Using a PDE solver for pricing multi-assets options

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In this projet a multidimensional PDE solver (ROC-HJ solver [2]) will be used in order to approximate a multi-asset option, such as European options. The aim of the project is to study the limit of a direct full grid approximation for some basic options in finance.

The ROC-HJ solver is a multi-dimensional PDE solver for solving linear PDE as well as Hamilon-Jacobi equations for stochastic optimal control. It is based on finite difference methods and Semi-Lagrangian methods. The semi-Lagrangian method is just an other form of finite difference methods. It is related to Markov chain approximations, coupled with a basic polynomial interpolation on a multi-dimensional cartesian grid. In this project a particular fully explicit semi-Lagrangian method will be considered.

1) In a first part we consider a multi-dimensional PDE of the form

$$-v_t - \frac{1}{2}Tr(\sigma\sigma^T D^2 v) - \sum_i r x_i v_{x_i} + rv \equiv -v_t + \mathcal{A}v = 0 \tag{1}$$

and

$$v(T,x) = \varphi(\sum \alpha_i x_i).$$

where $\alpha_i > 0$ are given, and $\varphi(y) := \max(K-y, 0)$ (put like form). First find coefficients (σ_{ij}) such that for u an analytical solution is known, of the form $v(t, x_1, \ldots, x_n) = w(t,y)$ where $y = \sum_i \alpha_i x_i$, and where v is solution of a one-dimensioal Black and Scholes equation. (The coefficients $\sigma_{ij} = x_i \beta_{ij}$ where β_{ij} are constants to be determined).

This particular problem is then used as a reference test case for multi-dimensional approximation.

2) A first order approximation method is then looked for. Let $u^n(x)$ represents an approximation of the value $v(t_n, x)$, where $t_n = nh$ and $h = T/N$ is a time step.

(i) We first look for u^n in the form:

$$u^N(x) = \varphi(x) \tag{2a}$$

and, for $n = N-1, \ldots, 0$:

$$u^n(x) = \frac{e^{-rh}}{2d} \sum_{k=1,\ldots,d} \sum_{\epsilon=\pm 1} u^{n+1}(x + b_k(x)h + \epsilon\sigma_k(x)\sqrt{\gamma_k h}) \tag{2b}$$

Show that for well chosen coefficients $\gamma_k > 0$ and vectors $b_k(x)$ (give simple equations they have to satisfy; $b_k(x)$ can be chosen independant of k, well chosen, and γ_k constant $\gamma_k = \gamma$, well chosen) we obtain a first order scheme in the sense that the following consistency error estimate holds, for any $v \in C^4([0,T] \times \mathbb{R}^+)$, with $v^n(x) = v(t_n, x)$, solution of (1),

$$\left|\frac{v^n(x) - (Sv^{n+1})(x)}{h}\right| \leq Ch.$$

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Deduce then that the true error is of first order in the sense that (assuming v sufficiently regular):

$$|u^n(x) - v^n(x)| \leq Ch.$$

(ii) The implemented scheme will be of the form

$$u^n(x) = \frac{e^{-rh}}{2d} \sum_{k=1,\dots,d} \sum_{\epsilon=\pm 1} [u^{n+1}](x + b_k(x)h + \epsilon\sigma_k(x)\sqrt{\gamma_k h}) \tag{3}$$

where $[\psi]$ denotes a $P1$ interpolation on a cartesian mesh (in particular $|[\psi](x) - \psi(x)| \leq \Delta x^2$ for any C^2 regular function ψ). Analyse again the consistency error as well a the true error of the scheme in this case, assuming that $\Delta x > 0$ is a uniform mesh step in any direction for the mesh grid. (Show a supplementary error term of the form $+\frac{\Delta x^2}{h}$, in the regular case).

3) The ROC-HJ library will be used in order to implement and test this scheme, check the accuracy order, CPU times, in the two-dimensional case.

4) The solver can then be test for higher-dimensions, to look for the limit of full grid approximation in this simple setting, in terms of CPU computational time, and in terms of memory limit capacity.

5) If time allows, higher order schemes will be studied, as well as approximation of other options (such as american).

References

[1] K. Debrabant et Epsen. R. Jakobsen, Preprint 2009. Voir `http://www.math.ntnu.no/~erj/publications.imf`

[2] The "ROC-HJ" solver: a parallel c++ library for Reachability and Optimal Control software using the Hamilton-Jacobi approach. `http://itn-sadco.inria.fr/software/ROC-HJ`