Convergence results

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Asymptotics results for CRR

Diener-Diener(MF 2002)

$$P_N^{\mathrm{CRR}} = P_{\mathrm{BS}} - \frac{Ke^{-rT}}{N} e^{-\frac{d_2^2}{2}} \sqrt{\frac{2}{\pi}} \left[\kappa_N(\kappa_N - 1)\sigma\sqrt{T} + D\right] + \mathcal{O}\left(\frac{1}{N^{3/2}}\right),$$

where

- D is a constant
- κ_N denotes the fractional part of $\frac{\log(\frac{K}{s_0})}{2\sigma}\sqrt{\frac{N}{T}} - \frac{N}{2}$.

In the at the money case i.e. that $K = s_0$, then one has $\kappa_N = 0$, and then for $N = 2m$ even, the Strike K coincides with the $(m+1)$-th final node of the (CRR) tree, and one has

$$P_{2m}^{\mathrm{CRR}} = P_{\mathrm{BS}} - \frac{DKe^{-rT}}{2m} e^{-\frac{d_2^2}{2}} \sqrt{\frac{2}{\pi}} + \mathcal{O}\left(\frac{1}{m^{3/2}}\right).$$

Richardson extrapolation

$$P^{\mathrm{CRR}}_{2m} = P_{\mathrm{BS}} - \frac{DKe^{-rT}}{2m} e^{-\frac{d_2^2}{2}} \sqrt{\frac{2}{\pi}} + \mathcal{O}\left(\frac{1}{m^{3/2}}\right).$$

In the approximation $2P^{\mathrm{CRR}}_{4m} - P^{\mathrm{CRR}}_{2m}$ of P_{BS} obtained using Richardson extrapolation, the term with order $1/N$ vanish.
As a consequence the rate of convergence of

$$2P^{\mathrm{CRR}}_{4m} - P^{\mathrm{CRR}}_{2m}$$

to

$$P_{\mathrm{BS}}$$

is

$$\mathcal{O}\left(\frac{1}{m^{3/2}}\right)$$

which explains the good numerical behaviour of this approximation in at the money case.

- The BIR method of Gaudenzi-Pressacco (DEF 2003), is Binomial Interpolated with Richardson extrapolation. The logic of the BI approach then is to create a set of computational options, each one with a computational Strike lying exactly on a final node of the tree. The value of the option with the contractual Strike is then obtained by interpolation of the values of the computational options. Furthermore, it is possible to exploit the recovered regularity a two-points Richardson extrapolation : this leads to the BIR method.
- The BBSR method introduced by Broadie and Detemple (RFS 96) replaces at any node of the last but one time before maturity, the binomial continuation value with the Black-Scholes European one.
- The Adaptive Mesh Model AMM introduced by Figlewski and Gao (JFE 99) resorts to refining the grid around the strike and at maturity.
- Moments and Strike Matching Tree MSM of Jourdain and Zanette

MSM

Matching Moments and Strike

When N is even, if the Strike K is equal to one of the final nodes $(s_0 e^{(2k-N)\sigma\sqrt{\Delta T}})_{0 \leq k \leq N}$ of the tree, then one has $\kappa_N = 0$ (so that the term with order $1/N$ vanish in Richardson extrapolation).
This justifies our interest in trees such that the Strike coincides with one of the final nodes.
MSM is based on two matching conditions :

- **Strike Condition** : the Strike K is equal to one of the final nodes of the tree.
- **Local Consistency Condition** : the tree is consistent with the Black-Scholes model in the limit of an infinite step number.

Strike Matching

The Strike K is equal to one of the final nodes of the tree .

Instead of requiring $u = \frac{1}{d}$ as in the Cox-Rubinstein model, we propose in the MSM method to ensure that the Strike K is the $(k+1)$-th (with $k \in \{1, \dots, N-1\}$) final node of the tree :

$$K = s_0 u^k d^{N-k}$$

which also writes

$$\frac{1}{N} \log(\frac{K}{s_0}) = q \log u + (1-q) \log d$$

where $q = \frac{k}{N}$.

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Moments Matching : Local consistency condition

The two first moments matching conditions read

$$\begin{cases} p_u \log u + (1 - p_u) \log d = (r - \frac{1}{2}\sigma^2)\Delta T \\ p_u (\log u)^2 + (1 - p_u)\,(\log d)^2 = \sigma^2 \Delta T. \end{cases}$$

Linear System

We want to find $(\log u, \log d, p_u)$ with $\log u > \log d$ and $p_u \in]0,1[$ solving the following system of equations with unknowns (x, y, p)

$$\begin{cases} qx + (1-q)y = \alpha \\ px + (1-p)y = \beta \\ px^2 + (1-p)y^2 = \gamma \end{cases}$$

where $\alpha = \frac{1}{N}\log(\frac{K}{s_0})$, $\beta = (r - \frac{\sigma^2}{2})\Delta T$ and $\gamma = \sigma^2 \Delta T$, $q = \frac{k}{N}$.

MSM

For $q = k/N$ with $k \in \{1, \dots, N-1\}$, the solutions $(p_i, x_i, y_i)_{i \in \{1,2\}}$ of the system provide two trees :

- **the first one** with $p_u = p_1$, $\log u = x_1$ and $\log d = y_1$ is such that the $(k+1)$-th final node $s_0 u^k d^{N-k}$ of the tree is equal to the Strike K,
- **the second one** with $p_u = 1 - p_2$, $\log u = y_2$ and $\log d = x_2$ is such that the $(N-k)$-th final node $s_0 u^{N-k} d^k$ of the tree is equal to the Strike K.

When N is even and $k = N/2$, both trees are equal.

Remark The tree is **recombining** since u and d remain constant within the tree but **not symmetric** .

Tree MSM parameters

For any $k \in \{1, \ldots, N-1\}$, there is a unique MSM tree with N steps and parameters $(p_u, \log u, \log d)$ (with $p_u \in]0,1[$ and $\log u > \log d$) satisfying the two first moment matching conditions and such that the strike K is equal to the $(k+1)$-th final node of the tree : $K = s_0 u^k d^{N-k}$.

$$\begin{cases} p_u = \frac{(\alpha-\beta)^2 + 2q(\gamma-\beta^2) - (\alpha-\beta)\sqrt{(\alpha-\beta)^2 + 4q(1-q)(\gamma-\beta^2)}}{2((\alpha-\beta)^2 + (\gamma-\beta^2))} \\ \log u = \alpha + (1-q)\frac{\beta-\alpha}{p_u - q} \\ \log d = \alpha - q\frac{\beta-\alpha}{p_u-q} \end{cases} .$$

where

$$\begin{cases} q = \frac{k}{N} \\ \alpha = \frac{1}{N}\log\left(\frac{K}{s_0}\right) \\ \beta = \left(r - \frac{\sigma^2}{2}\right)\frac{T}{N} \\ \gamma = \sigma^2 \frac{T}{N} \end{cases} .$$

Asymptotics results for MSM tree

Following Diener-Diener(MF 2002) results, we prove in the European case, when N is even and $k = N/2$,

Lemma 1 *As m tends to infinity,*

$$P_{2m} = P_{\mathrm{BS}} + \frac{C_P}{m} + \mathcal{O}\left(\frac{1}{m^{3/2}}\right)$$
$$\delta_{2m} = \delta_{\mathrm{BS}} + \frac{C_\delta}{m} + \mathcal{O}\left(\frac{1}{m^{3/2}}\right),$$

.

with $C_P = Ke^{-rT}\left(\eta\mathcal{N}(-d_2) + \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\left(\frac{d_1^3 + d_2 - d_2^3 - d_1}{8} + \nu - \mu\right)\right)$

and

with $C_\delta = \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left(\frac{d_1^3 - d_1}{8} + \nu\right)$

MSMR method

We have not been able to obtain asymptotic expansions for the price and delta of the American Put Option in the MSM tree with $2m$ steps and $k = m$. Nevertheless, because of the expansions obtained for the European Put option, we propose to use Richardson extrapolation even when computing the price and the delta of the American Put. This leads to MSMR method:

Price

$$2P^{\mathrm{A}}_{4m} - P^{\mathrm{A}}_{2m}$$

Delta

$$2\frac{v^{\mathrm{A}}_{4m}\left(1, s_0 u_{4m}\right) - v^{\mathrm{A}}_{4m}\left(1, s_0 d_{4m}\right)}{s_0\left(u_{4m} - d_{4m}\right)} - \frac{v^{\mathrm{A}}_{2m}(1, s_0 u_{2m}) - v^{\mathrm{A}}_{2m}(1, s_0 d_{2m})}{s_0(u_{2m} - d_{2m})}.$$

Numerical Results

American Put Options in Black-Scholes Model

- We compare our algorithm MSMR with the procedures we have mentionned (CRR,BIR,BBSR,AMM) for pricing and hedging American Put options in the Black-Scholes model.
- A sample of 5.000 options was extracted randomly from a population whose parameters are the ones used in Gaudenzi-Pressacco-Zanette-Ziani[04].
- Several options of the sample have been discarded for various reasons. 4.443 options survived.
- For each option of the sample a neutral reliable price benchmark was computed as the CRR at 96.000 steps.
- the errors for the whole sample are summarized by the Mean Relative Error (MRE) and by the Squared Root of the Mean Quadratic Relative Error (RMSRE).

Price

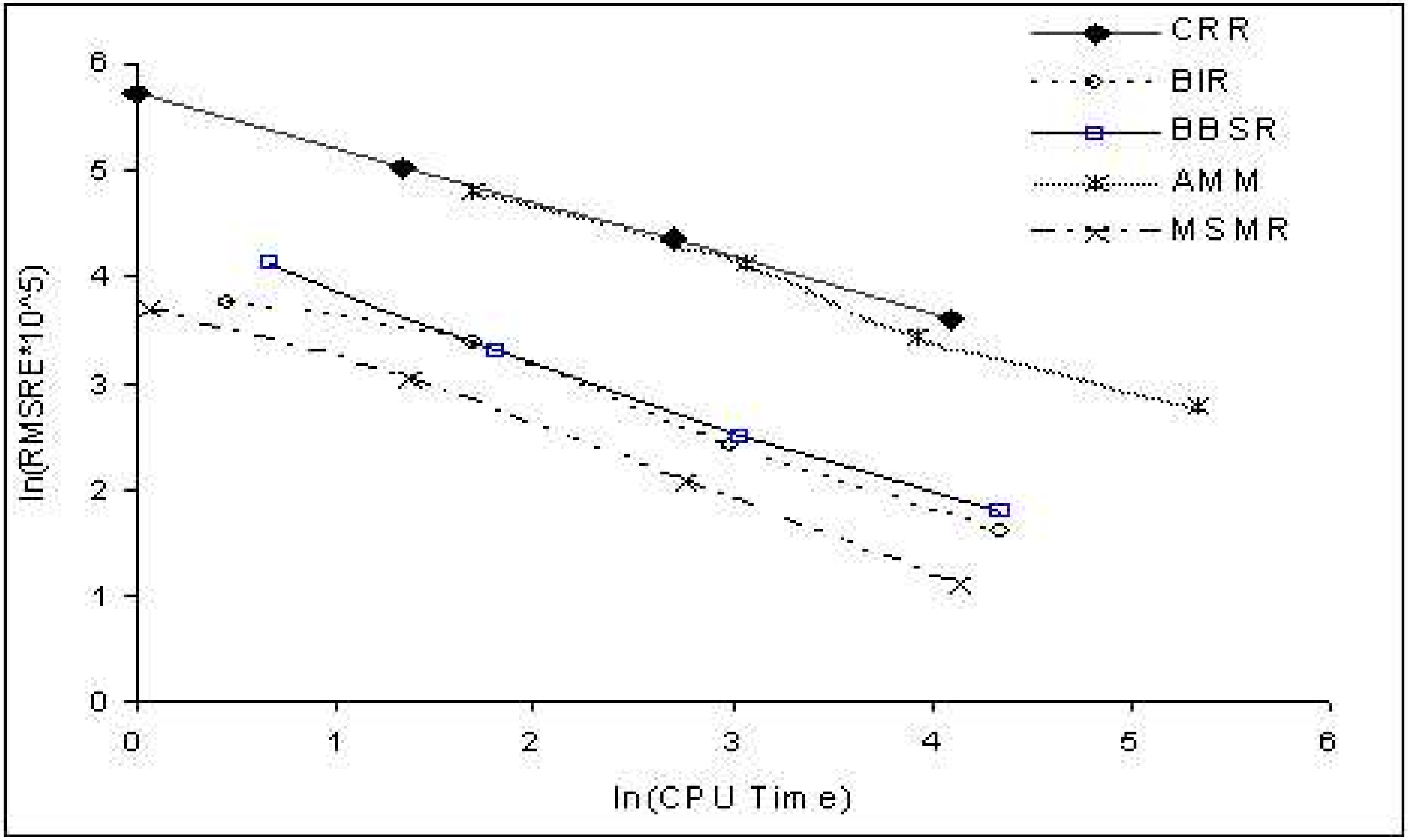

Figure 1: Price: speed-precision efficiency for tfhe 4.443 samples.

Delta

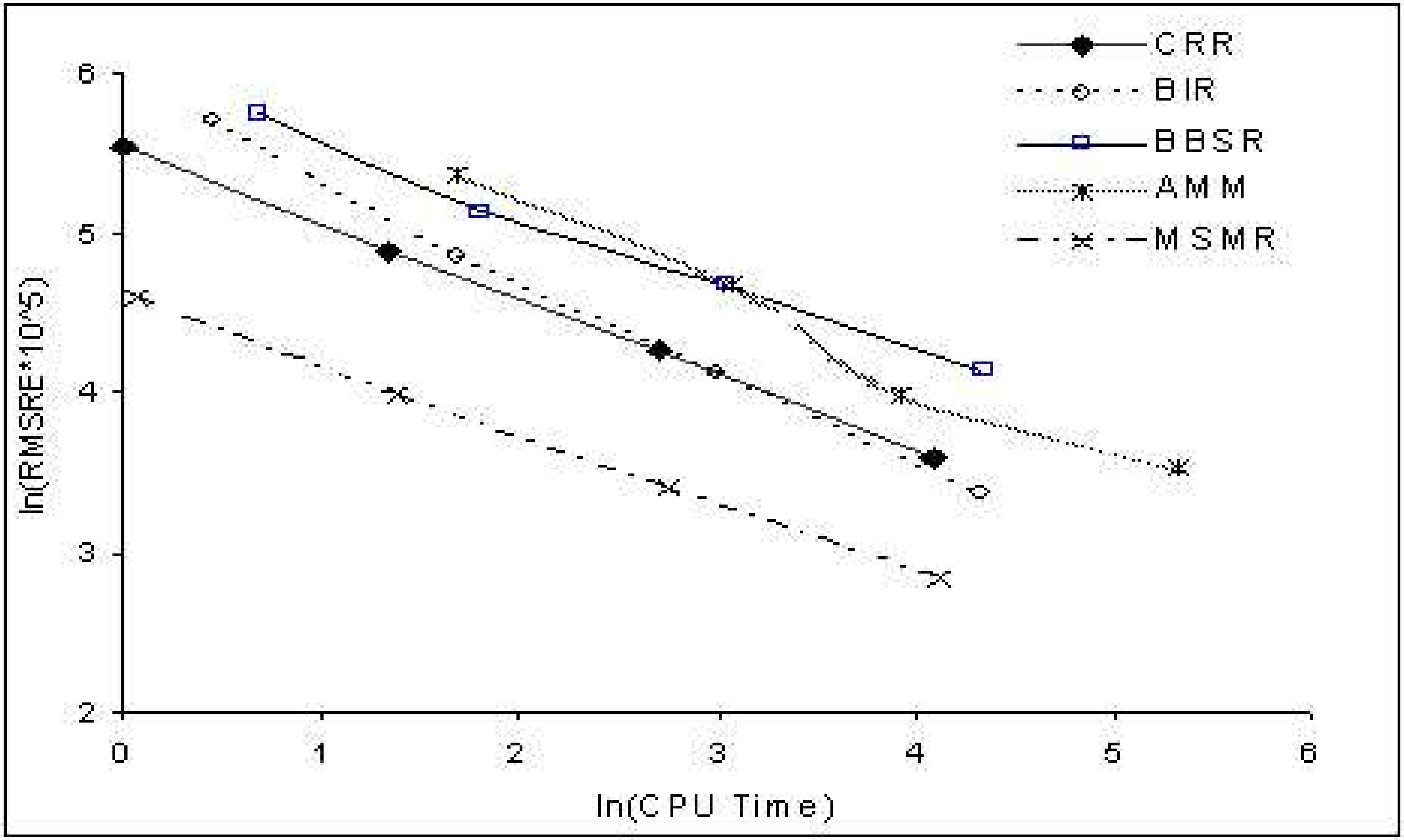

Figure 2: Delta: speed-precision efficiency for the 4.443 samples.

Conclusion

- New binomial lattice method : Moments and Strike Matching Tree(MSM)
- The method is very easy to implement since the parameters of the tree are explicitly given.
- Asymptotic expansions are obtained for the MSM European Put price and delta.
- Speed-Precision Efficiency of MSMR in the American Put case.
- Download HTTP://WWW.INRIA.FR/RRRT/RR-5569.HTML