# LSI for Kawasaki dynamics with superquadratic single-site potential

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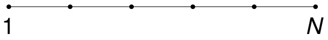
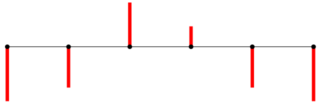
# Spin system

**State space** Euclidean space $\mathbb{R}^N$



**Hamiltonian** Function $H : \mathbb{R}^N \to \mathbb{R}$

$H$ specified later

**Gibbs measure** $\mu(dx) := Z^{-1} \exp\left(-H(x)\right)dx$

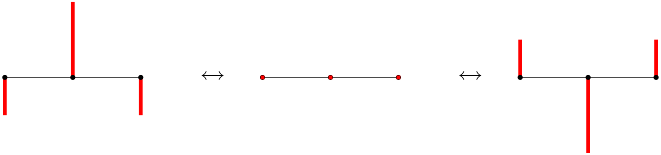
# Kawasaki dynamics

**Stochastic process** $\quad dX_t = -A\nabla H(X_t)dt + \sqrt{2A}dB_t, \quad X_t \in \mathbb{R}^N$

$A$ second order difference operator
f.e. for the 1-d periodic lattice

$$A \; := \; N^2 \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & & & & & \\ 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$$

**Spin exchange between nearest neighbors**

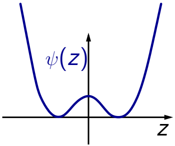


## Kawasaki dynamics: simulation

**System size** $N = 6$

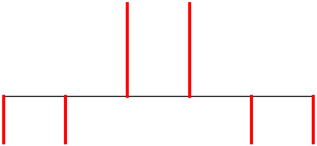
**Single-site potential** $\psi(z) = (z^2 - 1)^2$

**Hamiltonian** $H(x) = \sum_{i=1}^{6} \psi(x_i)$



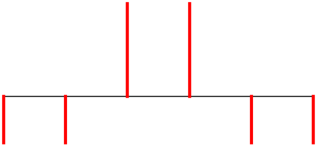

**Deterministic gradient flow**

$$dX_t = -A\nabla H(X_t)dt$$



**Stochastic perturbation**

$$+\sqrt{2A}dB_t$$



# Kawasaki dynamics

**Questions**

Is there an equilibrium state for $X_t$?
If yes, does $X_t$ approach the equilibrium?
In which sense and how fast?

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**Conservation of mean spin**

$\frac{1}{N}\sum_{i=1}^N x_i(t) = \frac{1}{N}\sum_{i=1}^N x_i(t=0) = m$

**New state space**

$\mathbb{R}^N \rightsquigarrow X_{N,m} = \left\{ x \in \mathbb{R}^N \mid \frac{1}{N}\sum_{i=1}^N x_i = m \right\}$

**Canonical ensemble**

$\mu \rightsquigarrow \mu = Z^{-1} \; \exp(-H(x)) \, \mathcal{H}_{\lfloor \{ \frac{1}{N}\sum_{i=1}^N x_i = m \}}(dx)$

$\mu$ is stationary for $X_t$.

# Logarithmic Sobolev inequality

**Definition** $\mu(dx)$ satisfies LSI($\rho$), if for all $f(x) \geq 0$

$$\mathrm{Ent}(f\mu|\mu) := \int f \log f d\mu - \left(\int f d\mu\right) \log \left(\int f d\mu\right) \leq \rho^{-1} \int \frac{|\nabla f|^2}{2f} d\mu.$$

**Observation 1** $|\nabla f|^2 \leq N^{-2} |\sqrt{A} \nabla f|^2$.

**Observation 2** Let $X_t$ be distributed as $f_t\mu$, then

$$\frac{d}{dt} \mathrm{Ent}(f_t\mu|\mu) = -\int \frac{|\sqrt{A}\nabla f_t|^2}{2f_t} d\mu.$$

**Consequence**

If $\mu$ satisfies LSI($\rho$), then $\mathrm{Ent}(f_t\mu|\mu) \leq \exp(-N^{-2}\rho\, t)\, \mathrm{Ent}(f_0\mu|\mu)$.

**Remark** LSI is well adapted to the hydrodynamic limit $N \to \infty$.

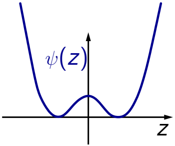
# LSI for Kawasaki: known results

## Quadratic case

**Proposition** (Landim, Panizo & Yau '02)

Assume

$$H(x) = \sum_{i=1}^{N} \psi(x_i), \quad \text{non-interacting}$$




$$\psi(z) = \frac{1}{2}z^2 + V_b(z), \quad z \in \mathbb{R} \quad \text{with} \quad |V_b|_\infty, |V_b'|_\infty, |V_b''|_\infty < \infty.$$

Then $\mu$ satisfies LSI($\varrho$).

The constant $\varrho > 0$ is independent of the system size $N$ and the mean spin $m$.

## Two different proofs

Landim, Panizo & Yau '02, Chafaï '03: Lu-Yau martingal method
Grunewald, Otto, Villani & Westdickenberg '09: Two-scale approach

## LSI for Kawasaki: known results

**Subquadratic case** counterexample of Barthe & Wolff '09

$$\psi(z) = \begin{cases} z, & \text{for } z > 0 \\ \infty, & \text{else} \end{cases} \quad \Longrightarrow \quad \begin{array}{l} \mathrm{SG}(\varrho) \approx \frac{1}{m^2} \\ \mathrm{LSI}(\varrho) \approx \frac{1}{Nm^2} \end{array}$$

**Superquadratic case** Question raised by Varadhan '93

$$\psi(z) = V_c(z) + V_b(z)$$

$$V_c''(z) \geq c > 0 \qquad \text{and} \qquad |V_b|_\infty, |V_b'|_\infty < \infty.$$

Conjecture by Landim, Panizo & Yau '02

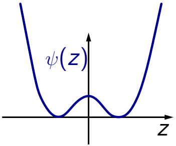
Partial answer for spectral gap by Caputo '02

# LSI for Kawasaki: superquadratic case

**Proposition** (M. & Otto '10).

Assume

$$H(x) = \sum_{i=1}^{N} \psi(x_i), \quad \text{non-interacting}$$




with $\psi$ is perturbed strictly convex i.e

$$\psi(z) = V_c(z) + V_b(z), \quad z \in \mathbb{R}$$
$$V_c''(z) \geq c > 0 \qquad \text{and} \qquad |V_b|_\infty, |V_b'|_\infty < \infty.$$

Then $\mu$ satisfies LSI($\varrho$), independent of $N$ and $m$.

**Remarks** Improves result of Caputo for spectral gap.

# LSI for Kawasaki: technical difficulties

**Standard Criteria for LSI fail:**

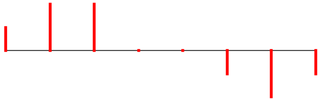
| | |
|---|---|
| **Tenzorization principle** | Restriction to $X_{N,m} \rightsquigarrow \mu$ is not product measure, |
| **Bakry & Émery** | $H(x)$ is not convex, |
| **Holley & Stroock** | LSI($\varrho$) independent of $N$. |

**Elaborated machinery for LSI:**

| | |
|---|---|
| **Lu & Yau Martingal method** | $\psi(z)$ bounded perturbation of Gaussian |
| **Two-scale approach** | requires $|\psi''|_\infty < \infty$. |
| **Strategy:** | **Adapt the two-scale approach.** |

# Two-scale approach

**Coarse graining**



$N =$ size of the system
$K =$ size of a block
$M =$ number of blocks

## Two-scale approach

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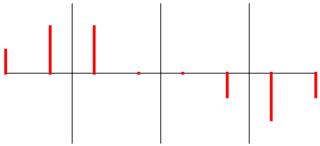

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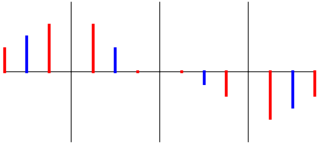

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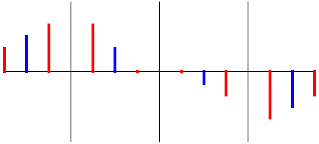

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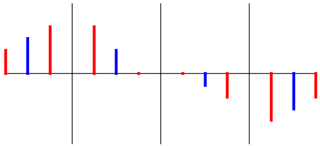


**Coarse graining operator**

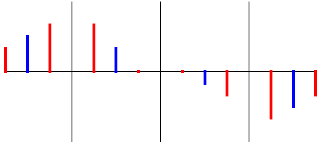
$$Px := (y_1, \ldots, y_M)$$
$$:= \left( \frac{1}{K} \sum_{i=1}^{K} x_i, \ldots, \frac{1}{K} \sum_{i=N-K+1}^{N} x_i \right)$$

**Decomposition of $\mu$**

$$\mu(dx) = \mu(dx | Px = y)\, \bar{\mu}(dy)$$

# Two-scale approach

**Coarse graining**



$N$ = size of the system
$K$ = size of a block
$M$ = number of blocks

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$$:= \left( \frac{1}{K} \sum_{i=1}^{K} x_i, \ldots, \frac{1}{K} \sum_{i=N-K+1}^{N} x_i \right)$$

**Decomposition of $\mu$** $\quad \mu(dx) = \mu(dx|Px = y)\, \bar{\mu}(dy)$

**Proposition** (Two-scale criterion for LSI, GOVW '09)

Assume that

- $\kappa := sup \left\{ \frac{\langle x, \text{Hess}\, H\, y \rangle}{|x|\ |y|}\ ;\ x \in \text{Ker}\, P,\ y \in (\text{Ker}\, P)^{\perp} \right\} < \infty,$
- $\mu(dx|Px = y)$ satisfies LSI($\varrho_1$) independent of N, m, and y, (Mikro LSI)
- $\bar{\mu}(dy)$ satisfies LSI($\varrho_1$) independent of N and m. (Makro LSI)

Then $\mu$ satisfies the LSI($\varrho$), independent of N and m.

## Adapted two-scale approach

**Problem** $\kappa \stackrel{!}{=} \infty$ for general $\psi = V_c + V_b$

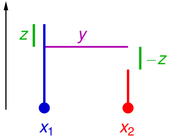
# Adapted two-scale approach

**Problem** $\kappa \overset{!}{=} \infty$ for general $\psi = V_c + V_b$

**Simplification** Coarse graining of pairs i.e. $K = 2$

$$\mu(dx|Px = y) = \bigotimes_{i=1}^{M} \mu\left(dx_{2i-1}, dx_{2i}| \frac{x_{2i-1} + x_{2i}}{2} = y_i\right)$$

**Fluctuations of a spin pair**




$$\mu\left(dx_1, dx_2 \,|\, \frac{x_1 + x_2}{2} = y\right) \approx \underbrace{\frac{\exp\left(-\psi(z+y) - \psi(-z+y)\right)}{Z} dz}_{=:\, \nu(dz|y)}$$

$\nu(dz|y)$ is an one-dimensional measure

# Crucial estimate for two-scale criterion

**Lemma** (Covariance estimate GOVW '09)

Assume that $\psi = V_c + V_b$. Then for $\int f(z)\nu(dz|y) = 1$

$$\operatorname{cov}_{\nu(dz|y)}\left(g(z), f(z)\right) \;\leq\; C \,\sup_z \left|g'(z)\right| \,\left(\int \frac{|f'(z)|^2}{2f(z)}\, \nu(dz|y)\right)^{\frac{1}{2}}.$$

**For the two-scale criterion:** $g(z) = V_c'(-z+y) + V_c'(z+y)$

**Case** $V_c(z) = z^2$: $g'(z) = 4 \quad \Rightarrow \quad \kappa = \sup_z |g'(z)| < \infty$

**Case** $V_c(z) = z^4$: $g'(z) = 48\, zy \quad \Rightarrow \quad \kappa = \sup_z |g'(z)| = \infty$

**Consequence** Improve the covariance estimate!

**Hope** Convexity is "good".

Use the additional convexity of $z^4 \longleftrightarrow z^2$!

## Crucial estimate: superquadratic case

**Proposition** (Brascamp & Lieb '76).

Assume $\mu(dz) = Z^{-1} \exp(-H(z))dz, \quad z \in \mathbb{R} \quad \text{and} \quad H''(z) > 0.$

Then $\mathrm{var}_\mu(f) \leq \int \frac{|f'(z)|^2}{H''(z)} \mu(dz).$

**Proposition** (Assymetric Brascamp & Lieb, M. & Otto '10).

Assume $\mu(dz) = Z^{-1} \exp(-H(z))dz, \quad z \in \mathbb{R} \quad \text{and} \quad H''(z) > 0.$

Then $\mathrm{cov}_\mu(g, f) \leq \sup_z \left| \frac{g'(z)}{H''(z)} \right| \int |f'(z)| \mu(dz).$

Elementary proof using calculus.

Multidimensional version by Carlen, Cordero-Erausquin & Lieb '11.

# Crucial estimate: superquadratic case

**Corollary** (Improved covariance estimate).

Assume that $\psi = V_c + V_b$. Then for $\int f(z)\nu(dz|y) = 1$

$$\begin{aligned} &\mathrm{cov}_{\nu(dz|y)}(g, f) \\ &\leq C \sup_z \left| \frac{g'(z)}{V_c''(-z+y) + V_c''(z+y)} \right| \left( \int \frac{|f'(z)|^2}{f(z)} \nu(dz|y) \right)^{\frac{1}{2}}. \end{aligned}$$

**Apply covariance estimate to** $\quad g(z) = V_c'(-z+y) + V_c'(z+y)$

$$\sup_z \left| \frac{g'(z)}{V_c''(-z+y) + V_c''(z+y)} \right| = \sup_z \left| \frac{V_c''(-z+y) - V_c''(z+y)}{V_c''(-z+y) + V_c''(z+y)} \right| \leq 2$$

## Crucial estimate: superquadratic case

**Corollary** (Improved covariance estimate).

Assume that $\psi = V_c + V_b$. Then for $\int f(z)\nu(dz|y) = 1$

$$\mathrm{cov}_{\nu(dz|y)}(g, f) \leq C \sup_z \left| \frac{g'(z)}{V_c''(-z+y) + V_c''(z+y)} \right| \left( \int \frac{|f'(z)|^2}{f(z)} \nu(dz|y) \right)^{\frac{1}{2}}.$$

**Apply covariance estimate to** $\quad g(z) = V_c'(-z+y) + V_c'(z+y)$

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**Two-scale criterion for LSI is satisfied for coarse-graining of pairs!**

# Open question

## Hamiltonian with interaction

**Proposition** (M '09).

Assume

$$H(x) = \sum_{i=1}^{N} (x_i^2 + V_b(x_i)) \quad + \quad \varepsilon \sum_{i \sim j} x_i x_j, \quad \text{interacting}$$

$$|V_b|_\infty, |V_b'|_\infty, |V_b''|_\infty < \infty.$$

If $\varepsilon > 0$ is small enough, then $\mu$ satisfies LSI($\varrho$).

In this case $\varrho > 0$ is independent of the system size $N$ and the mean spin $m$.

Generalize last statement to

$$H(x) = \sum_{i=1}^{N} (V_c(x_i) + V_b(x_i)) \quad + \quad \varepsilon \sum_{i \sim j} x_i x_j$$

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$$H(x) = \sum_{i=1}^{N} (V_c(x_i) + V_b(x_i)) \quad + \quad \varepsilon \sum_{i \sim j} x_i x_j$$

**Thank you for your attention.**

# Iterated of coarse graining of pairs

**Coarse grained system** $\bar{\mu}(dy) = Z^{-1} \exp\left(-\bar{H}(y)\right) \mathcal{H}(dy)$

**Coarse grained Hamiltonian** $\bar{H}(y) = \sum_{i=1}^{\frac{N}{2}} \mathcal{R}\psi(y_i)$

**Renormalization operator** $\mathcal{R} : \mathbb{R} \to \mathbb{R}$

$$\mathcal{R}\psi(z) = -\log \int \exp(-\psi(x_1) - \psi(x_2))\, \mathcal{H}_{\lfloor\left\{\frac{x_1+x_2}{2}=z\right\}}(dx_1, dx_2)$$

**Lemma** (Invariance under renormalization)

If $\psi$ is perturbed strictly convex then $\mathcal{R}\psi$ also is perturbed strictly convex.

**Lemma** (Convexification of $\psi$ by iterated renormalization)

There is $K$ such that $\mathcal{R}^K\psi$ is uniformly striclty convex.

Macroscopic LSI by iterated coarse-graining and Bakry & Émery

# Hierarchic criterion for LSI

**Decomposition of $\mu$** $\qquad \mu(dx) = \mu(dx|Px = y)\ \bar{\mu}(dy)$

**Decomposition of entropy** $\qquad \bar{f}(y) := \int f(x)\mu(dx|Px = y)$, then

$$\mathrm{Ent}(f\mu, \mu) = \int \mathrm{Ent}(f\mu(dx|y), \mu(dx|y))d\bar{\mu} + \mathrm{Ent}(\bar{f}\bar{\mu}, \bar{\mu})$$

**Microscopic scale** $\qquad \psi$ perturbed strictly convex

$\Rightarrow \quad \mu(dx|Px = y)$ satisfies LSI($\tilde{\varrho}$) i.e.

$$\int \mathrm{Ent}(f\mu(dx|y), \mu(dx|y))\bar{\mu}(dy) \le \frac{1}{\tilde{\varrho}} \int \frac{|\nabla f(x)|}{2f(x)} \underbrace{\mu(dx|Px = y)\bar{\mu}(dy)}_{=\mu(dx)}.$$

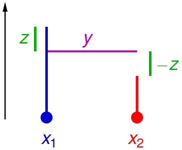
**Macroscopic scale** $\qquad$ By assumption $\bar{\mu}$ satisfies LSI($\varrho_1$) i.e.

$$\mathrm{Ent}(\bar{f}\bar{\mu}, \bar{\mu}) \le \frac{1}{\varrho_1} \int \frac{|\nabla \bar{f}(y)|}{2\bar{f}(y)}\ \bar{\mu}(dy).$$

## Hierarchic criterion for LSI

### Crucial estimate for hierachic criterion

$$\boxed{\int \frac{|\nabla \bar{f}(y)|}{2\bar{f}(y)}\,\bar{\mu}(dy) \overset{!}{\leq} C\int \frac{|\nabla f(x)|}{2f(x)}\,\mu(dx)}$$




**Distribution of a pair**

$$\mu\left(dx_1, dx_2 \mid \frac{x_1 + x_2}{2} = y\right) \approx \underbrace{\frac{\exp\left(-\psi(z+y) - \psi(-z+y)\right)}{Z}dz}_{=:\nu(dz|y)}$$

**Crucial estimate reduced to**

Recall $\psi = V_c + V_b$. For all $f:\mathbb{R}\to\mathbb{R}$ with $\int f(z)\nu(dz|y) = 1$,

$$\operatorname{cov}_{\nu(dz|y)}\left(V_c'(z+y) + V_c'(-z+y), f(z)\right) \overset{!}{\leq} C\left(\int \frac{|f'(z)|}{2f(z)}\,\nu(dz|y)\right)^{\frac{1}{2}}.$$

# From quadratic to superquadratic potentials

**Change the structure of the proof of GOVW '09**

| | | |
|---|---|---|
| two-scale argument (big blocks) | $\rightsquigarrow$ | iterative argument (block of pairs) |

**New tools to handle superquadratic potentials**

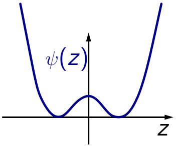
| | | |
|---|---|---|
| generalized covariance estimate | $\rightsquigarrow$ | Hierarchic criterion for LSI |
| | $\rightsquigarrow$ | Invariance of renormalization |
| generalized local Cramér theorem | $\rightsquigarrow$ | Convexification of $\mathcal{R}^K\psi$ |

# Result for spectral gap by Caputo

**Proposition** (Caputo '02).

Assume

$$H(x) = \sum_{i=1}^{N} \psi(x_i), \quad \text{non-interacting}$$




$$\psi(z) = V_c(z) + V_b(z), \quad z \in \mathbb{R} \quad \text{with} \quad |V_b|_\infty, |V_b'|_\infty, |V_b''|_\infty < \infty,$$

$$V_c''(z) \geq c > 0, \; \frac{1}{C} \leq \liminf \frac{V_c''(z)}{|z|^n} \leq \limsup \frac{V_c''(z)}{|z|^n} \leq C, \quad C > 0, \; n \in [0, \infty) .$$

Then $\mu$ satisfies SG($\varrho$).

The constant $\varrho > 0$ is independent of the system size $N$ and the mean spin $m$.

**Remark** Conservation of mean spin $\rightsquigarrow$ long-range interaction.

**Question** Similar result for LSI?