis spaces?

Reduced Basis Method: approximation stability

Reduced Basis (RB) approximation: given $\boldsymbol{\mu} \in \mathcal{D}$, find $(x_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \times Q_N$:

$$\begin{cases} \mathcal{A}(x_N(\boldsymbol{\mu}), w; \boldsymbol{\mu}) + \mathcal{B}(w, p_N(\boldsymbol{\mu}); \boldsymbol{\mu}) & = \langle F(\boldsymbol{\mu}), w \rangle \qquad \forall w \in X_N \\ \mathcal{B}(x_N(\boldsymbol{\mu}), q; \boldsymbol{\mu}) & = \langle G(\boldsymbol{\mu}), q \rangle \qquad \forall q \in Q_N \end{cases} \qquad (*)$$

How to define the reduced basis spaces? we have to provide a spaces pair $\{X_N, Q_N\}$ that guarantee the fulfillment of an equivalent parametrized Brezzi *inf-sup* condition [Negri *et al.*, 2012]

$$\beta_N(\boldsymbol{\mu}) = \inf_{q \in Q_N} \sup_{w \in X_N} \frac{\mathcal{B}(w, q; \boldsymbol{\mu})}{\|w\|_X \|q\|_Q} \geq \beta_0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

For the state and adjoint variables: aggregated spaces

$$Y_N \equiv Q_N = \text{span}\{y^{\mathcal{N}}(\boldsymbol{\mu}^n), \, p^{\mathcal{N}}(\boldsymbol{\mu}^n)\}_{n=1}^N$$

For the control variable:

$$W_N = \text{span}\{u^{\mathcal{N}}(\boldsymbol{\mu}^n)\}_{n=1}^N$$

Let $X_N = Y_N \times W_N$, we can prove that

- $\beta_N(\boldsymbol{\mu}) \geq \alpha^{\mathcal{N}}(\boldsymbol{\mu}) > 0$ being $\alpha^{\mathcal{N}}(\boldsymbol{\mu})$ the coercivity constant associated to the FE approximation of the PDE operator
- Brezzi theorem $\Longrightarrow$ for any $\boldsymbol{\mu} \in \mathcal{D}$, the RB approximation $(*)$ has a unique solution depending continuously on the data

RB method: Offline/Online decomposition

Algebraic formulation:

$$\underbrace{\begin{pmatrix} A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & 0 \end{pmatrix}}_{K_N(\boldsymbol{\mu})} \underbrace{\begin{pmatrix} \mathbf{x}_N(\boldsymbol{\mu}) \\ \mathbf{p}_N(\boldsymbol{\mu}) \end{pmatrix}}_{\mathbf{U}_N(\boldsymbol{\mu})} = \underbrace{\begin{pmatrix} \mathbf{F}_N(\boldsymbol{\mu}) \\ \mathbf{G}_N(\boldsymbol{\mu}) \end{pmatrix}}_{\mathbf{F}_N(\boldsymbol{\mu})}$$

affine decomposition:

$$K_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_b} \Theta_b^q(\boldsymbol{\mu}) K_N^q \qquad \mathbf{F}_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_f^q(\boldsymbol{\mu}) F_N^q$$

$$\sum_{q=1}^{Q_b} \Theta_b^q(\boldsymbol{\mu}) K_N^q \, \mathbf{U}_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_f^q(\boldsymbol{\mu}) F_N^q$$

- **Offline** pre-processing: compute and store the basis functions $\{\zeta_i, \; 1 \le i \le 5N\}$, store the matrices K_N^q and the vectors F_N^q

  Operation count: depends on N, Q_b, Q_f and $\mathcal{N}$

- **Online**: evaluate coefficients $\Theta_*^q(\boldsymbol{\mu})$, assemble the matrix $K_N(\boldsymbol{\mu})$ and the vector $\mathbf{F}_N(\boldsymbol{\mu})$ and solve the reduced system of **dimension** $5N \times 5N$

  Operation count: $O((5N)^3 + Q_b N^2 + Q_f N)$ independent of $\mathcal{N}$, $N \ll \mathcal{N}$

RB Method: a posteriori error estimation

Goal: provide rigorous, sharp and inexpensive estimators for the error on the solution variables and for the error on the cost functional

A posteriori error estimation on the solution variables

$$\|\mathsf{U}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{U}_N(\boldsymbol{\mu})\|_{\mathcal{X}} \leq \frac{\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'}}{\hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu})} := \Delta_N(\boldsymbol{\mu})$$

- $0 < \hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu}) \leq \hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ is a constructible *lower bound* of the Babuška inf-sup constant

  $$\hat{\beta}(\boldsymbol{\mu}) = \inf_{\mathsf{W}\in\mathcal{X}} \sup_{\mathsf{U}\in\mathcal{X}} \frac{\mathsf{B}(\mathsf{U},\mathsf{W};\boldsymbol{\mu})}{\|\mathsf{U}\|_{\mathcal{X}}\|\mathsf{W}\|_{\mathcal{X}}} \geq \hat{\beta}_0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}$$

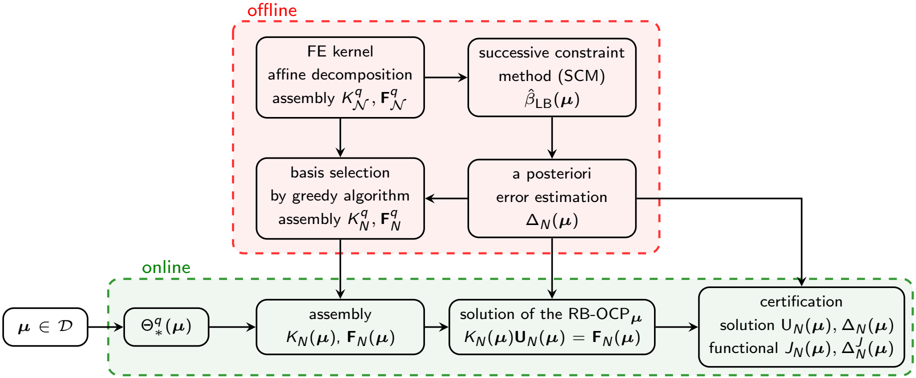
  given by the *successive constraint method* (SCM) (or by an *interpolant surrogate*); Offline/Online strategy

- residual of the optimality system: $\mathsf{r}(\mathsf{W};\boldsymbol{\mu}) = \mathsf{F}(\mathsf{W};\boldsymbol{\mu}) - \mathsf{B}(\mathsf{U}_N,\mathsf{W};\boldsymbol{\mu})$; we can provide the standard Offline/Online stratagem for the efficient computation of $\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'}$;

A posteriori error estimation on the cost functional

$$|\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu})| \leq \frac{1}{2}\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'}\|\mathsf{U}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{U}_N(\boldsymbol{\mu})\|_{\mathcal{X}} \leq \frac{1}{2}\frac{\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|^2_{\mathcal{X}'}}{\hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu})} := \Delta_N^J(\boldsymbol{\mu}).$$

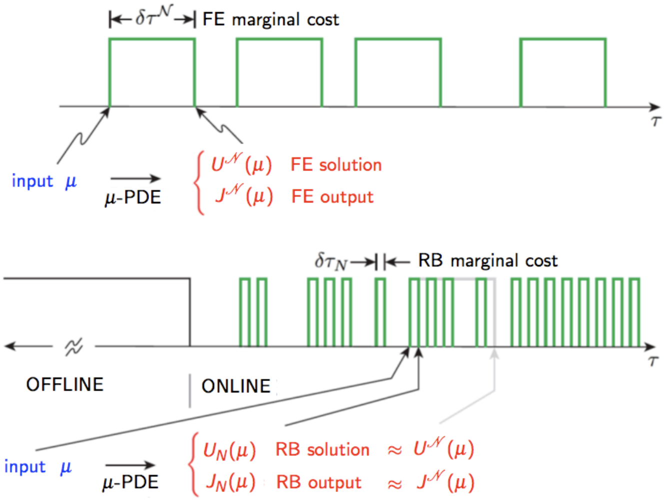
RB Method: the "complete game"



- Offline stage involves precomputation of FE structures required for the RB space construction and the certified error estimates.
- Online stage has complexity only depending on N and allows resolution of the Optimal Control Problem for any $\boldsymbol{\mu} \in \mathcal{D}$ with a certified error bound.

Implementation in MATLAB using `MLife` and `rbMIT` libraries.

Computational reduction



http://augustine.mit.edu

Geometrical Parametrization

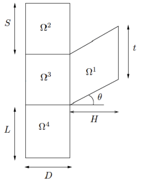
- ✓ RB framework requires a geometrical map $T(\cdot;\mu) : \Omega \to \Omega_o(\mu)$ in order to combine discretized solutions for the space construction
- ✓ This procedure enables to avoid shape deformation and remeshing (that, e.g. normally occur at each step of an iterative optimization procedure)
- ✓ Reduction in the complexity of parametrization: versatility, low-dimensionality, automatic generation of maps, capability to represent realistic configurations, ...

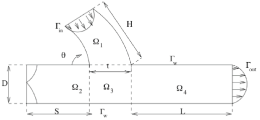
Left: Different carotid bifurcation specimens obtained by autopsy (adults aged 30-75); picture taken from Z. Ding et al., Journal of Biomechanics 34 (2001),1555-1562.
Right: Different carotid bifurcation obtained through radial basis functions techniques.

Shape Parametrization Techniques

Cartesian geometries:

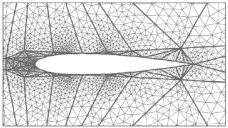
- Affine/nonaffine mapping "by hands"

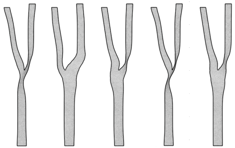


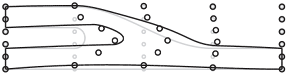


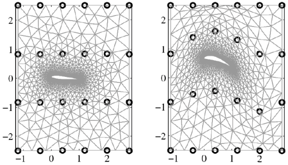
Complex realistic geometries:

- Automatic affine transformation (DD) `rbMIT`
- Free-shape nonaffine transformations based on control points (e.g. Free-Form Deformation [Sederberg & Parry], Radial Basis Functions [Bookstein, Buhmann])
- Transfinite Mappings [Gordon, Hall]

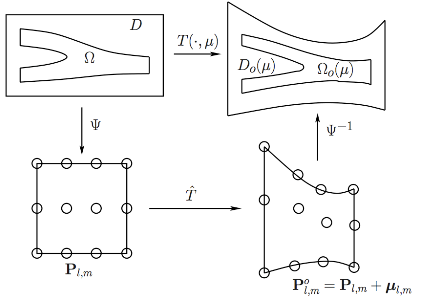


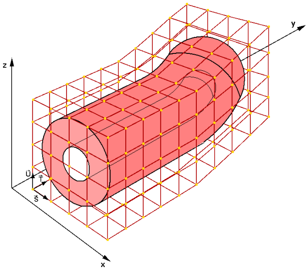






Free-Form Deformation (FFD) Techniques





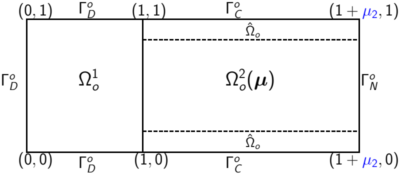
Construction:

- Parametric map: $T(\mathbf{x}, \boldsymbol{\mu}) = \sum_{l=0}^{L} \sum_{m=0}^{M} b_{l,m}^{L,M}(\Psi(\mathbf{x}))(\mathbf{P}_{l,m} + \boldsymbol{\mu}_{l,m})$ where

$$b_{\ell,m}^{L,M}(s,t) = b_\ell^L(s) b_m^M(t) = \binom{L}{\ell}\binom{M}{m}(1-s)^{L-\ell}s^\ell(1-t)^{M-m}t^m$$

  are tensor products of Bernstein basis polynomials
- FFD mapping defined as $\Omega_o(\boldsymbol{\mu}) = \Psi^{-1} \circ \hat{T} \circ \Psi(\Omega; \boldsymbol{\mu}) =: T(\Omega; \boldsymbol{\mu})$
- Parameters $\mu_1, \ldots, \mu_P$ are displacements of selected control points

L0 - Boundary control for a Graetz convection-diffusion problem



observation function: $y_d(\boldsymbol{\mu}) = \mu_3 \chi_{\hat{\Omega}_o}$

parameter domain:

$$\mathcal{D} = [6, 20] \times [1, 3] \times [0.5, 3]$$

We consider the following optimal control problem:

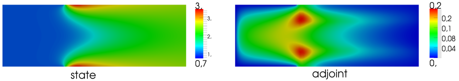
$$\text{minimize } J(y_o(\boldsymbol{\mu}), u_o(\boldsymbol{\mu}); \boldsymbol{\mu}) = \frac{1}{2}\|y_o(\boldsymbol{\mu}) - y_d(\boldsymbol{\mu})\|^2_{L^2(\hat{\Omega}_o)} + \frac{\alpha}{2}\|u_o(\boldsymbol{\mu})\|^2_{L^2(\Gamma_C^o)}$$

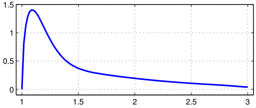
$$\text{s.t.} \begin{cases} -\dfrac{1}{\mu_1}\Delta y_o(\boldsymbol{\mu}) + x_{o2}(1 - x_{o2})\dfrac{\partial y_o(\boldsymbol{\mu})}{\partial x_{o1}} = 0 & \text{in } \Omega_o(\boldsymbol{\mu}) \\ y_o(\boldsymbol{\mu}) = 1 & \text{on } \Gamma_D^o \\ \dfrac{1}{\mu_1}\nabla y_o(\boldsymbol{\mu}) \cdot \mathbf{n} = u_o(\boldsymbol{\mu}) & \text{on } \Gamma_C^o(\boldsymbol{\mu}) \\ \dfrac{1}{\mu_1}\nabla y_o(\boldsymbol{\mu}) \cdot \mathbf{n} = 0 & \text{on } \Gamma_N^o(\boldsymbol{\mu}), \end{cases}$$

- the problem is mapped to a reference domain $\Omega = \Omega_o(\boldsymbol{\mu}_{\text{ref}})$ with $\boldsymbol{\mu}_{\text{ref}} = (\cdot, 1, \cdot)$
- we obtain an affine decomposition with $Q_B = 6$, $Q_F = 5$

Boundary control for a Graetz convection-diffusion problem

Representative solution for $\boldsymbol{\mu} = (12, 2, 2.5)$

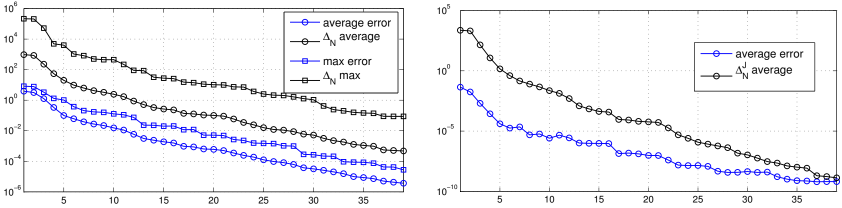




optimal control u_N on Γ_C

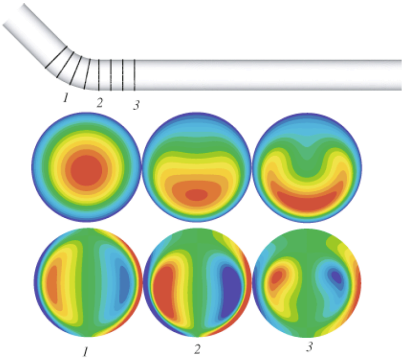
Number of FE dof $\mathcal{N}$	8915
Number of parameters P	3
Number of RB functions N	39
Dimension of RB linear system	$39 \cdot 5$
Affine operator components Q	6

Linear system dimension reduction	50:1
FE evaluation t_{FE} (s)	14.5
RB evaluation t_{RB}^{online} (s)	0.1
RB evaluation $t_{RB}^{offline}$ (s)	3970



Error estimation (●) and true error (●) for the solution (left) and the cost functional (right)

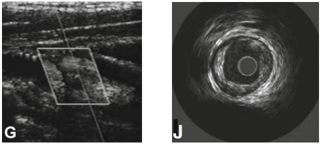
Towards reduced data reconstruction/assimilation



Sectional axial flow profile (top) and vorticity (bottom) and salient locations along a bend.
Picture taken from D. Doorly and S. Sherwin, Geometry and flow,
In Cardiovascular Mathematics, L. Formaggia, A. Quarteroni and A. Veneziani (Eds.)

L1 - Reduced data reconstruction/assimilation

- goal: to reconstruct, from areal data provided by eco-dopplers measurements, the blood velocity field in a section of a carotid artery
- surface estimation starting from scattered data: the reconstruction should take into account the shape of the domain and preserve the no-slip condition

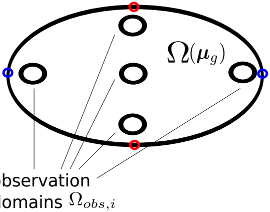


Duplex US image of a carotid artery bifurcation
Intravascular US image of a coronary artery (cross-section)

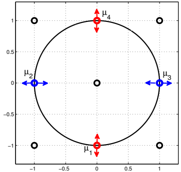
Surface estimation problem [Azzimonti *et al.*, 2011]

$$\min_{y,u} \; J(y,u;\boldsymbol{\mu}) = \sum_{i=1}^{m} \int_{\Omega_{obs,i}} |y(\boldsymbol{\mu}) - z_i|^2 d\Omega + \frac{\alpha}{2} \|u(\boldsymbol{\mu})\|^2_{L^2}$$

$$\text{s.t.} \quad \begin{cases} -\Delta y(\boldsymbol{\mu}) = u(\boldsymbol{\mu}) & \text{in } \Omega(\boldsymbol{\mu}_g) \\ y(\boldsymbol{\mu}) = 0 & \text{on } \partial\Omega(\boldsymbol{\mu}_g) \end{cases}$$



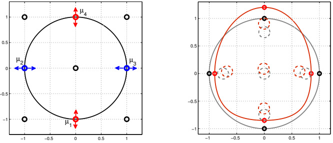
- Geometrical parametrization: Free Form Deformation
  $P = 4$ displacements of the control points • •,
  $\boldsymbol{\mu}_g \in (-0.15, 0.15)^4$ [Manzoni, Phd thesis]
- Parametrized observation values: $\mu^i_{obs} = z_i$, $1 \le i \le m = 5$



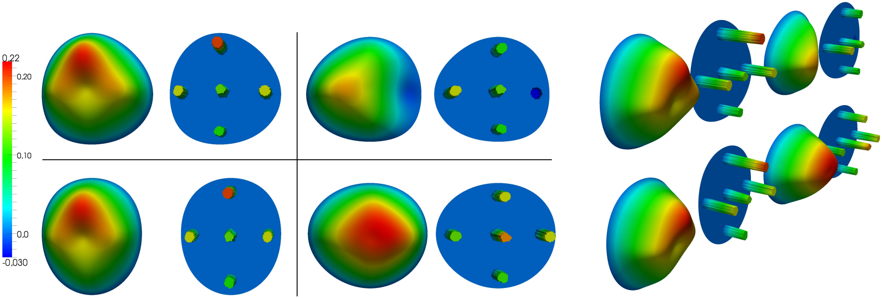
L1 - Reduced data reconstruction/assimilation [Rozza *et al.*, 2012, ECCOMAS]

Number of FE dof $\mathcal{N}$	$3.3 \cdot 10^4$
Regularization parameter α	10^{-4}
Number of parameters P	$4 + 5$
Number of RB functions N	42
Affine components Q_B	53
Linear system dimension red.	160:1
RB solution t_{RB}^{online}(s)	0.013
RB certification t_{Δ}^{online}(s)	0.98

To fulfill the affine parametric dependence assumption we rely on the Empirical Interpolation Method [Barrault *et al*, 2004]



Example of reconstructed profiles given different sets of (virtual) observation values:



Stokes constraint: how to extend the method

minimize $J(\mathbf{v}, \pi, \mathbf{u}; \boldsymbol{\mu}) = \frac{1}{2} m(\mathbf{v} - \mathbf{v}_d(\boldsymbol{\mu}), \mathbf{v} - \mathbf{v}_d(\boldsymbol{\mu}); \boldsymbol{\mu}) + \frac{\alpha}{2} n(\mathbf{u}, \mathbf{u}; \boldsymbol{\mu})$ subject to

$$\begin{cases} a(\mathbf{v}, \boldsymbol{\xi}; \boldsymbol{\mu}) + b(\boldsymbol{\xi}, \pi; \boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}), \boldsymbol{\xi} \rangle + c(\mathbf{u}, \boldsymbol{\xi}; \boldsymbol{\mu}) & \forall \boldsymbol{\xi} \in V, \\ b(\mathbf{v}, \tau; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), \tau \rangle & \forall \tau \in M, \end{cases}$$

Functional setting: $V = [H^1(\Omega)]^2$ $M = L^2(\Omega)$ velocity and pressure spaces

$Y = V \times M$ state space, $Q \equiv Y$ adjoint space, U control space

- two nested saddle-point
  - outer: optimal control
  - inner: Stokes constraint
- reduced basis functions computed by solving N times the FE approximation (with stable spaces pair for velocity and pressure variables)
- stability of the RB approximation of the Stokes constraint fulfilled by introducing suitable **supremizer operators** [Rozza & Veroy, 2007; Rozza *et al.*, n.d.]
- stability of the RB approximation of the whole optimal control problem fulfilled by defining suitable **aggregated spaces** for the state and adjoint variables [Negri *et al.*, 2013]

Stokes constraint: how to extend the method

minimize $J(\mathbf{v}, \pi, \mathbf{u}; \boldsymbol{\mu}) = \frac{1}{2} m(\mathbf{v} - \mathbf{v}_d(\boldsymbol{\mu}), \mathbf{v} - \mathbf{v}_d(\boldsymbol{\mu}); \boldsymbol{\mu}) + \frac{\alpha}{2} n(\mathbf{u}, \mathbf{u}; \boldsymbol{\mu})$ subject to

$$\begin{cases} a(\mathbf{v}, \boldsymbol{\xi}; \boldsymbol{\mu}) + b(\boldsymbol{\xi}, \pi; \boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}), \boldsymbol{\xi} \rangle + c(\mathbf{u}, \boldsymbol{\xi}; \boldsymbol{\mu}) & \forall \boldsymbol{\xi} \in V, \\ b(\mathbf{v}, \tau; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), \tau \rangle & \forall \tau \in M, \end{cases}$$

Functional setting: $V = [H^1(\Omega)]^2$ $M = L^2(\Omega)$ velocity and pressure spaces

$Y = V \times M$ state space, $Q \equiv Y$ adjoint space, U control space

Reminder: enrichment by supremizers operators for the Stokes equations

$$M_N = \text{span}\{\pi^{\mathcal{N}}(\boldsymbol{\mu}^n),\ n = 1, \dots, N\}, \qquad \text{pressure}$$

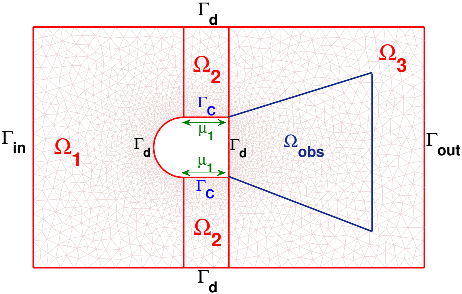
$$V_N^{\boldsymbol{\mu}} = \text{span}\{\mathbf{v}^{\mathcal{N}}(\boldsymbol{\mu}^n),\ T^{\boldsymbol{\mu}}(\pi^{\mathcal{N}}(\boldsymbol{\mu}^n)),\ n = 1, \dots, N\}, \qquad \text{velocity}$$

being $T^{\boldsymbol{\mu}} : M \to V$ the supremizer operator s.t.

$$(T^{\boldsymbol{\mu}} q, \mathbf{w})_V = b(q, \mathbf{w}; \boldsymbol{\mu}) \qquad \forall\ \mathbf{w} \in V,$$

so that $\{V_N^{\boldsymbol{\mu}}, M_N\}$ fulfill an equivalent RB Brezzi *inf-sup* stability condition [R., Veroy, et al.]

L2 - Vorticity minimization on the downstream portion of a bluff body



GOAL: minimize the vorticity in the wake of the body through suction/injection of fluid on the control boundary Γ_C

The state velocity and pressure variables $\{\mathbf{v}, \pi\}$ satisfy the Stokes equations in $\Omega(\boldsymbol{\mu}_1)$ with the following boundary conditions:

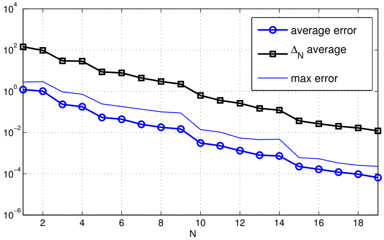
$$\begin{aligned} \mathbf{v} &= 0 && \text{on } \Gamma_D(\boldsymbol{\mu}_1), \\ \mathbf{v} &= \mathbf{g}(\boldsymbol{\mu}_2) && \text{on } \Gamma_{in}, \\ -\pi \mathbf{n} + \nu \nabla \mathbf{v}\, \mathbf{n} &= \mathbf{0} && \text{on } \Gamma_{out}(\boldsymbol{\mu}_1), \end{aligned} \qquad \begin{aligned} v_1 &= 0 && \text{on } \Gamma_C, \\ v_2 &= u && \text{on } \Gamma_C, \end{aligned}$$

where $\mathbf{g}(\boldsymbol{\mu}_2)$ is a parabolic inflow profile with peak velocity equal to $\boldsymbol{\mu}_2$.

The cost functional is given by:

$$\mathcal{J}(\mathbf{v}(\boldsymbol{\mu}), \mathbf{u}(\boldsymbol{\mu}); \boldsymbol{\mu}) = \frac{1}{2} \int_{\Omega_{obs}} |\nabla \times \mathbf{v}(\boldsymbol{\mu})|^2 \, d\Omega + \frac{\boldsymbol{\mu}_3}{2} \|u(\boldsymbol{\mu})\|^2_{H^1(\Gamma_C)}$$

L2 - Vorticity minimization on the downstream portion of a bluff body



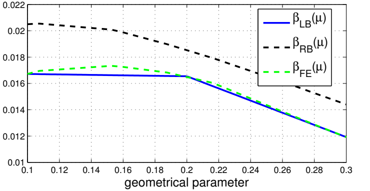
Average computed error and bound between the *truth* FE solution and the RB approximation.

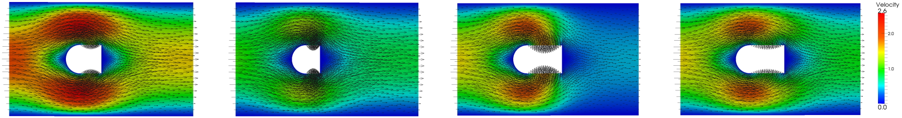
$\mu_1 \in [0.1, 0.3]$ $\mu_2 \in [0.5, 2]$ $\mu_3^{-1} \in [1, 200]$

Number of FE dof $\mathcal{N}$	$3.6 \cdot 10^4$
Number of parameters P	3
Number of RB functions N	19
Dimension of RB linear system	$19 \cdot 13$
Affine operator components Q	14

Linear system dim reduction	150:1
FE evaluation t_{FE} (s)	≈ 15
RB evaluation t_{RB}^{online} (s)	0.1

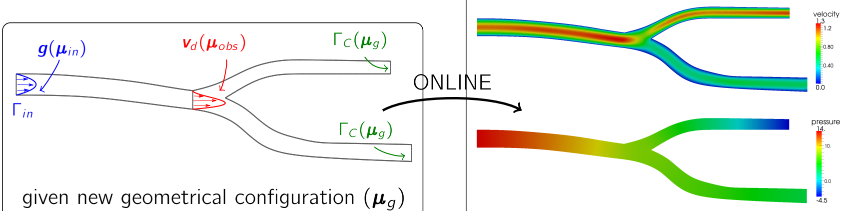
Stability factor: Babuška inf-sup w.r.t. to μ_1





L3 - An (idealized) application in haemodynamics: a data assimilation problem

- we consider an inverse boundary problem in hemodynamics, inspired by the work [D'Elia *et. al*, 2011]
- parametrized geometrical model of an arterial bifurcation (with FFD)
- we suppose to have a measured velocity profile on the red section, but not the Neumann flux on Γ_C that will be our control variable
- starting from the velocity measures we want to find the control variable in order to retrieve the velocity and pressure fields in the whole domain.



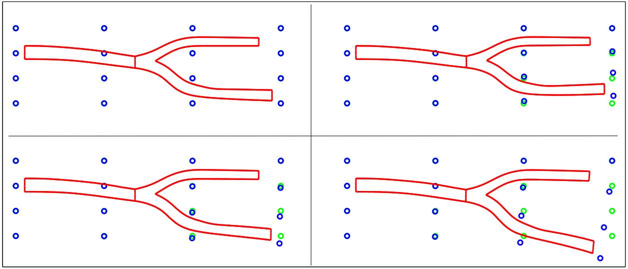
given new geometrical configuration $(\boldsymbol{\mu}_g)$ and parametrized measurements $\boldsymbol{\mu}_{obs}$ on the red section

find the unknown Neumann boundary condition on Γ_C and retrieve the whole velocity and pressure fields

An (idealized) application in haemodynamics: a data assimilation problem

Free Form Deformation

the geometrical parameter $\boldsymbol{\mu}_g$ is related to the angle of rotation of the lower branch



The state velocity and pressure variables $\{\mathbf{v}, \pi\}$ satisfy the following Stokes problem in $\Omega(\boldsymbol{\mu})$:

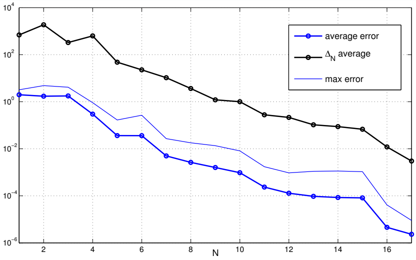
$$
\begin{aligned}
-\nu\Delta\mathbf{v} + \nabla\pi &= 0 && \text{in } \Omega(\boldsymbol{\mu}_g), \\
\operatorname{div}\mathbf{v} &= 0 && \text{in } \Omega(\boldsymbol{\mu}_g), \\
\mathbf{v} &= 0 && \text{on } \Gamma_D(\boldsymbol{\mu}_g),
\end{aligned}
\qquad
\begin{aligned}
\mathbf{v} &= \mathbf{g}(\boldsymbol{\mu}_{in}) && \text{on } \Gamma_{in}, \\
-\pi\mathbf{n} + \nu\frac{\partial\mathbf{v}}{\partial\mathbf{n}} &= \mathbf{u} && \text{on } \Gamma_C(\boldsymbol{\mu}_g),
\end{aligned}
$$

where $\mathbf{g}(\boldsymbol{\mu}_{in})$ is a parabolic inflow profile.

Then we consider the following parametrized cost functional to be minimized

$$
\mathcal{J}(\mathbf{v}, \pi, \mathbf{u}; \boldsymbol{\mu}) = \frac{1}{2}\int_{\Gamma_{obs}} |\mathbf{v} - \mathbf{v}_d(\boldsymbol{\mu}_{obs})|^2 \, d\Gamma + \text{regularization}(\mathbf{u})
$$

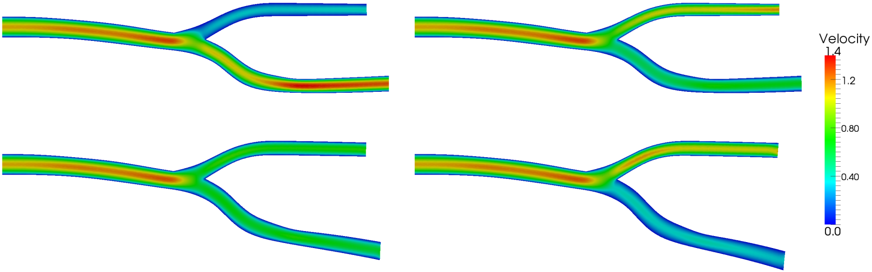
L3 - An (idealized) application in haemodynamics: a data assimilation problem



Average computed error and bound between the *truth* FE solution and the RB approximation.

Number of FE dof $\mathcal{N}$	$4 \cdot 10^4$
Number of parameters P	3
Number of RB functions N	17
Dimension of RB linear system	$17 \cdot 13$
Affine operator components Q	20

FE evaluation t_{FE} (s)	≈ 20
RB evaluation t_{RB}^{online} (s)	0.15



Boundary control of Navier-Stokes flow

Find $(\mathbf{v}, \pi, \boldsymbol{\mu})$ such that the cost functional

$$\mathcal{J}(\mathbf{v}, \pi, \mathbf{u}; \boldsymbol{\mu}) = \mathcal{F}(\mathbf{v}, \pi; \boldsymbol{\mu}) + \mathcal{G}(\mathbf{u}; \boldsymbol{\mu})$$

is minimized subject to the steady Navier-Stokes equations:

$$\begin{aligned}
-\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\pi &= \mathbf{f} && \text{in } \Omega(\boldsymbol{\mu})\\
\operatorname{div}\mathbf{v} &= 0 && \text{in } \Omega(\boldsymbol{\mu})\\
\mathbf{v} &= \mathbf{u} && \text{on } \Gamma_C(\boldsymbol{\mu})\\
\mathbf{v} &= \mathbf{0} && \text{on } \Gamma_D(\boldsymbol{\mu})\\
-\pi\mathbf{n} + \nu\nabla\mathbf{v}\cdot\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N(\boldsymbol{\mu}).
\end{aligned}$$

Possible choices for $\mathcal{F}$, viscous energy dissipation or velocity tracking type functionals:

$$\mathcal{F}(\mathbf{v}, \pi; \boldsymbol{\mu}) = \frac{\nu}{2}\int_{\Omega(\boldsymbol{\mu})} |\nabla\mathbf{v}|^2\, d\Omega, \qquad \mathcal{F}(\mathbf{v}, \pi; \boldsymbol{\mu}) = \frac{1}{2}\int_{\Omega_{\text{obs}}(\boldsymbol{\mu})} |\mathbf{v} - \mathbf{v}_d(\boldsymbol{\mu})|^2\, d\Omega$$

Regularization contribute:
$$\mathcal{G}(\mathbf{u}; \boldsymbol{\mu}) = \frac{\alpha}{2}\int_{\Gamma_C(\boldsymbol{\mu})} (|\nabla\mathbf{u}|^2 + |\mathbf{u}|^2)\, d\Gamma$$

[Gunzburger *et al.*, 1991], [Hou & Ravindran, 1999], [Biros & Ghattas, 1999, 2005]

Boundary control of Navier-Stokes flow

Find $(\mathbf{v}, \pi, \boldsymbol{\mu})$ such that the cost functional

$$\mathcal{J}(\mathbf{v}, \pi, \mathbf{u}; \boldsymbol{\mu}) = \mathcal{F}(\mathbf{v}, \pi; \boldsymbol{\mu}) + \mathcal{G}(\mathbf{u}; \boldsymbol{\mu})$$

is minimized subject to the steady Navier-Stokes equations:

$$\begin{aligned} -\nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla \pi &= \mathbf{f} && \text{in } \Omega(\boldsymbol{\mu}) \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega(\boldsymbol{\mu}) \\ \mathbf{v} &= \mathbf{u} && \text{on } \Gamma_C(\boldsymbol{\mu}) \\ \mathbf{v} &= \mathbf{0} && \text{on } \Gamma_D(\boldsymbol{\mu}) \\ -\pi \mathbf{n} + \nu \nabla \mathbf{v} \cdot \mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N(\boldsymbol{\mu}). \end{aligned}$$

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Regularization contribute:

$$\mathcal{G}(\mathbf{u}; \boldsymbol{\mu}) = \frac{\alpha}{2} \int_{\Gamma_C(\boldsymbol{\mu})} (|\nabla \mathbf{u}|^2 + |\mathbf{u}|^2) d\Gamma$$

[Gunzburger *et al.*, 1991], [Hou & Ravindran, 1999], [Biros & Ghattas, 1999, 2005]

Boundary control of Navier-Stokes flow: optimality system **quadratic nonlinearity**

State equation

$$-\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\pi = \mathbf{f}$$

$$\operatorname{div}\mathbf{v} = 0$$

$$\mathbf{v} = \mathbf{u} \quad \text{on } \Gamma_C \; + \text{other BCs}$$

Adjoint equation

$$-\nu\Delta\boldsymbol{\lambda} + (\nabla\mathbf{v})^T\boldsymbol{\lambda} - (\mathbf{v}\cdot\nabla)\boldsymbol{\lambda} + \nabla\eta = \nu\Delta\mathbf{v}$$

$$\operatorname{div}\boldsymbol{\lambda} = 0$$

$$\boldsymbol{\lambda} = \mathbf{0} \quad \text{on } \Gamma_C \; + \text{other BCs}$$

Optimality equation

$$-\alpha(\Delta_{\Gamma_C}\mathbf{u} + \mathbf{u}) = \eta\mathbf{n} - \nu(\nabla\boldsymbol{\lambda} + \nabla\mathbf{v})\cdot\mathbf{n} \qquad \text{on } \Gamma_C$$

Boundary control of Navier-Stokes flow: optimality system **quadratic nonlinearity**

State equation

$$-\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\pi = \mathbf{f}$$
$$\operatorname{div}\mathbf{v} = 0$$
$$\mathbf{v} = \mathbf{u} \quad \text{on } \Gamma_C \; + \text{other BCs}$$

Adjoint equation

$$-\nu\Delta\boldsymbol{\lambda} + (\nabla\mathbf{v})^T\boldsymbol{\lambda} - (\mathbf{v}\cdot\nabla)\boldsymbol{\lambda} + \nabla\eta = \nu\Delta\mathbf{v}$$
$$\operatorname{div}\boldsymbol{\lambda} = 0$$
$$\boldsymbol{\lambda} = \mathbf{0} \quad \text{on } \Gamma_C \; + \text{other BCs}$$

Optimality equation

$$-\alpha(\Delta_{\Gamma_C}\mathbf{u} + \mathbf{u}) = \eta\mathbf{n} - \nu(\nabla\boldsymbol{\lambda} + \nabla\mathbf{v})\cdot\mathbf{n} \qquad \text{on } \Gamma_C$$

- Variational formulation: find $U = (\mathbf{v}, \pi; \mathbf{u}; \boldsymbol{\lambda}, \eta) \in \mathcal{X}$ s.t.

$$G(U, W; \boldsymbol{\mu}) = 0 \qquad \forall W \in \mathcal{X},$$

- Newton method: for $k = 1, 2, \ldots$

$$dG[U^k](U^{k+1}, W; \boldsymbol{\mu}) = -G(U^k, W; \boldsymbol{\mu}) \qquad \forall W \in \mathcal{X}$$

where $dG[U](V, W; \boldsymbol{\mu})$ denotes the Fréchet derivative of $G(\cdot, \cdot; \boldsymbol{\mu})$

RB approximation and BRR error bound

As in the Stokes case:

- reduced basis functions computed by solving N times the FE approximation
- stability of the RB approximation: **supremizer operators** + **aggregated spaces** for the state and adjoint variables

Nonlinear ingredients:

- Galerkin projection on $\mathcal{X}_N$ + Newton method: for $k = 1, 2, \dots$ until convergence

$$dG[U_N^k](U_N^{k+1}, W_N; \boldsymbol{\mu}) = -G(U_N^k, W_N; \boldsymbol{\mu}) \qquad \forall W_N \in \mathcal{X}_N$$

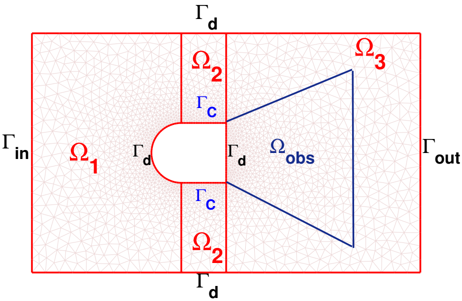
- Brezzi-Rappaz-Raviart error bound:

$$\text{if} \quad \tau_N(\boldsymbol{\mu}) = 4\frac{\gamma(\boldsymbol{\mu})\varepsilon_N(\boldsymbol{\mu})}{\hat{\beta}^2(\boldsymbol{\mu})} < 1 \qquad \text{where} \quad \varepsilon_N(\boldsymbol{\mu}) = \|G(U_N, \cdot; \boldsymbol{\mu})\|_{\mathcal{X}'_{\mathcal{N}}}$$

then

$$\|U^{\mathcal{N}}(\boldsymbol{\mu}) - U_N(\boldsymbol{\mu})\|_{\mathcal{X}} \le \Delta_N(\boldsymbol{\mu}) := \frac{\hat{\beta}(\boldsymbol{\mu})}{2\gamma(\boldsymbol{\mu})}\left(1 - \sqrt{1 - \tau_N(\boldsymbol{\mu})}\right)$$

NL1 - Vorticity minimization on the downstream portion of a bluff body



GOAL: minimize the vorticity in the wake of the body through suction/injection of fluid on the control boundary Γ_C

$$\mu_1^{-1} \in [5, 80] \qquad \mu_2 \in [10, 60]$$

The geometry is fixed. The parameters are the regularization constant μ_1 in the functional (tuning the size of the control) and the Reynolds number μ_2.

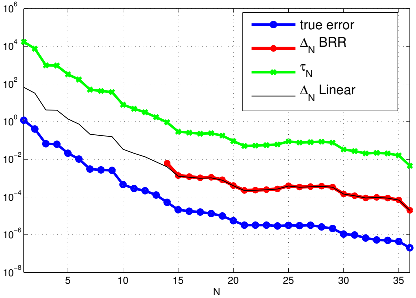
$$\text{minimize } \mathcal{J}(\mathbf{v}, \mathbf{u}; \mu) = \frac{1}{2} \int_{\Omega_{obs}} |\nabla \times \mathbf{v}|^2 \, d\Omega + \frac{\mu_1}{2} \|u\|^2_{H^1(\Gamma_C)}$$

$$\text{s.t.} \begin{cases} -\dfrac{1}{\mu_2} \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla \pi = 0 & \text{in } \Omega \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = \mathbf{u} & \text{on } \Gamma_C \\ + \text{ other boundary conditions} \end{cases}$$

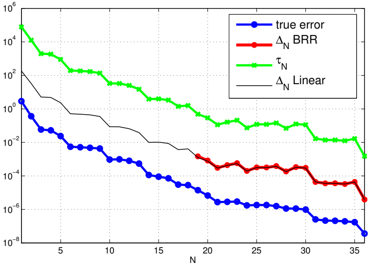
NL1 - Vorticity minimization on the downstream portion of a bluff body

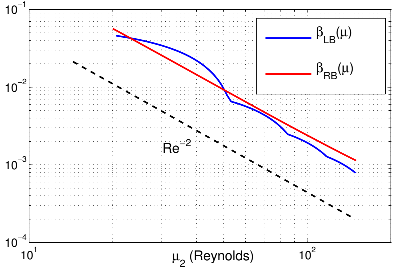
Results: no greedy algorithm (due to computational limitations), computation of reduced basis in randomly chosen parameter points.

Error bound for low Reynolds.



Sharpness of the error bounds depends on Reynolds number through $\hat{\beta}(\boldsymbol{\mu})$:

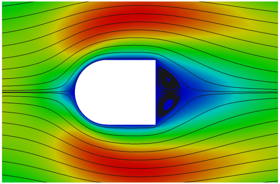


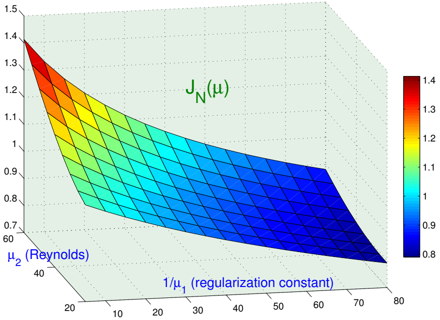


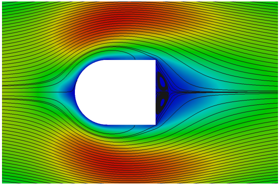
NL1 - Vorticity minimization on the downstream portion of a bluff body

FE evaluation t_{FE} (s)	≈ 60
RB evaluation t_{RB}^{online} (s)	0.9
Number of RB functions N	35

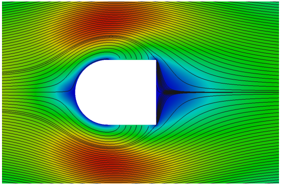
Uncontrolled solution



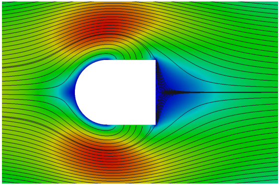




$\boldsymbol{\mu} = [1/10, 45]$



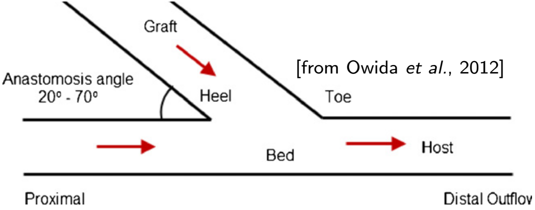
$\boldsymbol{\mu} = [1/55, 30]$



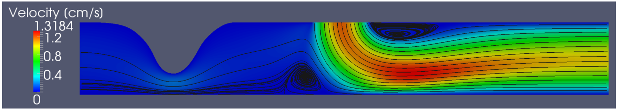
$\boldsymbol{\mu} = [1/80, 45]$

NL2 - Arterial bypass design: minimize restenosis risk

- Arterial bypass grafts tend to fail after some years due to the development of intimal thickening (restenosis).
- Restenosis formation is usually characterized by abnormally high or low values of shear stress, high values of its gradient, recirculation regions and graft deformation.
- The WSS, its gradient (WSSG) and the vorticity downstream the anastomosis are indicators of the restenosis risk.



[from Owida *et al.*, 2012]



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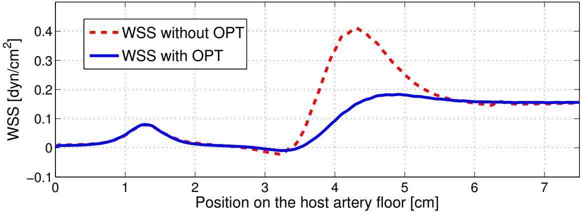
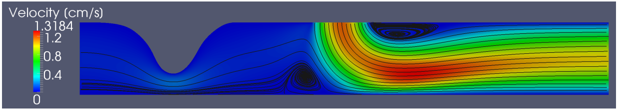




NL2 - Arterial bypass design via boundary optimal control

Find $(\mathbf{v}, \pi, \mathbf{u})$ such that the cost functional

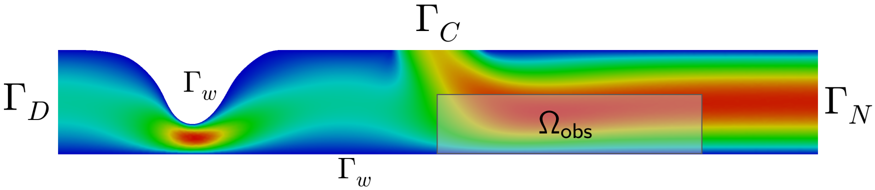
$$\mathcal{J}(\mathbf{v}, \pi, \mathbf{u}; \boldsymbol{\mu}) = \frac{1}{2} \int_{\Omega_{\text{obs}}(\boldsymbol{\mu})} |\nabla \times \mathbf{v}|^2 \, d\Omega + \frac{\alpha}{2} \int_{\Gamma_C(\boldsymbol{\mu})} |\nabla \mathbf{u}|^2 d\Gamma$$

is minimized subject to the steady Navier-Stokes equations:

$$-\frac{1}{\text{Re}} \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla \pi = \mathbf{0} \quad \text{in } \Omega(\boldsymbol{\mu}) \qquad \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_w(\boldsymbol{\mu})$$

$$\text{div}\, \mathbf{v} = 0 \quad \text{in } \Omega(\boldsymbol{\mu}) \qquad \mathbf{v} = \mathbf{g}_{\text{res}}(\boldsymbol{\mu}) \quad \text{on } \Gamma_D$$

$$-\pi \mathbf{n} + \frac{1}{\text{Re}} \nabla \mathbf{v} \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_N \qquad \mathbf{v} = \mathbf{u} \quad \text{on } \Gamma_C(\boldsymbol{\mu}).$$



[Lassila, Manzoni, Quarteroni, Rozza] [Gunzburger *et al.*, 91; Hou & Ravindran, 99; Biros & Ghattas, 99, 05]

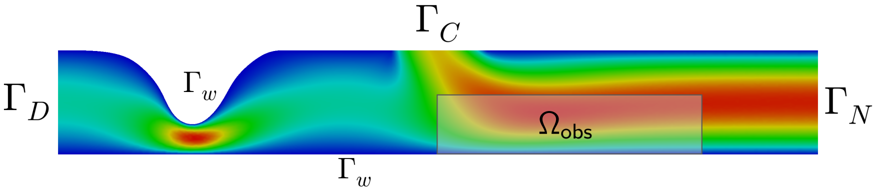
NL2 - Arterial bypass design via boundary optimal control – parameters

We consider the following parameters:

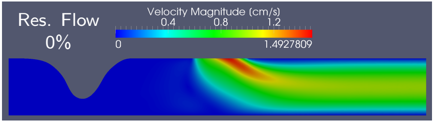
- $\mu_1 \in [40, 100]$: Reynolds number
- $\mu_2 \in [0, 40]$: percentage of residual flow $\mathbf{g}_{\text{res}}(\mu_2) = \mu_2/25\; y(1-y)$
- $\mu_3 \in [0.05, 10]$: penalization parameter α in the cost functional
- $\mu_4 \in [0.5, 1.2]$: length of the control boundary (graft diameter)

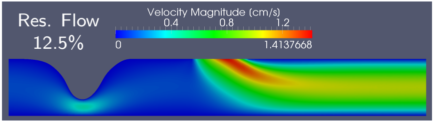
Total conservation of fluxes $\Longrightarrow$ additional constraint on the control variable:

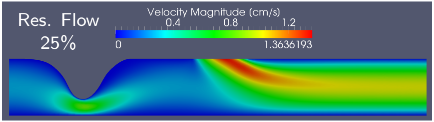
$$\int_{\Gamma_C} \mathbf{u} \cdot \mathbf{n}\, d\Gamma = Q_C(\mu_2) \quad \left(:= Q_{TOT} - \int_{\Gamma_D} \mathbf{g}_{\text{res}}(\mu_2)\, d\Gamma \right)$$

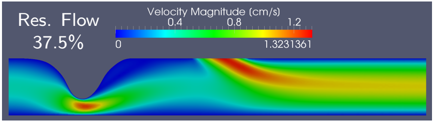


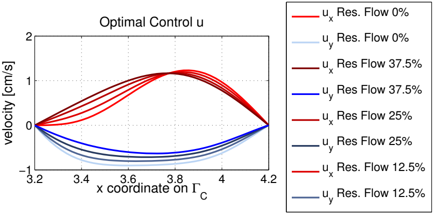
NL2 - Bypass design: sensitivity to the residual flow





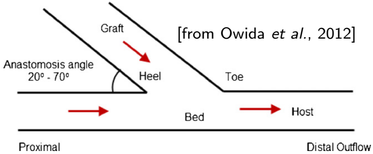




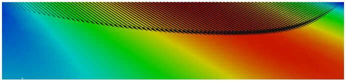


A link between boundary control velocity and shape of the bypass anastomosis:

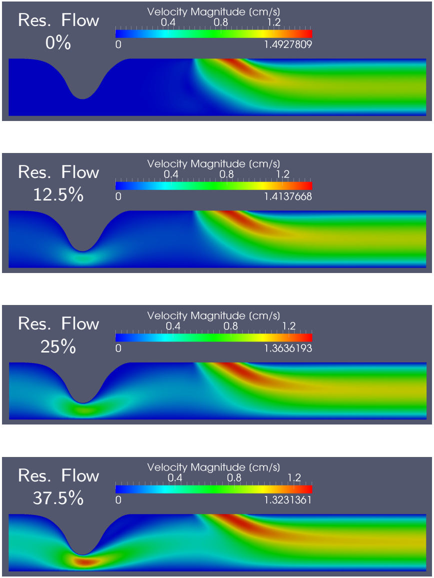
$$\theta(x) = \arctan\left(-\frac{u_y}{u_x}\right)$$

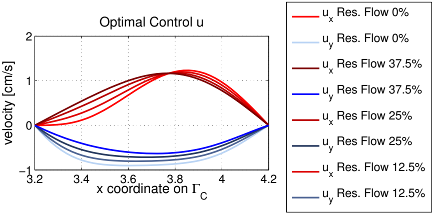


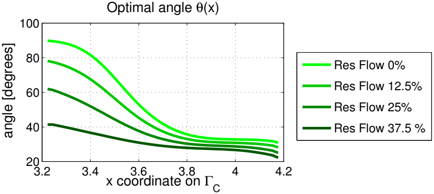
[from Owida *et al.*, 2012]

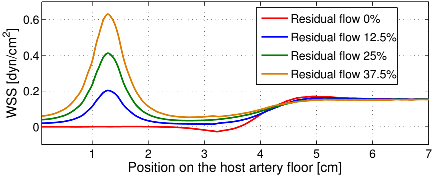


NL2 - Bypass design: sensitivity to the residual flow

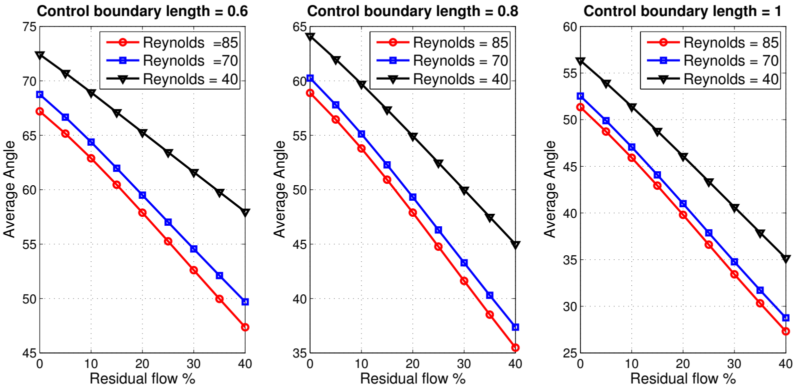








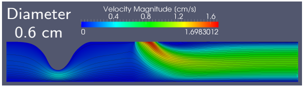
NL2 - Bypass design: sensitivity to the parameters

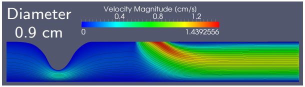


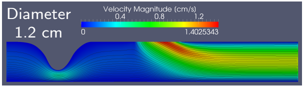
Number of FE dof $\mathcal{N}$	80 000
Number of parameters P	4
FE evaluation t_{FE} (s)	$60 - 250$
Affine terms Q	27

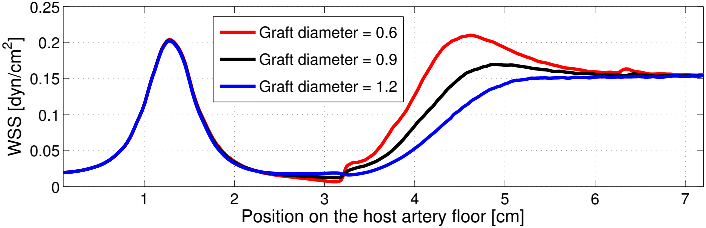
DOFs reduction	300:1
Number of RB functions N	20
Dimension of RB linear system	$20 \cdot 13$
RB evaluation t_{RB}^{online} (s)	1

NL2 - Bypass design: sensitivity to the parameters









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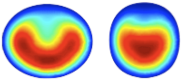
Conclusions and perspectives

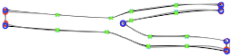
Data reconstruction/assimilation

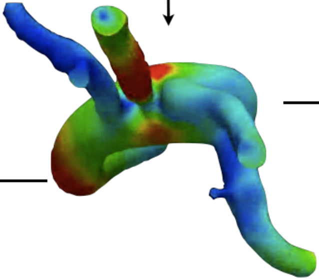
(e.g. boundary data for blood flow simulations)



Shape reconstruction

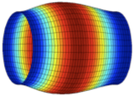
from patient-dependent configurations



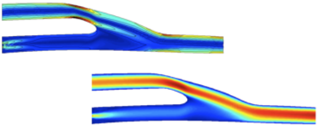


Many-query context

Reduced FSI problems

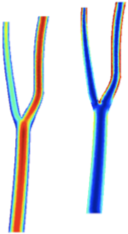


Shape optimization/design of cardiovascular devices



Real-time context

Output evaluation



Thank you for your attention! - http://people.sissa.it/grozza

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