Optimal execution and price manipulations in Limit Order Books

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Structure of the talk

What is a limit order book ?

The limit order book of an asset gather all the buy and sell orders.

- Orders are made on a common discretized price grid.
- Transactions are made as soon as they can.
- Highest waiting buy order : *bid* price. Lowest waiting sell order : *ask* price. Asset value $mid = \frac{bid+ask}{2}$.
- When a possible buy (resp. sell) order is made, it is executed with the cheapest waiting sell orders (resp. most expensive waiting buy orders) according to a FIFO rule for orders at the same price.
- Different transaction costs may be applied between the waiting orders and the orders that are immediately executed.

How to take into account this when modeling ?

Limit order book are very complex objects that are rather difficult to model in their wholeness. The liquidity risk it implies is mainly treated according two points of view :

- Hedging derivatives and portfolio management : how does liquidity risk impact the hedging strategies ? What is the extra cost it induces ? Many works on extensions to usual asset models.
- Order Execution : Once an order is made (amount and deadline), what is the optimal way to execute it ? Standard approach : statistical studies and **LOB modeling**.

The problem addressed in this talk

- We want to solve the following problem : to buy a large number of shares X_0 with deadline T, find an **optimal buy/sell strategy** $\xi_0, \dots, \xi_N$ executed on a time-grid $t_0 < t_1 < \cdots < t_n \leq T$ such that $\sum_{n=0}^{N} \xi_n = X_0$, and **that minimizes the whole expected transaction cost**. What is the optimal time grid ?
- We study this problem in a rather simple LOB model, with an intuitive parametrization.
- This problem is related to the viability of the market, which we will discuss.

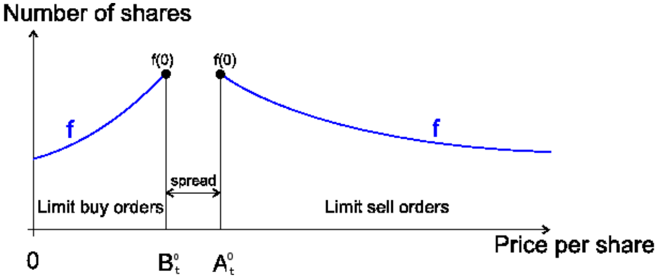
LOB Model assumptions

- We assume that there is one large trader that aim to buy X_0 shares.
- When the large trader is not active, we assume that the ask (resp. bid) price is given by $(A_t^0, t \geq 0)$ (resp. $(B_t^0, t \geq 0)$).
- We assume that $(A_t^0, t \geq 0)$ is a martingale and $(B_t^0, t \geq 0)$ is such that $\forall t \geq 0, B_t^0 \leq A_t^0 a.s.$ (mg assumption on B_t^0 if we consider a sell order).
- The LOB is model as follows : the number of sell orders between prices $A_t^0 + x$ and $A_t^0 + x + dx$ ($x \geq 0$) is given by :

$$f(x)dx,$$

and the number of buy orders between $B_t^0 + x$ and $B_t^0 + x + dx$ ($x < 0$) is also $f(x)dx$. The function $f : \mathbb{R} \to \mathbb{R}_+^*$ is called the *shape function* of the LOB and is assumed to be continuous.

The LOB at time t without any trade from the large trader :



Model for large buy/sell order

We will denote by $D_t^A \geq 0$ (resp. $D_t^B \leq 0$) the extra-shift on the ask (resp. bid) price by the large trader at time t. We assume that $D_0^A = D_0^B = 0$, and we set

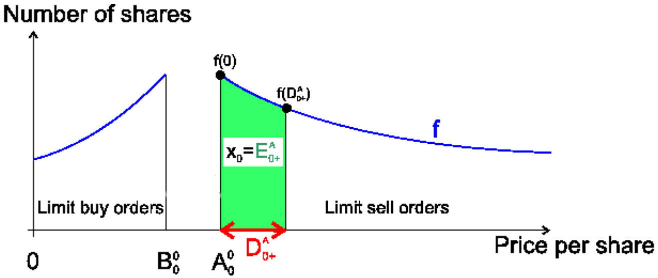
$$A_t = A_t^0 + D_t^A \text{ (resp. } B_t = B_t^0 + D_t^B.)$$

This means that all the shares in the LOB between prices A_t^0 and A_t (resp. B_t and B_t^0) have been consumed by previous trades.

When the large trader buys $x_t > 0$ shares at time t, he will consume the cheapest one between A_t and A_{t+} where $\int_{D_t^A}^{D_{t+}^A} f(x)dx = x_t$: the ask price is shifted from $A_t = A_t^0 + D_t^A$ to $A_{t+} = A_t^0 + D_{t+}^A$ and the transaction cost is

$$\int_{D_t^A}^{D_{t+}^A} (x + A_t^0) f(x)dx = A_t^0 x_t + \int_{D_t^A}^{D_{t+}^A} x f(x)dx.$$

Similarly, a sell order of $-x_t > 0$ shares moves the bid price from B_t to B_{t+} where $\int_{D_t^B}^{D_{t+}^B} f(x)dx = x_t$.



The LOB just after a buy order of size $x_0 = \int_0^{D^A_{0+}} f(x)dx$ at time 0.

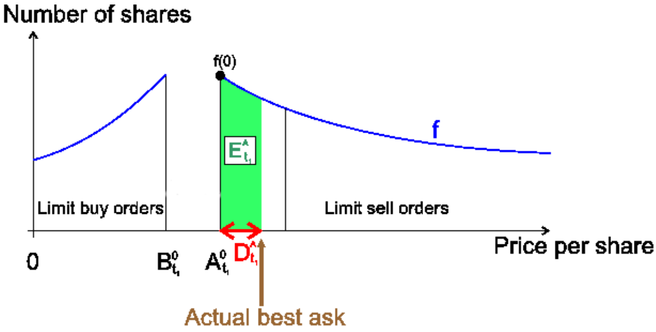
LOB dynamics without large trade

Now, we specify how new orders regenerate. We set $F(x) = \int_0^x f(u)du$ and denote $E^A_t = F(D^A_t)$ (resp. $E^B_t = F(D^B_t)$) the number of sell (resp. - up to the sign - buy) orders already eaten up at time t.
If the large investor is inactive on $[t, t+s[$, we assume :

- either **(Model 1) :** $E^A_{t+s} = e^{-\int_t^{t+s} \rho_u du} E^A_t$ (resp. $E^B_{t+s} = e^{-\int_t^{t+s} \rho_u du} E^B_t$).
- or **(Model 2) :** $D^A_{t+s} = e^{-\int_t^{t+s} \rho_u du} D^A_t$ (resp. $D^B_{t+s} = e^{-\int_t^{t+s} \rho_u du} D^B_t$).

$\rho_t > 0$ is assumed to be deterministic and is called the *resilience speed* of the LOB.
Rem : for $f(x) = q$ and $\rho_t = \rho$ both models coincide (Obizhaeva and Wang model).

Same example with no trade on $]0, t_1]$:



The cost minimization problem

At time t, a buy market order $x_t \geq 0$ moves D^A_t to D^A_{t+} s.t. $\int_{D^A_t}^{D^A_{t+}} f(x)\,dx = x_t$ and the cost is :

$$\pi_t(x_t) := \int_{D^A_t}^{D^A_{t+}} (A^0_t + x) f(x)\,dx = A^0_t x_t + \int_{D^A_t}^{D^A_{t+}} x f(x)\,dx.$$

Similarly, the cost of a sell order $x_t \leq 0$ is $\pi_t(x_t) := B^0_t x_t + \int_{D^B_t}^{D^B_{t+}} x f(x)\,dx$.
A admissible trading strategy is a sequence $\mathcal{T} = (\tau_0, \dots, \tau_N)$ of stopping times such that $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_N = T$ and a sequence of adapted trades $(\xi_0, \dots, \xi_n)$ s.t. $\sum_{n=0}^N \xi_n = X_0$. The *average cost* $\mathcal{C}(\xi)$ to minimize is :

$$\boxed{\mathcal{C}(\xi) = \mathbb{E}\Big[\sum_{n=0}^{N} \pi_{\tau_n}(\xi_n)\Big].}$$

The simplified model I

We consider the following model $S_t := A_t^0 + D_t$, where :

- $E_0 = D_0 = 0$ and

$$E_t = F(D_t) \qquad \text{and} \qquad D_t = F^{-1}(E_t).$$

- For $n = 0, \dots, N$, **regardless of the sign** of ξ_n,

$$E_{\tau_n+} = E_{\tau_n} + \xi_n \qquad \text{and} \qquad D_{\tau_n+} = F^{-1}\left(\xi_n + F\left(D_{\tau_n}\right)\right).$$

- For $k = 0, \dots, N-1$,

$$E_{\tau_{k+1}} = e^{-\int_{\tau_k}^{\tau_{k+1}} \rho_s ds} E_{\tau_k+} \text{ in Model 1, } D_{\tau_{k+1}} = e^{-\int_{\tau_k}^{\tau_{k+1}} \rho_s ds} D_{\tau_k+} \text{ in Model 2.}$$

The simplified model II

We clearly have $E^B_t \leq E_t \leq E^A_t$, $D^B_t \leq D_t \leq D^A_t$, and $S_t \leq A_t$.
The *simplified price* of ξ_n at time τ_n is given by

$$\overline{\pi}_{\tau_n}(\xi_n) := A^0_{\tau_n}\xi_n + \int_{D_{\tau_n}}^{D_{\tau_n+}} xf(x)\,dx.$$

We have $\overline{\pi}_{\tau_n}(\xi_n) \leq \pi_{\tau_n}(\xi_n)$ with equality if $\xi_k \geq 0$ for all $k \leq n$, therefore, an optimal strategy for the simplified cost

$$\overline{\mathcal{C}}(\xi) := \mathbb{E}\Big[\sum_{n=0}^{N} \overline{\pi}_{\tau_n}(\xi_n)\Big]$$

is optimal for the LOB model if it consists in only buy orders.

Reduction to det. strategies in the simplified model I

We have $\sum_{n=0}^N \overline{\pi}_{\tau_n}(\xi_n) = \sum_{n=0}^N A^0_{\tau_n}\xi_n + \sum_{n=0}^N \int_{D_{\tau_n}}^{D_{\tau_n+}} xf(x)\,dx$ and denote $X_t := X_0 - \sum_{\tau_k<t}\xi_k$ for $t \leq T$ and $X_{\tau_{N+1}} := 0$ (bounded and predictable for admissible strategies).
$\sum_{n=0}^N A^0_{\tau_n}\xi_n = -\sum_{n=0}^N A^0_{\tau_n}(X_{\tau_{n+1}} - X_{\tau_n}) = X_0A_0 + \sum_{n=1}^N X_{\tau_n}(A^0_{\tau_n} - A^0_{\tau_{n-1}})$, and since in each model $i \in \{1,2\}$, there is a determinisitic function $C^{(i)}$ s.t. $\sum_{n=0}^N \int_{D_{t_n}}^{D_{t_n+}} xf(x)\,dx = C^{(i)}(\mathcal{T}, \xi_0, \dots, \xi_N)$ we get

$$\overline{\mathcal{C}}(\xi) = A_0X_0 + \mathbb{E}\big[\,C^{(i)}(\xi_0, \dots, \xi_N, \mathcal{T})\,\big].$$

Reduction to det. strategies in the simplified model II

- If there is a unique minimizer of $C^{(i)}$ in $\left\{ (x_0, \dots, x_N), (t_0, \dots, t_N) \in \mathbb{R}^{N+1} \mid \sum_{n=0}^{N} x_n = X_0,\ 0 = t_0 \leq t_1 \dots \leq t_N = T \right\}$, the problem is solved and the optimal strategy is deterministic.
- Moreover, if this strategy is made with only buy trades, it is also optimal for the corresponding bid/ask model.

$\Longrightarrow$ Thus, we will only work in the sequel in this simplified model.

The cost function on a fixed time-grid $\mathcal{T} = (t_0, \dots, t_N)$

We have $E_t = qD_t$ and $E_{t_n^+} = \sum_{k=0}^n x_{t_k} \exp(-\int_{t_k}^{t_n} \rho_u du)$. The cost is :

$$
\begin{aligned}
C(\mathbf{x}, \mathcal{T}) &= \sum_{n=0}^{N} \int_{D_{t_n}}^{D_{t_n^+}} qx dx = \frac{q}{2} \sum_{n=0}^{N} (D_{t_n^+}^2 - D_{t_n}^2) \\
&= \frac{1}{2q} \sum_{n=0}^{N} x_{t_n} \left(x_{t_n} + 2 \sum_{k=0}^{n-1} x_{t_k} \exp(-\int_{t_k}^{t_n} \rho_u du) \right)
\end{aligned}
$$

Cost minimization on a fixed time-grid $\mathcal{T} = (t_0, \dots, t_N)$

We set :

$$a_n = e^{-\alpha_n} \text{ with } \alpha_n = \int_{t_{n-1}}^{t_n} \rho_s \, ds, n = 1, \dots, N.$$

We have $C(\mathbf{x}, \mathcal{T}) = \frac{\mathbf{1}}{\mathbf{2q}} \langle \boldsymbol{x}, \mathbf{M}(\boldsymbol{\alpha})\boldsymbol{x} \rangle$, where

$$M(\boldsymbol{\alpha})_{n,m} = \exp\left(-\int_{t_{n\wedge m}}^{t_{n\vee m}} \rho_u du\right) = \exp\left(-\Big|\sum_{i=1}^{n} \alpha_i - \sum_{j=1}^{m} \alpha_j\Big|\right), \qquad 0 \le n, m \le N.$$

Optimal strategy s.t. $\sum_{i=0}^{N} x_i = X_0$:

$$\boldsymbol{x}^*(\boldsymbol{\alpha}) = \frac{X_0}{\langle \mathbf{1}, M(\boldsymbol{\alpha})^{-1} \mathbf{1} \rangle} M(\boldsymbol{\alpha})^{-1} \mathbf{1}.$$

Inversion of $M(\boldsymbol{\alpha})$

$$M(\boldsymbol{\alpha})^{-1} = \begin{bmatrix} \frac{1}{1-a_1^2} & \frac{-a_1}{1-a_1^2} & 0 & \cdots & 0 \\ \frac{-a_1}{1-a_1^2} & \left(\frac{1}{1-a_1^2} + \frac{a_2^2}{1-a_2^2}\right) & \frac{-a_2}{1-a_2^2} & 0 \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \frac{-a_{N-1}}{1-a_{N-1}^2} & \left(\frac{1}{1-a_{N-1}^2} + \frac{a_N^2}{1-a_N^2}\right) & \frac{-a_N}{1-a_N^2} \\ 0 & \cdots & 0 & \frac{-a_N}{1-a_N^2} & \frac{1}{1-a_N^2} \end{bmatrix}.$$

The optimal strategy on a fixed time-grid $(t_0, \ldots, t_N)$

There exists a unique optimal strategy $\xi^* = (\xi_0^*, \ldots, \xi_N^*)$ in the class of all admissible strategies. With the notation $\lambda_0 := \frac{X_0}{\frac{2}{1+a_1} + \sum_{n=2}^{N} \frac{1-a_n}{1+a_n}}$, the initial market order is

$$\xi_0^* = \frac{\lambda_0}{1+a_1},$$

the intermediate market orders are given by

$$\xi_n^* = \lambda_0 \left(\frac{1}{1+a_n} - \frac{a_{n+1}}{1+a_{n+1}} \right), \qquad n = 1, \ldots, N-1,$$

and the final market order is

$$\xi_N^* = \frac{\lambda_0}{1+a_N}.$$

It is deterministic and s.t. $\xi_n^* > 0$ for all n. It does not depend on q.

Optimization under affine constraints I

Thanks to the Kuhn-Tucker theorem, it is easy to calculate the optimal strategy under additional affine constraints
Example : assume that we want to trade at least $\alpha \times 100$ percent of shares during the opening and the closure of the market.

$$\boxed{\xi_0 + \xi_N \geq \alpha X_0 \ (= \sum_{i=0}^{N} \xi_i)}.$$

We assume that the resilience is such that $a_1 = a_N = a$ and for regular trading $a_2 = \cdots = a_{N-1} = b$.
After calculations, one gets when the constraint is active $x_0^* = x_N^* = \frac{\alpha X_0}{2}$ that

$$x_2^* = \cdots = x_{N-2}^* = X_0 \cdot \frac{(1-\alpha(1-a))(1-b)}{N-1-(N-3)b} > 0, \ x_1^* = x_{N-1}^* = X_0 \cdot \frac{1-\alpha\left(\frac{1}{2}a(1-b)(N-3)+1\right)}{N-1-(N-3)b}.$$

Optimization under affine constraints II

If $\alpha\left(\frac{1}{2}a(1-b)(N-3)+1\right) \leq 1$, then $\mathbf{x}^*$ is also optimal in the LOB model.

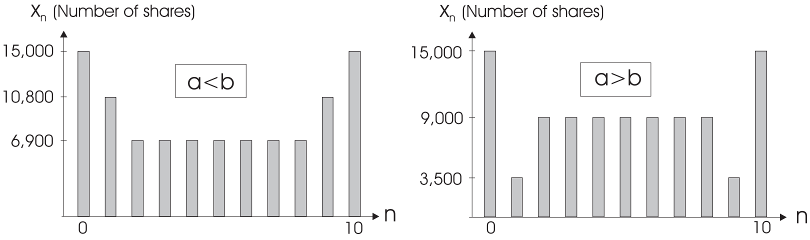


FIGURE: $X_0 = 100,000$, $\alpha = 0.3$, and resilience coefficients are $a = e^{-5}$ and $b = e^{-1}$ in the first graph and $a = e^{-1}$ and $b = e^{-5}$ in the second graph.

The optimal time-grid

We want to minimize the cost on $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}_+^N$ s.t. $\sum_{i=1}^N \alpha_i = \int_0^T \rho_s ds$.
Optimal cost :

$$\begin{aligned}\min_{\boldsymbol{x} \in \Xi} C(\boldsymbol{x}, \boldsymbol{\alpha}) &= C(\boldsymbol{x}^*(\boldsymbol{\alpha}), \boldsymbol{\alpha}) = \frac{X_0^2}{2\langle \mathbf{1}, M(\boldsymbol{\alpha})^{-1}\mathbf{1}\rangle} \\ &= \frac{X_0^2}{2}\left(\sum_{n=1}^N \frac{2}{1+e^{-\alpha_n}} - (N-1)\right)^{-1}.\end{aligned}$$

The function $x \mapsto \frac{2}{1+e^{-x}}$ is strictly concave in $x > 0$. Hence,

$$\sum_{n=1}^N \frac{2}{1+e^{-\alpha_n}} \leq \frac{2N}{1+e^{-\frac{1}{N}\sum_{n=1}^N \alpha_n}} = \frac{2N}{1+e^{-\frac{1}{N}\int_0^T \rho_u du}},$$

with equality if and only if $\boldsymbol{\alpha} = \boldsymbol{\alpha}^*$, where $\boldsymbol{\alpha}^*$ corresponds to the homogeneous time spacing $\mathcal{T}^* : \int_{t_{n-1}}^{t_n} \rho_u du = \frac{1}{N}\int_0^T \rho_u du$.

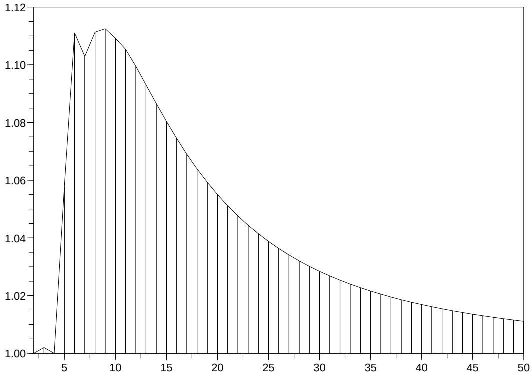


FIGURE: Relative gain between the extra liquidity cost of the optimal strategy on the optimal grid $\mathcal{T}^*$ and the optimal strategy on the equidistant grid as a function of N, when $T = 2$ and $\rho_t = 10 + 8\cos(t/(2\pi))$.

Assumptions and notations

We consider a general LOB shape function f, and assume from now that $\lim_{x\uparrow\infty} F(x) = \infty$ and $\lim_{x\downarrow-\infty} F(x) = -\infty$.
For a fixed N, we set $\alpha = \frac{1}{N}\int_0^T \rho_u du$ and consider the time grid that is regular w.r.t. the resilience :

$$\mathcal{T}^* = (t_0^*, \dots, t_N^*), \text{ where } \int_{t_{i-1}^*}^{t_i^*} \rho_u du = \alpha.$$

We also set $a^* = \exp(-\alpha)$.
We now present results obtained with A. Fruth and look at the optimal strategy on the homogeneous grid $\mathcal{T}^*$.

Optimal strategy on $\mathcal{T}^*$ for model 1

Suppose $h_1(u) := F^{-1}(u) - a^* F^{-1}(a^* u)$ one-to-one. Then there exists a unique optimal strategy $\xi^{(1)} = (\xi_0^{(1)}, \dots, \xi_N^{(1)})$.
$\xi_0^{(1)}$: unique solution of the equation

$$F^{-1}\left(X_0 - N\xi_0^{(1)}(1 - a^*)\right) = \frac{h_1(\xi_0^{(1)})}{1 - a^*},$$

the intermediate orders are given by

$$\xi_1^{(1)} = \cdots = \xi_{N-1}^{(1)} = \xi_0^{(1)}(1 - a^*),$$

the final order is determined by

$$\xi_N^{(1)} = X_0 - \xi_0^{(1)} - (N-1)\xi_0^{(1)}(1 - a^*).$$

It is deterministic and s.t. $\xi_n^{(1)} > 0$ for all n.

Optimal strategy on $\mathcal{T}^*$ for model 2

Suppose $h_2(x) := x\frac{f(x)-(a^*)^2 f(a^*x)}{f(x)-a^*f(a^*x)}$ one-to-one, and $\lim_{|x|\to\infty} x^2 \inf_{y\in[a^*x,x]} f(y) = \infty$. Then there exists a unique optimal strategy $\xi^{(2)} = (\xi_0^{(2)}, \dots, \xi_N^{(2)})$.

$\xi_0^{(2)}$: unique solution of the equation

$$F^{-1}\left(X_0 - N\big[\xi_0^{(2)} - F\big(a^* F^{-1}(\xi_0^{(2)})\big)\big]\right) = h_2\big(F^{-1}(\xi_0^{(2)})\big),$$

the intermediate orders are given by

$$\xi_1^{(2)} = \dots = \xi_{N-1}^{(2)} = \xi_0^{(2)} - F\big(a^* F^{-1}(\xi_0^{(2)})\big)$$

the final order is determined by

$$\xi_N^{(2)} = X_0 - N\xi_0^{(2)} + (N-1)F\big(a^* F^{-1}(\xi_0^{(2)})\big).$$

It is deterministic and s.t. $\xi_n^{(2)} > 0$ for all n.

Comments

- Optimal strategies have a clear interpretation in both models : the first trade shifts the ask price to the best trade-off between price and attracting new orders.
- One can show that h_1 is one-to-one if f is increasing on $\mathbb{R}_-$ and decreasing on $\mathbb{R}_+$. There is no such simple characterization for h_2.
- In the case $f(x) = q$ (block-shaped LOB), both theorems give the following optimal strategy :

$$\xi_0^* = \xi_N^* = \frac{X_0}{(N-1)(1-a^*)+2} \quad \text{and} \quad \xi_1^* = \cdots = \xi_{N-1}^* = \frac{X_0 - 2\xi_0^*}{N-1}.$$

It does not depend on q.

Example

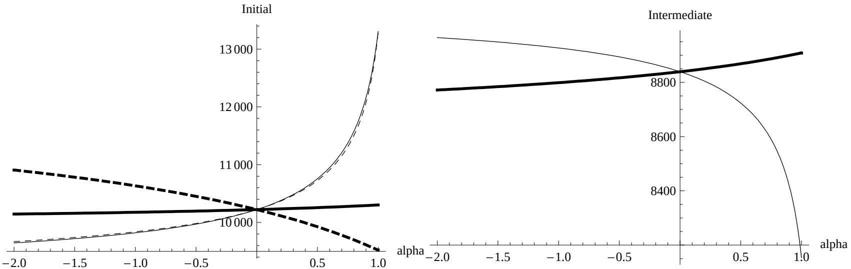


FIGURE: The plots show the optimal strategies for $f(x) = q/(|x|+1)^{\alpha}$. We set $X_0 = 100,000$ and $q = 5,000$ shares, $\rho = 20$, $T = 1$ and $N = 10$. In the left figure we see $\xi_0^{(1)}$, $\xi_N^{(1)}$ (thick lines) and $\xi_0^{(2)}$, $\xi_N^{(2)}$. The figure on the right hand side shows $\xi_1^{(1)}$ (thick line) and $\xi_1^{(2)}$.

Cost minimization for model 1 I

Define $\tilde{F}(x) = \int_0^x uf(u)du$, $G(x) = \tilde{F}(F^{-1}(x))$ and $a = a^*$. We have :

$$\begin{aligned}
C^{(1)}(x_0, \dots, x_N) &= \sum_{n=0}^{N} \int_{D_{t_n}}^{D_{t_n+}} xf(x)dx \\
&= \sum_{n=0}^{N} \left(\widetilde{F}\left(F^{-1}\left(E_{t_n+}\right)\right) - \widetilde{F}\left(F^{-1}\left(E_{t_n}\right)\right) \right) \\
&= \sum_{n=0}^{N} \left(G\left(E_{t_n} + x_n\right) - G\left(E_{t_n}\right) \right) \\
&= \quad G\left(x_0\right) - G\left(0\right) \\
&\quad + G\left(ax_0 + x_1\right) - G\left(ax_0\right) \\
&\quad + G\left(a^2 x_0 + ax_1 + x_2\right) - G\left(a^2 x_0 + ax_1\right) \\
&\quad + \dots \\
&\quad + G\left(a^N x_0 + \dots + x_N\right) - G\left(a^N x_0 + \dots + ax_{N-1}\right).
\end{aligned}$$

Cost minimization for model 1 II

We can show $C^{(1)}(x_0,\dots,x_N)\longrightarrow+\infty$ for $|(x_0,\dots,x_N)|\to\infty$. At the optimum $\xi^*=(x_0^*,\dots,x_N^*)\in\Xi$, there is a Lagrange multiplier $\nu\in\mathbb{R}$ s.t. $\frac{\partial}{\partial x_j}C^{(1)}(x_0^*,\dots,x_N^*)=\nu$ for $j=0,\dots,N$. Since

$$\begin{aligned}\frac{\partial}{\partial x_j}C^{(1)}(x_0,\dots,x_N) &= a\left[\frac{\partial}{\partial x_{j+1}}C^{(1)}(x_0,\dots,x_N)-G'\left(a(a^jx_0+\dots+x_j)\right)\right]\\ &+ G'\left(a^jx_0+\dots+x_j\right)\end{aligned}$$

and $G'=F^{-1}$, we get $h_1\left(a^jx_0^*+\dots+x_j^*\right)=\nu\,(1-a)$ for $j=0,\dots,N-1$. Since h_1 is one-to-one we must have

$$\begin{aligned}x_0^* &= h_1^{-1}\left(\nu\,(1-a)\right)\\ x_j^* &= x_0^*\,(1-a)\qquad\text{for } j=1,\dots,N-1\\ x_N^* &= X_0-x_0^*-(N-1)x_0^*\,(1-a)\,.\end{aligned}$$

The continuous time limit (T fixed, $N \to +\infty$)

- **Model 1 :** If $F^{-1}(X_0 - \int_0^T \rho_u du x) = h_1^\infty(x) := F^{-1}(x) + \frac{x}{f(F^{-1}(x))}$ has a unique solution $\xi_0^{(1),\infty}$, the optimal strategy consists in an initial block order of $\xi_0^{(1),\infty}$ shares at time 0, continuous buying at the rate $\rho_t \xi_0^{(1),\infty}$ during $]0, T[$, and a final block order of $\xi_T^{(1),\infty} = X_0 - \xi_0^{(1),\infty}(1 + \int_0^T \rho_u du)$ shares at time T.
  This result has been recently extended by Predoiu, Shaikhet and Shreve in a model where $F(x) = \mu([0, x))$ (positive measure) and $dE_t = -h(E_t)dt$ instead of $dE_t = -\rho_t E_t dt$
- **Model 2 :** Idem with an initial trade solution of $F^{-1}(X_0 - \int_0^T \rho_u du F^{-1}(x) f(F^{-1}(x))) = h_2^\infty(F^{-1}(x))$ where $h_2^\infty(x) := x(1 + \frac{f(x)}{f(x)+xf'(x)})$, continuous buying rate $\rho_t F^{-1}(\xi_0^{(2),\infty}) f(F^{-1}(\xi_0^{(2),\infty}))$ on $]0, T[$, and a final block order $\xi_T^{(2),\infty} := X_0 - \xi_0^{(2),\infty} - \int_0^T \rho_u du F^{-1}(\xi_0^{(2),\infty}) f(F^{-1}(\xi_0^{(2),\infty}))$.

Time-grid optimization in Model 1

Assumption : In Model 1, we assume that f is nondecreasing on $\mathbb{R}_-$ and nonincreasing on $\mathbb{R}_+$ or that $f(x) = \lambda|x|^\alpha$, $\lambda, \alpha > 0$.

Proposition 1

Suppose that an admissible sequence of trading times $\mathcal{T} = (t_0, t_1, \ldots, t_N)$ is given. There exists a $\mathcal{T}$-admissible trading strategy $\boldsymbol{\xi}^{(1),\mathcal{T}}$, unique (up to equivalence), that minimizes the cost among all $\mathcal{T}$-admissible trading strategies. Moreover, it consists only of nontrivial buy orders, i.e., $\xi_i^{(1),\mathcal{T}} > 0$ $\mathbb{P}$-a.s. for all i up to equivalence.

Theorem 2

There is a unique optimal strategy $(\boldsymbol{\xi}^{(1)}, \mathcal{T}^*)$ consisting of homogeneous time spacing $\mathcal{T}^*$ and the deterministic trading strategy $\boldsymbol{\xi}^{(1)}$ defined in slide 27.

Time-grid optimization in Model 2

Assumption : In Model 2, we assume that $f(x) = \lambda|x|^{\alpha}$, $\lambda, \alpha > 0$ or that f is $\mathcal{C}^2$ on $\mathbb{R}\backslash\{0\}$, $\nearrow$ on $\mathbb{R}_-$ and $\searrow$ on $\mathbb{R}_+$, and :

$$x \mapsto xf'(x)/f(x) \text{ is } \nearrow \text{ on } \mathbb{R}_-, \searrow \text{ on } \mathbb{R}_+, \text{ and } (-1, 0]\text{-valued},$$

$$1 + x\frac{f'(x)}{f(x)} + 2x^2\left(\frac{f'(x)}{f(x)}\right)^2 - x^2\frac{f''(x)}{f(x)} \geq 0 \qquad \text{for all } x \geq 0.$$

Analogous proposition and

Theorem 3

Under the above assumption, there is a unique optimal strategy $(\boldsymbol{\xi}^{(2)}, \mathcal{T}^*)$, consisting of homogeneous time spacing $\mathcal{T}^*$ and the deterministic trading strategy $\boldsymbol{\xi}^{(2)}$ defined in slide *28**.*

Example : $f(x) = q/(1 + \lambda|x|)^{\alpha}$ satisfy this condition.

Sketch of the proof for model 1 I

(Analogous but more technical proof for model 2)
Define $\tilde{F}(x) = \int_0^x uf(u)du$, $G(x) = \tilde{F}(F^{-1}(x))$ and $a = a^*$. We have :

$$\begin{aligned} C^{(1)}(\boldsymbol{x}, \boldsymbol{\alpha}) &= \sum_{n=0}^{N} \int_{D_{t_n}}^{D_{t_n+}} xf(x)dx \\ &= \sum_{n=0}^{N} \left(G\left(E_{t_n} + x_n\right) - G\left(E_{t_n}\right) \right), \end{aligned}$$

with $E_{t_n} = \sum_{i=0}^{n-1} x_i e^{-\sum_{k=i+1}^{n} \alpha_k}$.
When $\boldsymbol{\alpha}$ is fixed, we can show $\boldsymbol{x} \mapsto C^{(1)}(\boldsymbol{x}, \boldsymbol{\alpha})$ is convex and $C^{(1)}(\boldsymbol{x}, \boldsymbol{\alpha}) \longrightarrow +\infty$ for $|\boldsymbol{x}| \to \infty$. Lagrange multiplier at the minimum gives Prop 1.

Sketch of the proof for model 1 II

Thanks to this Proposition, a minimum of $C^{(1)}$ is on $\{\boldsymbol{x} \in \mathbb{R}_+^{N+1}, \alpha \in \mathbb{R}_+^{N+1}, \sum_{i=0}^N x_i = X_0, \sum_{i=0}^N \alpha_i = \int_0^T \rho_u du\}$, which is compact. Using once again the proposition, a minimum satisfies $\boldsymbol{x} \in (\mathbb{R}_+^*)^{N+1}, \alpha \in (\mathbb{R}_+^*)^{N+1}$.
Lagrange multipliers (λ, ν) satisfy for $i = 1, \ldots, N$:

$$\nu = \frac{F^{-1}(E_{t_{i-1}} + x_{i-1}) - e^{-\alpha_i} F^{-1}(e^{-\alpha_i}(E_{t_{i-1}} + x_{i-1}))}{1 - e^{-\alpha_i}},$$

$$\lambda = e^{-\alpha_i}(E_{t_{i-1}} + x_{i-1}) \frac{F^{-1}(E_{t_{i-1}} + x_{i-1}) - F^{-1}(e^{-\alpha_i}(E_{t_{i-1}} + x_{i-1}))}{1 - e^{-\alpha_i}},$$

$\implies \alpha_1 = \cdots = \alpha_N$ and $x_0 = E_{t_1} + x_1 = \cdots = E_{t_{N-1}} + x_{N-1}$, which gives strategy 27.

Remark on the assumption on f

The assumption made on f is sufficient to get :

1. For each $a \in (0,1)$, $h_{1,a} : y \in \mathbb{R} \mapsto F^{-1}(y) - aF^{-1}(ay) \in \mathbb{R}$ is strictly increasing.
2. For all $a, b \in (0,1)$ and $\nu > 0$, we have the inequalities

$$h_{1,a}^{-1}\big(\nu(1-a)\big) > b \cdot h_{1,b}^{-1}\big(\nu(1-b)\big)$$

$$b \cdot h_{1,b}^{-1}\big(\nu(1-b)\big) < F(\nu).$$

3. $H_1 : (y,a) \in (0,\infty) \times (0,1) \longmapsto \left(\frac{F^{-1}(y)-aF^{-1}(ay)}{1-a}, ay\frac{F^{-1}(y)-F^{-1}(ay)}{1-a}\right) \in \mathbb{R}^2$ is one-to-one.

These conditions are sufficient to get Theorem 2.

Price manipulation strategies I

A *round trip* is an admissible strategy $(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}})$ such that $\sum_{i=0}^{N} \overline{\xi}_i = 0$.
A *price manipulation strategy* (Huberman and Stanzl) is a round trip $(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}})$ s.t. $\mathcal{C}(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}}) < 0$.

Corollary 4

Under the respective assumptions, any nontrivial round trip has a strictly positive average cost in Model 1 and 2. In particular, there are no price manipulation strategies.

Price manipulation strategies II

This is in contrast with the result by Gatheral :

$$S_t = A_t^0 + \int_0^t \varphi(\dot{x}_s) e^{-\rho(t-s)} ds$$

has no PMS iff φ is linear.
As a comparison, the continuous version of our Model 1 is for a constant resilience ρ :

$$S_t = A_t^0 + F^{-1}\left(\int_0^t \dot{x}_s e^{-\rho(t-s)} ds\right).$$

The model

We consider a block-shape LOB so that the price impact is proportional to the trade size.
When the strategy $\boldsymbol{\xi} = (\xi_{t_0}, \xi_{t_1}, \dots, \xi_{t_N})$ is applied, the price at time t is

$$S_t = A_t^0 + \sum_{t_n < t} \xi_{t_n} G(t - t_n), \quad (1)$$

where G is a nonincreasing function on the time axis $[0, \infty)$, the *resilience function*.
Three types of price impact :

- The *instantaneous impact* is $\xi_{t_n}(G(0) - G(0+))$, where $G(0+)$ denotes the righthand limit of G at $t = 0$.
- The *permanent impact* is $\xi_{t_n} G(\infty)$, where $G(\infty) := \lim_{t \nearrow \infty} G(t)$.
- The remaining part, $\xi_{t_n}(G(0+) - G(\infty))$, is called the *transient impact*.

The cost function

$\mathcal{C}(\boldsymbol{\xi}) := \mathbb{E}\Big[\sum_{n=0}^{N} \int_{S_{t_n}}^{S_{t_n+}} yG(0)^{-1}\, dy \Big] = \frac{1}{2G(0)}\mathbb{E}\Big[\sum_{n=0}^{N} \big(S_{t_n+}^2 - S_{t_n}^2\big) \Big].$
Since $S_{t_n+}^2 - S_{t_n}^2 = 2S_{t_n}\xi_{t_n} + \xi_{t_n}^2$, we get

$$\mathcal{C}(\boldsymbol{\xi}) = X_0 S_0 + \mathbb{E}[\, C(\boldsymbol{\xi})\,],$$

with
$C(\boldsymbol{x}) := \frac{1}{2}\sum_{i,j=0}^{N} x_i x_j G(|t_i - t_j|) = \frac{1}{2}\langle \boldsymbol{x}, M\boldsymbol{x}\rangle, \ \ \boldsymbol{x} = (x_0, \dots, x_N) \in \mathbb{R}^{N+1}.$
The function G is said *positive definite* if $C(.) \geq 0$ and is *strictly definite positive* when $C(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$.
When G is **strictly definite positive**, the optimal strategy on $(t_0, \dots, t_N)$ is :

$$\boldsymbol{x}^* = \frac{X_0}{\mathbf{1}^\top M^{-1}\mathbf{1}} M^{-1}\mathbf{1}.$$

and **there is no Price manipulation strategies**.

Bochner's theorem (1932)

A continuous resilience function G is positive definite if and only if the function $x \to G(|x|)$ is the Fourier transform of a positive finite Borel measure μ on $\mathbb{R}$. If, in addition, the support of μ is not discrete, then G is strictly positive definite.

In particular, when G is convex and nonconstant, it is strictly positive definite (Caratheodory (1907), Toeplitz (1911) and Young (1913)).

Example : $G(t) = (1+t)^{-0.4}$

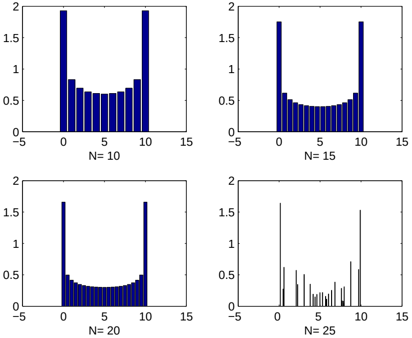


FIGURE: Optimal strategies for power-law resilience $G(t) = (1+t)^{-0.4}$ and various values of N. For $N = 25$ we use randomly chosen trading times.

Example : $G(t) = (1 - \rho t)^+$, $\rho \leq 1/T$

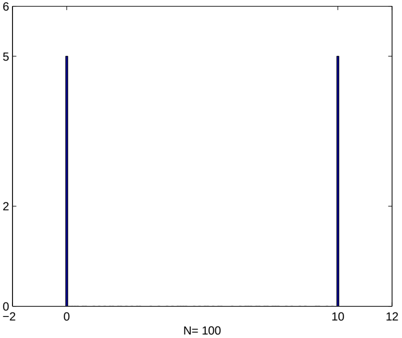


FIGURE: Optimal strategy for linear resilience $G(t) = (1 - \rho t)^+$ with $\rho \leq 1/T$ and arbitrary time grid.

Example : $G(t) = \cos \rho(t \wedge T) + 0.01e^{-t}$

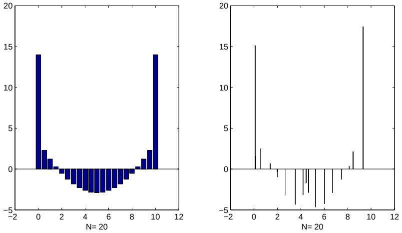


FIGURE: Optimal strategies for trigonometric resilience, $G_{0.01}(t) = \cos \rho(t \wedge T) + 0.01e^{-t}$, with $\rho = \pi/2T$ and equidistant and randomly chosen trading dates.

Example : $G(t) = e^{-t^2}$

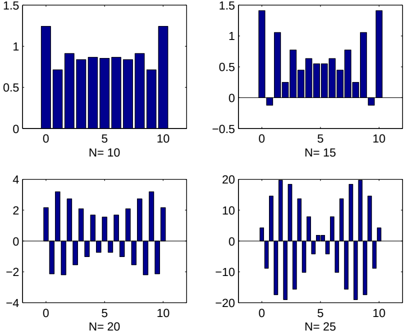


FIGURE: Optimal strategies for Gaussian resilience $G(t) = e^{-t^2}$.

Example : $G(t) = 1/(1+t^2)$

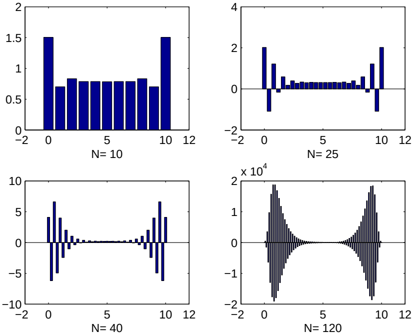


FIGURE: Optimal strategies for $G(t) = 1/(1+t^2)$.

Transaction-triggered price manipulations

These examples motivate the following definition :
A market impact model admits *transaction-triggered price manipulation* if the expected execution costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

- Weaker notion of manipulation strategy :

$$\text{No TTPMS} \implies \text{No PMS.}$$

- In the previous LOB model with exponential resilience, there is no TTPMS since the optimal strategy has only positive trades.

Theorem 5

For a convex resilience function G there are no transaction-triggered price manipulation strategies. If G is even strictly convex, then all trades in an optimal execution strategy are strictly positive for a buy program and strictly negative for a sell program.

We also have the following partial converse to the preceding theorem.

Proposition 6

Suppose that

$$\text{there are } s,\ t>0,\ s\neq t, \text{ such that} \qquad G(0)-G(s)<G(t)-G(t+s). \quad (2)$$

Then the model admits transaction-triggered price manipulation strategies.

Sum up

- We have proposed a simple LOB model with a general shape function and exponenial resilience.
- In that model, there is under general conditions a unique optimal strategy to buy X_0 shares that consists in deterministic buy trades. In particular there is no PMS.
- We have looked at a simple model with a block shape LOB and a general resilience function.
- We have introduced the notion of TTPMS and shown that convex resilience functions exclude this kind of manipulation strategies

Some further questions

- Optimal time-grid for general resilience functions.
- To analyse the impact of the bid/ask spread on the existence of TTPMS in a block-shape LOB with a general resilience. Modelling of the bid/ask.
- To consider a general LOB shape **and** a general resilience function.
- ...