TP : Tree methods

We consider the CRR model

$$S_{n+1} = S_n U_{n+1}$$

$(U_n, n \geq 0)$ a sequence of i.i.d. random variables with $\mathbf{P}(U_n = u) = q$, $\mathbf{P}(U_n = d) = 1 - q$. We choose u and d in order to guarantee the convergence in distribution to the Black-Scholes model. Let $\Delta T = T/N$ the time discretization step,

$$u = e^{\sigma\sqrt{\Delta T}}$$

$$d = e^{-\sigma\sqrt{\Delta T}}$$

$$q = \frac{e^{r\Delta T} - d}{u - d}$$

We consider the following parameters of the Black-Scholes model : $r = 0.1, S_0 = K = 100$, $\sigma = 0.2$, $T = 1$
This is the algorithm for pricing an European option with CRR method in the BS model :

```
function[y]=price(T,N,sigma,r,K,x)
//up and down factors
h=T/N;
u=exp(sigma*sqrt(h));
d=1/u;

//risk neutral probability
pu=(exp(r*h)-d)/(u-d);
pd=1-pu;
P=zeros(1,N+1);

//maturity
for j=0:N do
P(j+1)=max(0,K-x*u^(N-2*j));
end;


//Backward induction
for i=1:N do
for j=0:(N-i) do
P(j+1)=exp(-r*h)*(pu*P(j+1)+pd*P(j+2));
end;
end;
y=P(1);
endfunction.



function [Y]=main()
  r=0.1;   // r
  sigma=0.2 // sigma
  S0=100;    // S0
  T=1;     // maturity
  K=100;     // strike
  N =100;    // Number of Step
  Y=price(T,N,sigma,r,K,S0);
endfunction
```

1. **Plot a graph for $(P)_n$ with n even and n odd in at-the-money and not at-the-money case.**

2. **Compute the complexity of the program.**
3. **Modify the program in order to compute the delta and the gamma parameters.**
4. **Modify the program in order to compute American options.**

Let $(X_i, i \geq 1)$ a sequence of i.i.d. random variables $\mathbf{P}(X_i = \pm 1) = 1/2$.
Let $S_n = X_1 + \cdots + X_n$.
Let $\Delta T = T/N$ the time discretization step. Set :

$$B_N = \sqrt{\Delta T} S_N.$$

Then , B_N converges in distribution to B_T.Then

$$\mathbf{E}_{\mathbf{P}}\Big[f(B_N)\Big] \sim \mathbf{E}_{\mathbf{P}}\Big[f(B_T)\Big]$$

.

1. **Write the algorithm for the random walk discretization of B.M. in the European and American cases.**

Kamrad and Ritchken choose to take a symmetric 3-points approximation to $\log\left(\frac{S_{n\Delta T}}{S_0}\right)$

$$\log S_{(n+1)\Delta T} = \begin{cases} \log S_{n\Delta T} + \log u & \text{with} \quad p_u \\ \log S_{n\Delta T} & \text{with} \quad p_m \\ \log S_{n\Delta T} + \log d & \text{with} \quad p_d \end{cases} \tag{1}$$

1. **Write the algorithm for the trinomial tree of Kamrad-Ritchken in the European and American cases .**
2. **Compute the complexity of the program.**