A spatial version of the Itô-Stratonovich correction

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Joint work with Martin Hairer and Hendrik Weber

Introduction

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How to deal with spatially rough stochastic PDEs?

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We study equations of the form

$$\partial_t u = \partial_x^2 u + g(u)\partial_x u + \xi$$

where

- $u = u(x,t) \in \mathbb{R}^n$, $x \in [0,1]$, $t \geq 0$, periodic boundary conditions.
- $g : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ smooth
- ξ: space-time white noise: Gaussian process with covariance

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Remark

- We can deal with an reaction term $f(u)$ as well. (Easy)
- We can treat multiplicative noise as well. (Not so easy)

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- Natural extensions of Burgers equation ($g(u) = -u$).
- Path-sampling problems for stochastic ODEs
- Spatial regularity is at the borderline where 'classical' techniques break down. Techniques apply to even rougher equations (KPZ).

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Structure of the talk

- Part I: $g = DG$ is a gradient
  - Existence & Uniqueness: well-known (Da Prato & Debussche & Temam ’94; Gyöngy ’98)
  - Approximations: (Hairer & M.)
- Part II: g is not a gradient
  - Existence & Uniqueness: (Hairer; Hairer & Weber)
  - Approximations: (Hairer & M. & Weber)

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- Problem: how to make sense of the nonlinearity $g(u)\partial_x u$?

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Definition

u is a mild solution if it satisfies the 'variation of constants formula':

$$u(\cdot, t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \Big[g(u(\cdot, s))\partial_x u(\cdot, s) \Big] \, \mathrm{d}s + \int_0^t e^{(t-s)\Delta} \, \mathrm{d}W(s)$$

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- Moral: After fixing ψ, the equation for v becomes deterministic;
- $\rightsquigarrow$ can be solved pathwise by a standard fixed-point argument in C^1.

Approximations: numerical results

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where $\partial_t \bar{u} = \partial_x^2 \bar{u} + g(\bar{u}) \partial_x \bar{u} - C\,(\nabla \cdot g)(\bar{u}) + \xi$.

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Numerics 2 (Hairer & Voss)

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$$\text{where } \partial_t \bar{u} = \partial_x^2 \bar{u} + g(\bar{u}) \partial_x \bar{u} - C\,(\nabla \cdot g)(\bar{u}) + \xi.$$

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Theorem (Hairer - M. '10)

$$\lim_{\varepsilon \to 0} \mathbb{P}\Big(\sup_{t \le \tau_\varepsilon} \|u^\varepsilon(t) - \bar{u}(t)\|_{L^\infty} > \varepsilon^{\frac{1}{6} - \kappa} \Big) = 0 \qquad \forall \kappa > 0.$$

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- Correction term arises due to the lack of spatial regularity.
- It is exactly the local cross-variation in space : $d[g(u), u]$ (kind of Itô-Stratonovich correction).

Part II: g is not a gradient

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- Solution (following Hairer, Weber): use Lyons' Rough Paths Theory.

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Making sense of $\int_y^z Y(x)\,\mathrm{d}X(x)$ for suitable $X, Y \in C^\alpha$ with $\alpha < \frac{1}{2}$.

Problem: different Riemann sum approximations $\rightarrow$ different limits.

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Rough Path Theory (Lyons, Gubinelli)

- Postulate the value of 'area process'

$$\int_y^z X(x) - X(y)\,\mathrm{d}X(x) := \mathbf{X}(y,z)$$

- If Y is 'controlled' by X, i.e.,

$$Y(y) - Y(x) \;=\; Y'(x)\,\big(X(y) - X(x)\big) + \;\{smooth\},$$

  one can define $\int_y^z Y(x)\,\mathrm{d}X(x)$

  by 'second order Riemann sum approximation' involving $\mathbf{X}$.

- Note: the value of $\int_y^z Y(x)\,\mathrm{d}X(x)$ depends on the choice of $\mathbf{X}$!

Back to the SPDE... (following Hairer '11)

Agenda:

1. Construct an area process Ψ for the linearised solution ψ.
2. Make sense of mild solutions using rough integrals.
3. Construct solutions using a fixed point argument.
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Theorem (Friz & Victoir '05; Hairer '10)

Let ψ be the solution to the stochastic heat equation. For fixed t there is a "canonical" area process Ψ such that $(\psi(t,\cdot), \Psi(t,\cdot,\cdot))$ is a rough path.

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Note however: $\tilde{\Psi}(t,x,y) := \Psi(t,x,y) + c(y-x)$ would be admissible as well (for any $c \in \mathbb{R}$)!

Making sense of weak solutions

Let ψ be the solution to the stochastic heat equation.

Definition

A continuous stochastic process u is a solution to (SPDE) if $v := u - \psi$ belongs to C^1 (in space) and satisfies

$$v(\cdot, t) = \int_0^t \int_0^1 p_{t-s}(x-y)\Big[g\big(u(y,s)\big)\partial_x v(\cdot, s)\Big]\,\mathrm{d}y\,\mathrm{d}s + \int_0^t \int_0^1 p_{t-s}(x-y)\Big[g\big(u(y,s)\big)\Big]\,\mathrm{d}_y\psi(y,s)\,\mathrm{d}s.$$

First integral: no problem, v is differentiable.
Second integral: interpretation in the sense of rough paths.

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Remarks

- Notion of solution depends on the choice of the area process Ψ!
- Once Ψ has been fixed, everything is deterministic!

Results

Existence and uniqueness (Hairer; Hairer & Weber)

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- If $g = DG$, then solutions coincide with “classical solutions”, provided Ψ is the canonical area process.

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Thank you!