constant volatility σ): find $p : \mathcal{D} \to \mathbb{R}$ such that, for all $q : \mathcal{D} \to \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{D}} pq - \int_{\mathcal{D}} \frac{\sigma^2 S^2}{2}\frac{\partial p}{\partial S}\frac{\partial q}{\partial S} - \int_{\mathcal{D}} \frac{\sigma^2 S^2}{2}\frac{\partial p}{\partial S}\frac{\partial q}{\partial M} + \int_{\mathcal{D}} \frac{\sigma^2 S^2}{2}\frac{\partial p}{\partial M}\frac{\partial q}{\partial S} \\ + \int_{\mathcal{D}} \sigma^2 S \frac{\partial p}{\partial M} q + \int_{\mathcal{D}} \left(r - \sigma^2\right) S \frac{\partial p}{\partial S} q - r\int_{\mathcal{D}} pq = 0, \\ p(T,S,M) = \phi(S,M), \end{aligned} \tag{43}$$

where $\mathcal{D} = \{(S,M) \in \mathbb{R}^2,\, 0 \le S \le M\}$. The boundary condition $\partial p/\partial M(t,S,S) = 0$ is naturally contained in this variational formulation since, by integration by parts over $\mathcal{D}$:

$$\begin{aligned} -\int_{\mathcal{D}} \frac{\sigma^2 S^2}{2}\frac{\partial p}{\partial S}\frac{\partial q}{\partial S} - \int_{\mathcal{D}} \frac{\sigma^2 S^2}{2}\frac{\partial p}{\partial S}\frac{\partial q}{\partial M} + \int_{\mathcal{D}} \frac{\sigma^2 S^2}{2}\frac{\partial p}{\partial M}\frac{\partial q}{\partial S} + \int_{\mathcal{D}} \sigma^2 S \frac{\partial p}{\partial M} q - \int_{\mathcal{D}} \sigma^2 S \frac{\partial p}{\partial S} q \\ = \int_{\mathcal{D}} \frac{\sigma^2 S^2}{2}\frac{\partial^2 p}{\partial S^2} q + \frac{1}{\sqrt{2}}\int_{\{S=M\}} \frac{\sigma^2 S^2}{2}\frac{\partial p}{\partial M} q. \end{aligned}$$

The first term corresponds to the diffusion term in (11). The second term is an integral over the boundary $\{S = M\}$ of $\mathcal{D}$ and naturally enforces the boundary condition $\partial p/\partial M(t,S,S) = 0$. On Figure 2, we represent the price of a fixed strike call, obtained using the formulation (11), an implicit Euler scheme and $P1$ finite elements.

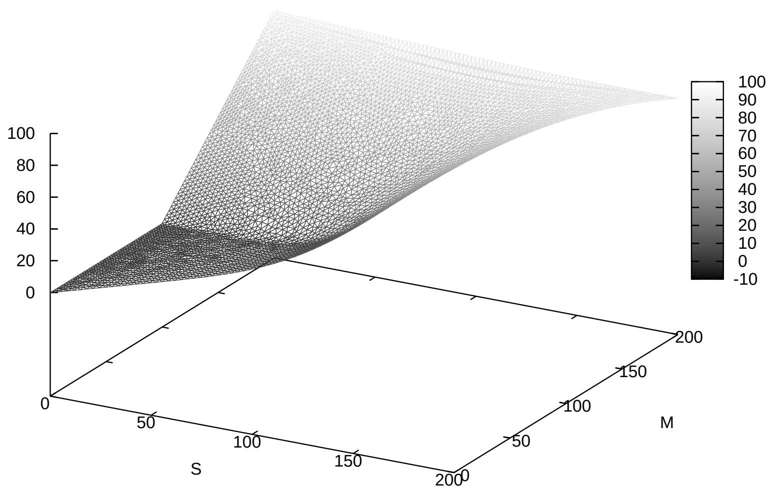

Figure 2: Price of a lookback option for a fixed strike call: $\phi(S,M) = (M - K)_+$. The parameters are: $\sigma = 0.3$, $r = 0.1$, $K = 100$, $T = 1$. Computations made with freeFEM++ [1].

Remark 8 (A posteriori error estimates) *A frequently mentioned advantage of the Monte Carlo methods is that they naturally provide* a posteriori *error bounds through an interval of confidence, typically built upon the central limit theorem. It is also possible to obtain such* a posteriori *error estimates in the framework of the finite*