Tree Methods in Finance

Antonino Zanette
Finance Department
University of Udine

antonino.zanette@uniud.it

Tree methods

Probabilitic methods for option pricing

- Plain vanilla options(European and American).
- Exotic options.

Markov chain

Let $(X_n, n \geq 0)$ be a sequence of random variables taking values in a finite or countable set E. X_n is a Markov chain if:

$$\mathbb{P}\left(X_{n+1} = y | X_0 = x_0, \ldots, X_n = x_n\right) = \mathbb{P}\left(X_{n+1} = y | X_n = x_n\right).$$

The Markov chain is said *time homogenous* if $\mathbb{P}\left(X_{n+1} = y | X_n = x\right)$ does not depend on n. One then sets:

$$P(x, y) = \mathbb{P}\left(X_{n+1} = y | X_n = x\right).$$

The matrix $(P(x, y))_{x \in E, y \in E}$ is called the transition matrix of the Markov chain

Remark

∀x,y,P(x,y) ≥ 0 and, ∀x,$\sum_{y \in E} P(x, y) = 1$.

Random walks

Example

Binomial random walk Let $(X_i, i \geq 1)$ a sequence of i.i.d. random variables with $\mathbb{P}(X_i = \pm 1) = 1/2$. Then $S_n = X_1 + \cdots + X_n$ is a homogenous Markov chain with transition matrix $P(x, x+1) = P(x, x-1) = 1/2$, $P(x, y) = 0$ otherwise.

Example

Trinomial random walk. Let $(X_i, i \geq 1)$ a sequence of i.i.d. random variables $\mathbb{P}(X_i = \pm 1) = \lambda/2$ and $\mathbb{P}(X_i = 0) = 1 - \lambda$, with $0 < \lambda \leq 1$. The transition matrix is given by $P(x, x+1) = P(x, x-1) = \lambda/2$, $P(x, x) = 1 - \lambda$, $P(x, y) = 0$ otherwise.

Example

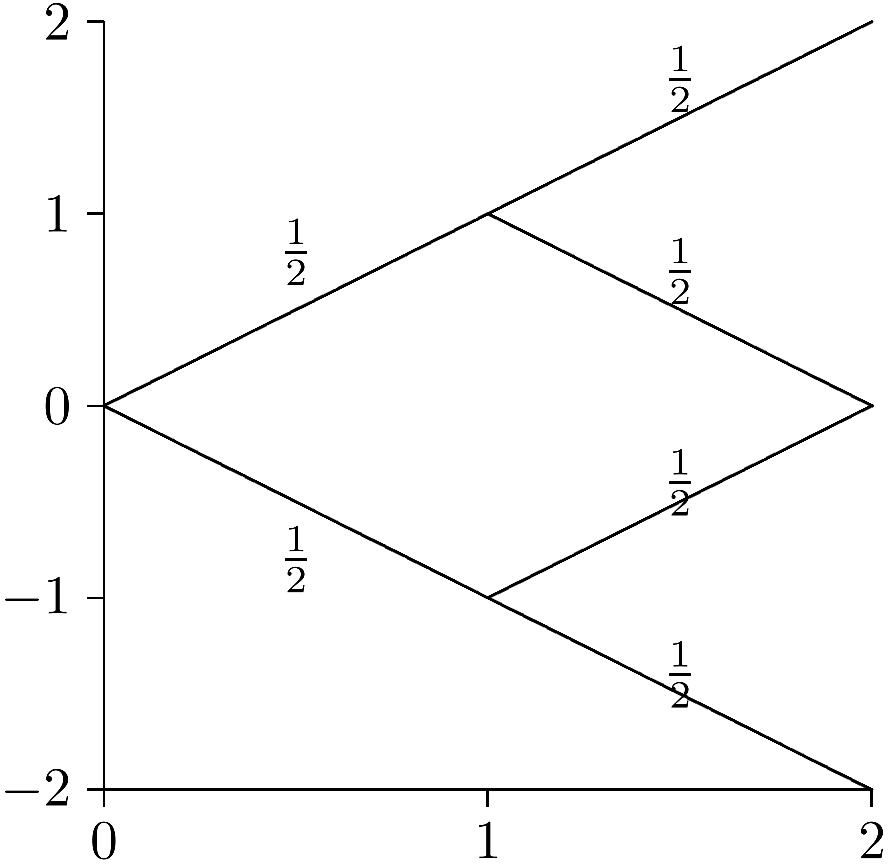
Random walk of Cox Ross-Rubinstein. Let $(U_n, n \geq 0)$ a sequence of i.i.d. random variables with $\mathbb{P}(U_n = u) = p$, $\mathbb{P}(U_n = d) = 1 - p$ and $0 < p < 1$, u and d real numbers.Let $S_0 = x$ and :

$$S_{n+1} = S_n U_{n+1}.$$

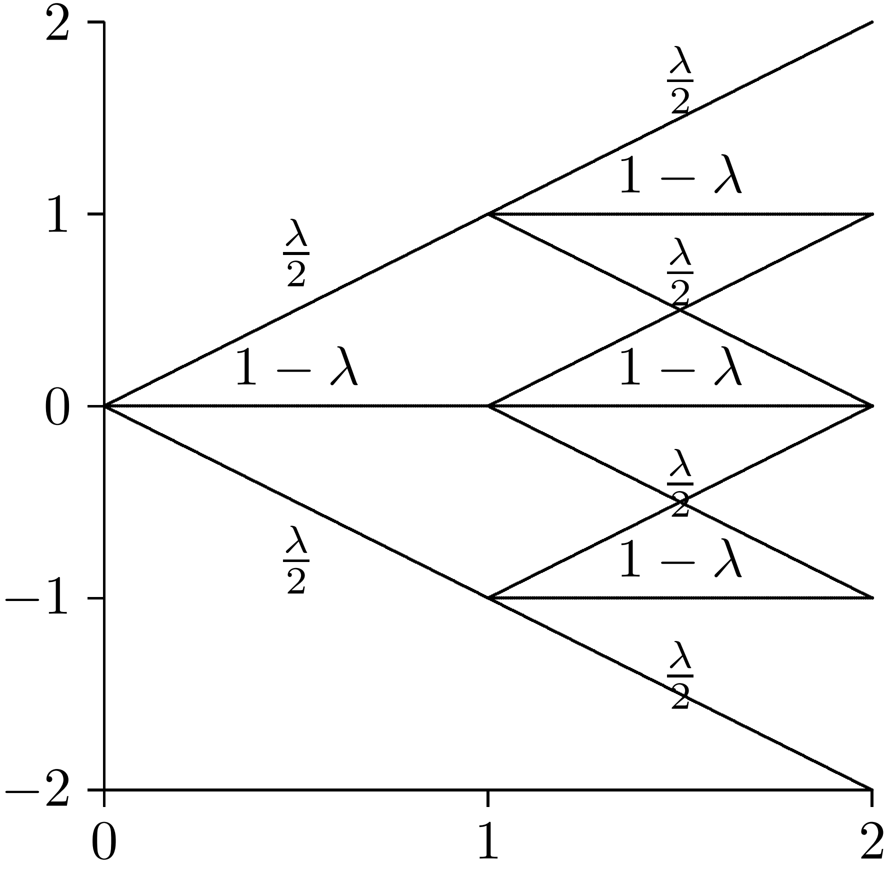
Let $S_n = x \prod_{i=1}^n U_i$. S_n is a homogenous Markov chain with transition matrix:

$$\begin{array}{lllll}
P(x, xu) & = & \mathbb{P}\left(S_{n+1} = xu | S_n = x\right) & = & p \\
P(x, xd) & = & \mathbb{P}\left(S_{n+1} = xd | S_n = x\right) & = & 1 - p \\
P(x, y) & = & 0 & \text{otherwise} &
\end{array}$$

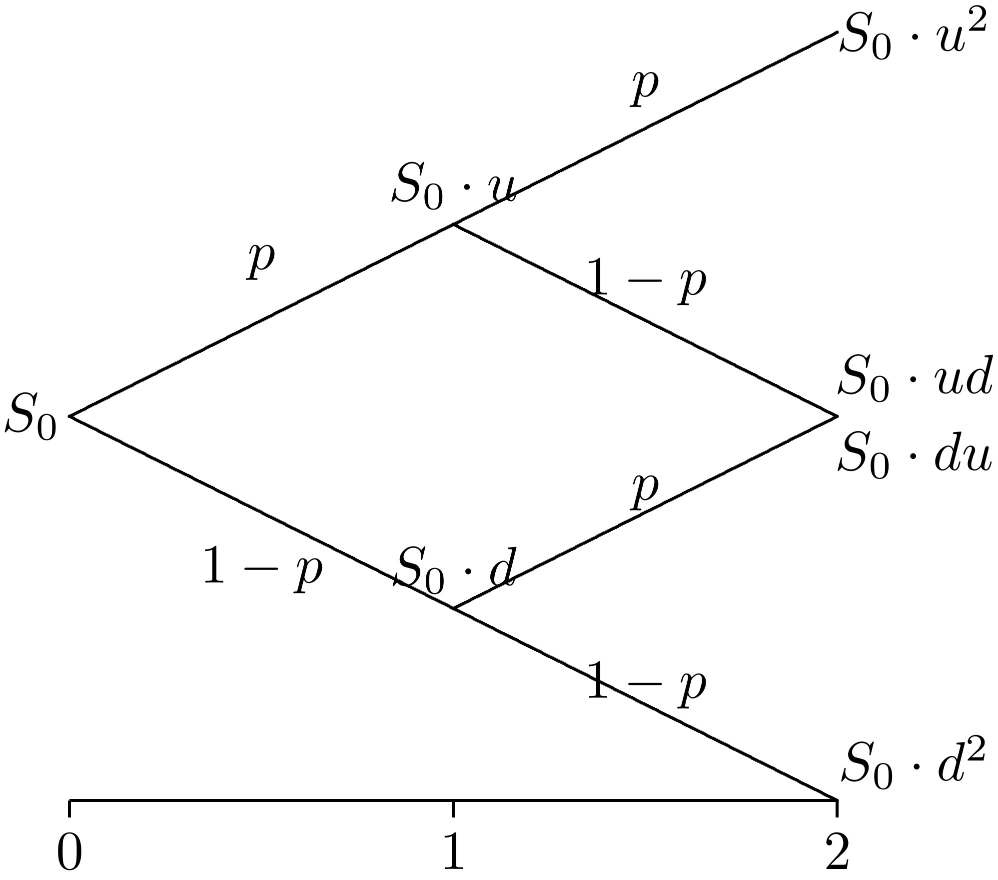
Binomial random walk



Trinomial random walk



Random walk of Cox Ross-Rubinstein



More general definition of Markov chain

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(\mathcal{F}_n, n \geq 1)$ *a filtration.* A process $(X_n, n \geq 1)$ taking values in a finite or countable set E is a Markov chain with the family of transition matrices (P_n) if :

- For all n, X_n is $\mathcal{F}_n$-misurable.
- For any bounded function f, we :

$$\mathbb{E}\left(f(X_{n+1})|\mathcal{F}_n\right) = \mathbb{E}\left(f(X_{n+1})|X_n\right) = \sum_{y \in E} P(x, y) f(y)$$

European option pricing

In the discrete models the European price could be written :

$$V_0 = \mathbb{E}\left(\frac{1}{(1+R)^N} f(X_N)\right),$$

where $(X_n, n \geq 0)$ is a Markov chain et r the intereste rate.

$$V_n = \mathbb{E}\left(\frac{1}{(1+R)^{N-n}} f(X_N) | \mathcal{F}_n\right).$$

Dynamic programming

Proposition

Let $f(x)$ a bounded function. Let $(X_n, n \geq 0)$ a Markov chain with transition matrix P.

Problem: compute

$$\mathbb{E}\left(f(X_N)\right).$$

Let u the unique solution of:

$$\left\{ \begin{array}{l} u(N,x) = f(x), \\ u(n,x) = \sum_{y \in E} P(x,y) u(n+1,y). \end{array} \right. \tag{1}$$

Then :

$$\mathbb{E}\left(f(X_N) | \mathcal{F}_n\right) = u(n, X_n),$$

In particular:

$$\mathbb{E}\left(f(X_N) | \, X_0 = x\right) = u(0,x).$$

Corollary

Let $f(x)$ a bounded function. Let $(X_n, n \geq 0)$ a Markov chain with transition matrix P.

Problem: compute

$$\mathbb{E}\left(\frac{1}{(1+R)^N} f(X_N)\right).$$

Let u the unique solution of:

$$\left\{ \begin{array}{l} v(N,x) = f(x), \\ v(n,x) = \sum_{y \in E} \frac{P(x,y)}{1+R} u(n+1,y). \end{array} \right. \tag{2}$$

Then :

$$\mathbb{E}\left(\frac{1}{(1+R)^{N-n}} f(X_N) | \mathcal{F}_n\right) = v(n, X_n),$$

In particular:

$$\mathbb{E}\left(\frac{1}{(1+R)^N} f(X_N) \middle| X_0 = x\right) = v(0,x).$$

Example

Binomial random walk S_n is a Markov chain with transition matrix $P(x, x+1) = P(x, x-1) = 1/2$. We have $u(0, x)) = \mathbb{E}(f(S_N)|S_0 = x)$, where u satisfies :

$$\begin{cases} u(N, x) = f(x), \\ u(n, x) = \frac{1}{2}u(n+1, x+1) + \frac{1}{2}u(n+1, x-1). \end{cases}$$

Example

Trinomial random walk S_n is a Markov chain with transition matrix $P(x, x+1) = P(x, x-1) = \lambda/2$, $P(x, x) = 1 - \lambda$.We have $u(0, x)) = \mathbb{E}(f(S_N)|S_0 = x)$, where u satisfies :

$$\begin{cases} u(N, x) = f(x), \\ u(n, x) = \frac{\lambda}{2}u(n+1, x+1) + (1-\lambda)u(n+1, x) + \frac{\lambda}{2}u(n+1, x-1). \end{cases}$$

Example

Cox-Ross-Rubinstein random walk

$$V_0 = \mathbb{E}\left(\frac{1}{(1+R)^N} f(S_N)\right),$$

we have $V_0 = v(0, S_0)$, where v satisfies :

$$\begin{cases} v(N,x) = f(x), \\ v(n,x) = \dfrac{p}{1+R} v(n+1, xu) + \dfrac{1-p}{1+R} v(n+1, xd). \end{cases}$$

$p = q$ Risk neutral measure

Tree algorithm
European option in discrete model

```
/*Risk neutral probability*/
pu=((1+R)-d)/(u-d);
pd=1-pu;

/* Condition at maturity*/
for (j=0;j<=N;j++)
    P[j]=MAX(0.,K-S_0*pow(u,N-j)*pow(d,j));

/* Backward induction */
  for (i=1;i<=N;i++)
    for (j=0;j<=N-i;j++)
 P[j]=pow(1.+R,-1.)*(pu*P[j]+pd*P[j+1]);

/*  E(f(S_N)|S_0=x) is given in P[0] */
```

European options in Black-Scholes continous model

Problem: compute European put options

$$P = e^{-rT}\mathbb{E}_Q\left[(K - S_T)_+\right].$$

where (S_t) is a geometric brownian motion.

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, S_0 = x$$

$$S_T = xe^{(r-\frac{1}{2}\sigma^2)T+\sigma B_T}$$

G.M.B. and random walk

Let $S_{0\leq t\leq T}$ a g.b.m. Then :

Proposition

Let $(X_i, i \geq 1)$ *a sequence of i.i.d. random variables* $\mathbb{P}(X_i = u) = q$ *e* $\mathbb{P}(X_i = d) = 1 - q$. *Let* $S_0 = x$ *and :*

$$S_{n+1} = S_n X_{n+1}.$$

Let $\Delta T = T/N$ *the time discretization step,*

$$u = e^{\sigma\sqrt{\Delta T}}$$

$$d = e^{-\sigma\sqrt{\Delta T}}$$

$$q = \frac{e^{r\Delta T} - d}{u - d}$$

Then with this choiche of parameters, S_N *converges in law* S_T*., i.e. if* f *is continous and bounded function,* $\mathbb{E}_q\Big[f(S_N)\Big]$ *approximates* $\mathbb{E}_Q\Big[f(S_T)\Big]$.

$$P = e^{-rT}\mathbb{E}_Q\ \Big[(K - S_T)_+\ \Big].$$

is approximated by $e^{-rT}\mathbb{E}_q\Big[(K - S_N)_+\Big]$

European price with CRR random walk

To compute $e^{-rT}\mathbb{E}_q\Big[f(S_N)\Big]$ we have to solve:

$$\left\{\begin{array}{l} u(N\Delta T, x) = f(x), \\ u(n\Delta T, x) = e^{-r\Delta T}\Big[qu\Big((n+1)\Delta T, xu\Big) + (1-q)u\Big((n+1)\Delta T, xd\Big)\Big]. \end{array}\right.$$

For the put option, $f(S_N) = (K - S_N)_+$ we obtain the following :

CRR Algorithm for put option in BS model

```
/*Up-Down factors*/
h=T/N;
u=exp(sigma*sqrt(h));
d=1./u;

/*Risk neutral probability*/
pu=(exp(r*h)-d)/(u-d);
pd=1-pu;

/* Condition at maturity */
for (j=0;j<=N;j++)
    P[j]=MAX(0.,K-x*pow(u,N-j)*pow(d,j));

/* Backward induction */
  for (i=1;i<=N;i++)
    for (j=0;j<=N-i;j++)
 P[j]=exp(-r*k)*(pu*P[j]+pd*P[j+1]);

/* E(f(S_N)|S_0=x) is given in P[0] */
```

Let $S_0 = 1$ and $h = \Delta T$. for $\lambda \in \mathbb{R}$

$$\begin{aligned} & E_q\left[\exp\left(i\lambda \ln S_N\left(h\right)\right)\right] \\ = \quad & E_q\left[\exp\left(i\lambda \ln \prod_{n=0}^{N-1} \frac{S_{n+1}\left(h\right)}{S_n\left(h\right)}\right)\right] \\ = \quad & E_q\left[\exp\left(i\lambda \ln \frac{S_1\left(h\right)}{S_0}\right)\right]^N \\ = \quad & \left(q \exp\left(i\lambda\sigma\sqrt{h}\right) + (1-q)\exp\left(-i\lambda\sigma\sqrt{h}\right)\right)^N \end{aligned}$$

and since $q = \frac{e^{rh}-d}{u-d} \sim \frac{1}{2} + \frac{\left(r-\frac{\sigma^2}{2}\right)}{2\sigma}\sqrt{h}$

$$\begin{aligned} E_q\left[\exp\left(i\lambda \ln S_N\left(h\right)\right)\right] \quad & \sim \quad \left(1 + \left[i\lambda\left(r - \tfrac{\sigma^2}{2}\right) - \lambda^2 \tfrac{\sigma^2}{2}\right] \tfrac{T}{N}\right)^N \\ & \to \quad \exp\left(\left[i\lambda\left(r - \tfrac{\sigma^2}{2}\right) - \lambda^2 \tfrac{\sigma^2}{2}\right] T\right) \\ & = \quad E_Q\left[\exp\left(i\lambda\left(\left(r - \tfrac{\sigma^2}{2}\right) T + \sigma B_T\right)\right)\right] \\ & = \quad E_Q\left[\exp\left(i\lambda \ln S_T\right)\right] \end{aligned}$$

$$S_N\left(h\right) \to S_T$$

in law for $N \to \infty$.

Local consistency

Kushner's theorem says that the local consistency conditions, that is the matching at the first order of the first and second moments of the logarithmic increments of the approximating chain with those of the continuous-time limit grant the convergence of the expectations of smooth functionals.

DISCRETE CONTINUOS

$S_{(n+1)\Delta T} = S_{n\Delta T} X_{(n+1)\Delta T}$ $dS_t = S_t dt + \sigma S_t dB_t$

$$qu + (1-q)d = \mathbb{E}_q[\frac{S_{(n+1)\Delta T}}{S_{n\Delta T}}] \approx \mathbb{E}_Q[\frac{S_{t+\Delta t}}{S_t}] = e^{r\Delta T}$$

$$qu^2 + (1-q)d^2 = E_q[(\frac{S_{(n+1)\Delta T}}{S_{n\Delta T}})^2] \approx \mathbb{E}_Q[(\frac{S_{t+\Delta t}}{S_t})^2] = e^{2r\Delta t + \sigma^2 \Delta t}$$

Brownian motion and discrete random walk

Proposition

Let $(X_i, i \geq 1)$ *a sequence of i.i.d. random variables* $\mathbb{P}(X_i = \pm 1) = 1/2$.
Let $S_n = X_1 + \cdots + X_n$.
Let $\Delta T = T/N$ *the time discretization step. Set:*

$$B_N = \sqrt{\Delta T} S_N.$$

Then , B_N *converges in law to* B_T. *Then*

$$\mathbb{E}_{\mathbb{P}}\Big[f(B_N)\Big] \sim \mathbb{E}_{\mathbb{P}}\Big[f(B_T)\Big]$$

.

In particular for the put options case, $h(B_T) = (K - S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma B_T})_+$ is approximated by $h(B_N) = (K - S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma B_N})_+$

$$\begin{cases} u(N\Delta T, x) = f(x), \\ u(n\Delta T, x) = e^{-r\Delta T}\Big[\dfrac{1}{2}u((n+1)\Delta T, x+1) + \dfrac{1}{2}u((n+1)\Delta T, x-1)\Big]. \end{cases}$$

M.B.G. and Kamrad Ritchken tree

Kamrad and Ritchken choose to take a symmetric 3-points approximation to $\log\left(\frac{S_{n\Delta T}}{S_0}\right)$

$$\log S_{(n+1)\Delta T} = \begin{cases} \log S_{n\Delta T} + \log u & \text{with} \quad p_u \\ \log S_{n\Delta T} & \text{with} \quad p_m \\ \log S_{n\Delta T} + \log d & \text{with} \quad p_d \end{cases} \tag{3}$$

In order to obtain the convergence, they match the 2 first moments of $\log\left(\frac{S_{n\Delta T}}{S_0}\right)$. By replacing $\log u$ by $\lambda\sigma\sqrt{\Delta T}$ this leads to

$$\begin{aligned} p_u &= \frac{1}{2\lambda^2} + \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\Delta T}}{2\lambda\sigma} \\ p_m &= 1 - \frac{1}{\lambda^2} \\ p_d &= \frac{1}{2\lambda^2} - \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\Delta T}}{2\lambda\sigma} \end{aligned}$$

The parameter λ appears as a free parameter of the geometry of the tree, which may be useful for some purposes. It is called the stretch parameter. The value $\lambda = 1.22474$ which corresponds to $p_m = \frac{1}{3}$ is reported to be a good choice for an at the money Call (or Put) option.

Trinomial algorithm of Kamrad Ritchen

To compute $e^{-rT}\mathbb{E}_q\Big[f(S_N)\Big]$, one has to solve :

$$\begin{cases} u(N\Delta T, x) = f(x), \\ u(n\Delta T, x) = e^{-r\Delta T}\Big[p_u u\Big((n+1)\Delta T, xu\Big) + p_m u\Big((n+1)\Delta T, x\Big) + p_d u\Big((n+1)\Delta T, xd\Big)\Big]. \end{cases}$$

In particular if, $f(S_N) = (K - S_N)_+$ then :

Trinomial model of Kamrad Ritchken

```
/*Up-Down factors*/
h=T/N;
lambda=1.22474;
u=exp(lambda*sigma*sqrt(h));
d=1./u;
/*Probabilities*/
z=r-SQR(sigma)/2.;
pu=(1./(2.*SQR(lambda))+z*sqrt(h)/(2.*lambda*sigma));
pm=(1.-1./SQR(lambda));
pu=(exp(r*k)-d)/(u-d);

/* Condition at maturiy */
for (j=0;j<=2*N;j++)
    P[j]=MAX(0.,K-x*pow(u,N-j)  induction */
  for (i=1;i<=N;i++)
    for (j=0;j<=2*N-2*i;j++)
 P[j]=exp(-r*k)*(pu*P[j]+pm*P[j+1]+pd*P[j+2]);

/* E(f(S_N)) is given in P[0] */
```

American option

The value at time $t = 0$ of an American Put option on the risky underlying, with maturity T and payoff function $\psi(x) = (K - x)_+$, is, in the connection with Optimal Stopping Theory, given by:

$$v(0, s_0) = \sup_{\tau \in \mathcal{T}_{0,T}} E_Q \left(e^{-r\tau} \psi(S_\tau) \right)$$

where $\mathcal{T}_{0,T}$ is the set of all stopping times with values in $[0, T]$.

CRR algorithm

$$u = e^{\sigma\sqrt{\Delta T}}$$

$$d = e^{-\sigma\sqrt{\Delta T}}$$

$$q = \frac{e^{r\Delta T} - d}{u - d}$$

The price of an American out v_0 is obtained solving:

$$\begin{cases} v(N, x) = (K - x)_+, \\ v(n, x) = MAX\left(e^{-r\Delta T} q v(n+1, xu) + (1-q) v(n+1, xd), (K - x)_+\right). \end{cases}$$

Binomial algorithm
American Put option

```
/*Up-Down factors*/
h=T/N;
u=exp(sigma*sqrt(h));
d=1./u;
/*Risk neutral probability*/
pu=(exp(r*h)-d)/(u-d);
pd=1-pu;

/*Intrinsic values*/
for (j=0;j<=2*N;j++)
    InV[j]=max(0.,K-xpow(u,N-j));
/*Terminal condition*/
for (j=0;j<=N;j++)
    P[j]=InV[2*j];

/*Dynamic programming*/
  for (i=1;i<=N;i++)
    for (j=0;j<=N-i;j++)
 P[j]=MAX(exp(-r*k)*(pu*P[j]+pd*P[j+1]),InV[i+2*j]);

  /* Price in P[0] */
```

Greek

Delta

$$\Delta = \frac{\partial C}{\partial x} = \frac{v(\Delta T, xu) - v(\Delta T, xd)}{xu - xd}$$

Gamma

Let $h = \frac{1}{2}(xu^2 - xd^2)$. Then

$$\Gamma = \frac{\partial^2 C}{\partial x^2} = \frac{\frac{\left(v(2\Delta T, xu^2) - v(2\Delta T, x)\right)}{\left(xu^2 - x\right)} - \frac{\left(v(2\Delta T, x) - v(2\Delta T, xd^2)\right)}{\left(x - xd^2\right)}}{h}$$