Direct simulation of dense suspensions

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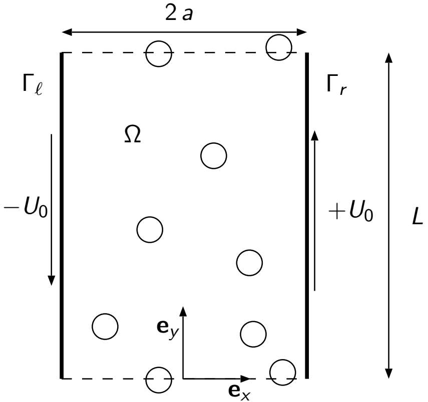

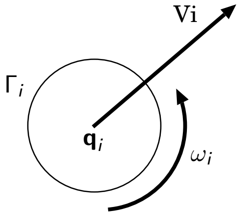

$\mathbf{q}_i = \mathbf{q}_i(t)$: center of particle i.

Fluid domain : $\Omega_F(t) = \Omega \setminus \bigcup_{i=1}^N B_i$.

Stokes equations in the moving domain $\Omega_F(t)$,

$$\left\{ \begin{array}{rcl} -\mu\triangle\mathbf{u} + \nabla p & = & 0, \\ \nabla\cdot\mathbf{u} & = & 0, \end{array} \right.$$

Balance of forces for the particles

$$\left\{ \begin{array}{l} \displaystyle\int_{\Gamma_i} \sigma\cdot\mathbf{n} = \mathbf{F}_i, \\ \displaystyle\int_{\Gamma_i} \mathbf{r}_i \times \sigma\cdot\mathbf{n} = 0, \end{array} \right.$$

$$\sigma = \mu(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}) - p\, I_d.$$

No-slip condition :

$$\mathbf{u}(\mathbf{x}) = \mathbf{V}_i + \omega_i \times \mathbf{r}_i \text{ on } \Gamma_i \text{ for } 1 \le i \le N,$$

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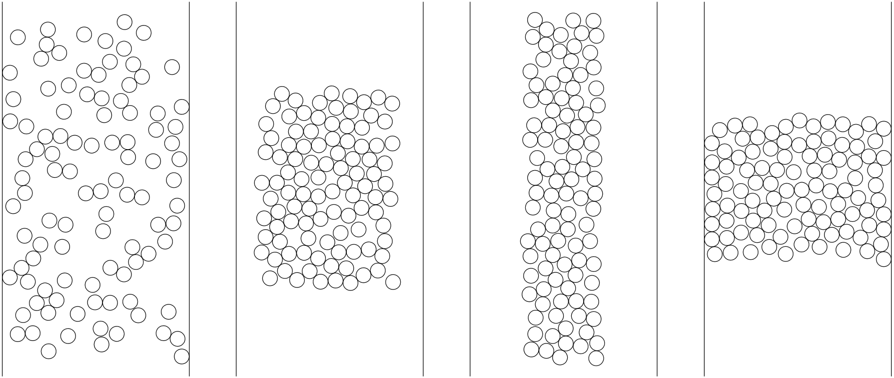
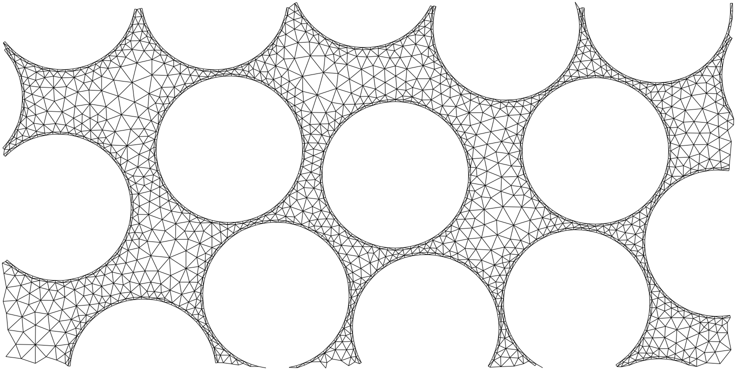
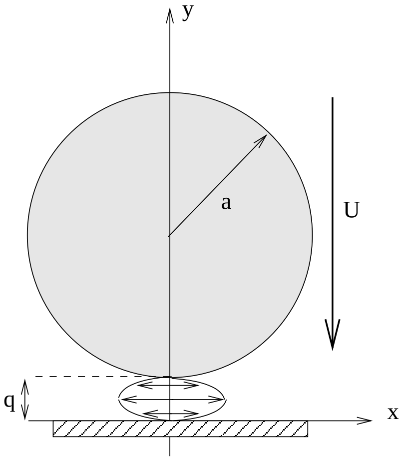
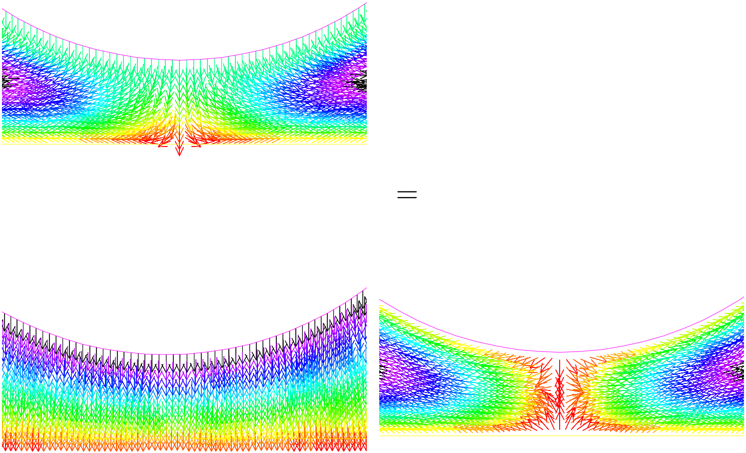


Apparent viscosities : 2.0, 2.45, 1.62, and 6.54.

Direct estimation of lubrication forces




y
a
U
q
x



Theoretical results

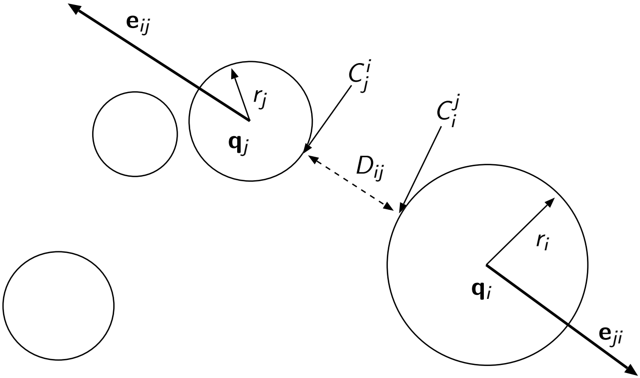
In 2D : no contact in finite time (M. Hillairet, D. Gérard-Varet) for sufficiently smooth surface

In 3D : non contact in finite time for smooth surfaces (M. Hillairet, T. Takahashi)

Asymptotic behaviour (Brenner, Kim)

$$F_{lub} \sim -6\pi\mu a^2 \frac{U}{q} \mathbf{e}_y.$$

Notations



$$\mathbf{F}_i^j = -\mathbf{F}_j^i = -\kappa\left(\mathrm{D}_{ij}\right)\ \left[(\dot{\mathbf{C}}_i^j - \dot{\mathbf{C}}_j^i)\cdot \mathbf{e}_{ij}\right]\ \mathbf{e}_{ij}\ ,$$

which can be written

$$\mathbf{F}_i^j = \left[-\kappa\left(\mathrm{D}_{ij}\right)\mathbf{e}_{ij}\otimes\mathbf{e}_{ij}\right]\cdot(\dot{\mathbf{C}}_i^j - \dot{\mathbf{C}}_j^i)\ ,\ \ \kappa\left(d\right) = \mu\ 1/d.$$

$$m_i\,\ddot{\mathbf{q}}_i = \mathbf{\Phi}_i + \sum_{j\neq i}\ \mathbf{F}_i^j(\dot{\mathbf{C}}_i^j, \dot{\mathbf{C}}_j^i).$$

$$M\,\ddot{\mathbf{q}} + \sum_{i<j}\left[\kappa\left(\mathrm{D}_{ij}\right)\mathbf{G}_{ij}\otimes\mathbf{G}_{ij}\right]\cdot\dot{\mathbf{q}} = \Phi.$$

The velocity $\mathbf{u} = \dot{\mathbf{q}}$ verifies the associated energy balance

$$\frac{d}{dt}\ \left(\frac{1}{2}\,M\mathbf{u}\cdot\mathbf{u}\right) + \Psi(\mathbf{u},\mathbf{u}) = \Phi\cdot\mathbf{u}\ ,$$

with Ψ symmetric, nonnegative bilinear form.
(in its kernel : rigid motion of clusters)

Numerical strategy : decoupling of **q** and the distances.

$$\dot{\mathrm{D}}_{pq} = \mathbf{G}_{pq} \cdot \dot{\mathbf{q}} \; , \quad \ddot{\mathrm{D}}_{pq} = \mathbf{G}_{pq} \cdot \ddot{\mathbf{q}} + \dot{\mathbf{G}}_{pq} \cdot \dot{\mathbf{q}} \; ,$$

$$\begin{aligned} \Longrightarrow \ddot{\mathrm{D}}_{pq} \quad = \quad & \mathbf{G}_{pq} \cdot \mathrm{M}^{-1}\Phi - \mu \, \frac{\dot{\mathrm{D}}_{pq}}{\mathrm{D}_{pq}} \, \mathbf{G}_{pq} \cdot \mathrm{M}^{-1}\mathbf{G}_{pq} \\ & -\frac{1}{2}\sum_{i \neq j} \; \kappa \, (\mathrm{D}_{ij})(\mathbf{G}_{ij} \cdot \mathbf{u}) \, (\mathbf{G}_{pq} \cdot \mathrm{M}^{-1}\mathbf{G}_{ij}) \end{aligned}$$

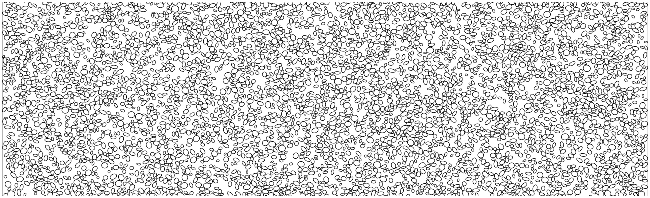
One keeps D_{pq} implicit, the other distances explicit
$\longrightarrow \ddot{D} = -\mu \dot{D}/D + f$

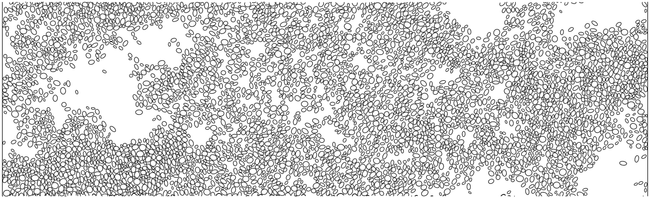
1) Compute the distances (ODE).
2) Compute the matrix

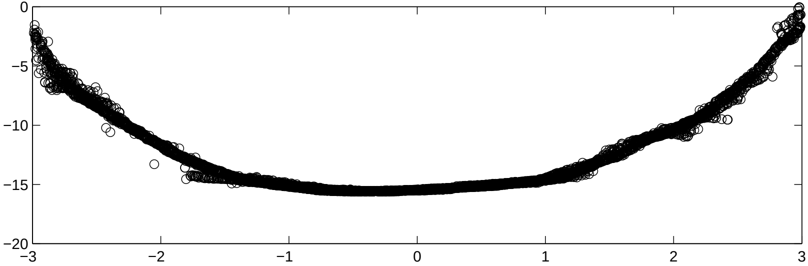
$$A = \sum_{i<j} \left[\kappa\left(\mathrm{D}_{ij}\right) \mathbf{G}_{ij} \otimes \mathbf{G}_{ij}\right]$$

3) Compute velocities and positions

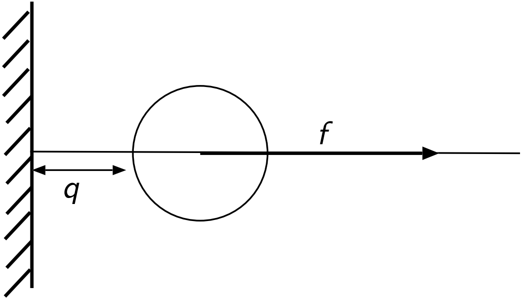
$$M \frac{\mathbf{u}^{n+1} - \mathbf{u}^{n}}{h} + A\mathbf{u}^{n+1} = \Phi$$






0
-5
-10
-15
-20
-3
-2
-1
0
1
2
3

Vanishing viscosity limit



$$\left| \begin{array}{l} \ddot{q}_\varepsilon = -\varepsilon \dfrac{\dot{q}_\varepsilon}{q_\varepsilon} + f(t), \\ q_\varepsilon(0) = q^0 > 0 \ , \ \ \dot{q}_\varepsilon(0) = u^0, \end{array} \right.$$

THEOREM : Let $\varepsilon > 0$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ be given.

$$\exists ! q_\varepsilon \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+).$$

PROOF. There exists a maximal solution defined on $[0, \tau[$.
If $\tau < \infty$, then necessarily q_ε goes to 0 as t goes to τ^-. But

$$\dot{q}_\varepsilon(t) = u^0 - \varepsilon \ln\left(\frac{q_\varepsilon(t)}{q^0}\right) + \int_0^t f(s)\, ds,$$

so that $q_\varepsilon \to 0$ implies $\dot{q}_\varepsilon(t) \longrightarrow +\infty$, hence a contradiction.

Asymptotic behaviour

$$
\left|
\begin{array}{l}
q \in W^{1,\infty}(I) \text{ with } \dot{q} \in BV(I)\,, \;\; \gamma \in BV(I)\,, \;\; \mu \in \mathcal{M}(I)\,, \quad \text{s. t.} \\
\ddot{q} = f + \lambda \text{ in } \mathcal{M}(I)\,, \\
\operatorname{supp}(\lambda) \in \{t,\, q(t) = 0\}\,, \\
\dot{q}^+ = P_{C_q} \dot{q}^-\,, \\
\dot{\gamma} = -\lambda\,, \;\; \gamma \leq 0\,, \;\; q \geq 0\,, \;\; q\gamma = 0 \quad \text{a.e. in } I\,, \\
q(0) = q^0 > 0\,, \;\; \dot{q}(0) = u^0,
\end{array}
\right.
$$

$$
\text{with } C_q = \left|
\begin{array}{lll}
\mathbb{R} & \text{if} & q > 0\,, \\
\mathbb{R}^+ & \text{if} & q = 0 \text{ and } \gamma^- = 0\,, \\
\{0\} & \text{if} & q = 0 \text{ and } \gamma^- < 0.
\end{array}
\right.
$$

Alternative formulation

(equivalent for a finite number of contacts)

$$
\left|\begin{array}{l}
q \in W^{1,\infty}(I)\ ,\ \ \gamma \in L^{\infty}(I),\ \text{such that} \\
\dot{q} + \gamma = \tilde{u} = u^0 + \int_0^t f(s)\,ds \qquad , \\
\gamma \leq 0\ ,\ \ q \geq 0\ ,\ \ q\gamma = 0\ , \\
q(0) = q^0 > 0\ ,\ \ \dot{q}(0) = u^0.
\end{array}\right.
$$

Theorem

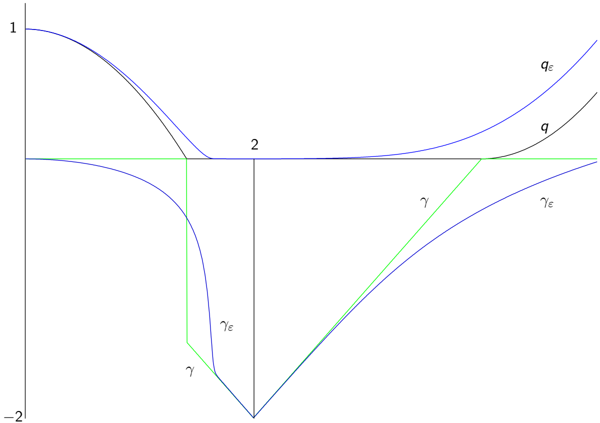
$q^0 > 0$, $u^0 \in \mathbb{R}$, $I =]0, T[$, $f \in L^1(I)$.
We denote by $q_\varepsilon \in W^{1,\infty}(I)$ the unique solution in $\overline{I}$, and we set $\gamma_\varepsilon = \varepsilon \ln q_\varepsilon$.
When ε goes to 0, there exists a subsequence, still denoted by (q_ε), $q \in W^{1,\infty}(I)$, $\gamma \in L^\infty(I)$, such that

$$\begin{array}{cccl} q_\varepsilon & \longrightarrow & q & \text{uniformly ,} \\ \gamma_\varepsilon & \stackrel{\star}{\rightharpoonup} & \gamma & \text{in } L^\infty \text{ weak} - \star, \end{array}$$

and the couple (q, γ) is a solution to the limit problem.

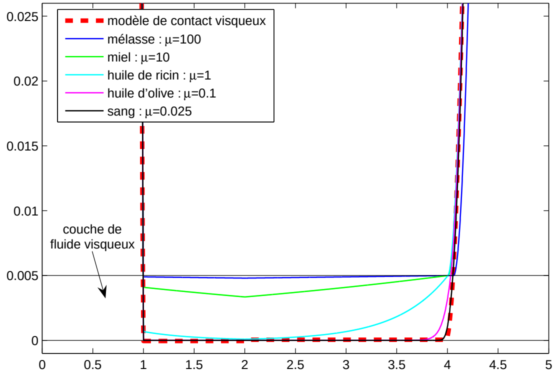
Numerical scheme

$$q^n > 0 \begin{cases} u^{n+1} = u^n + hf(t^{n+1}) \,, \\ \tilde{q}^{n+1} = q^n + hu^{n+1} \,, \\ \text{if } \tilde{q}^{n+1} < 0 \left| \begin{array}{l} q^{n+1} = 0 \,, \\ \gamma^{n+1} = u^{n+1} \end{array} \right. \\ \text{if } \tilde{q}^{n+1} \geq 0 \left| \begin{array}{l} q^{n+1} = \tilde{q}^{n+1} \,, \\ \gamma^{n+1} = 0 \,, \end{array} \right. \end{cases} , \quad q^n = 0 \begin{cases} \gamma^{n+1} = \gamma^n + hf(t^{n+1}) \,, \\ \text{if } \gamma^{n+1} \leq 0 \left| \begin{array}{l} q^{n+1} = 0 \\ u^{n+1} = 0 \end{array} \right. \\ \text{if } \gamma^{n+1} > 0 \left| \begin{array}{l} u^{n+1} = \gamma \\ q^{n+1} = q \end{array} \right. \end{cases}$$

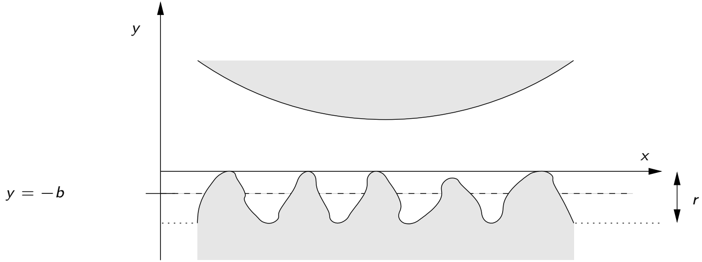

1
−2
2
q_ε
q
γ
γ_ε
γ_ε
γ

Modelling issues

Paradox : modeling of contacts with a highly viscous interstitial fluid obtained as a vanishing viscosity limit.



In real life : rough walls. Contact actually occurs at distance $\delta > 0$. If the distance below which the models produces significant forces is $\delta_\varepsilon < \delta$, the model does not make sense.



Numerically : cut-off applied to γ below some threshlod value.

Many-body situation

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} & = & \mathbf{f} + \displaystyle\sum_{i<j} \lambda_{ij} \mathbf{G}_{ij} \\ \mu_{ij} \in \mathcal{M}(I) \ , \ \ \mathrm{supp}(\mu_{ij}) & \subset & \{t \ , \ \ D_{ij}(\mathbf{q}(t)) = 0\}, \\ \mathbf{u}^+ & = & P_{\mathcal{C}_{\mathbf{q}}} \mathbf{u}^-, \\ \dot{\gamma}_{ij} & = & -\lambda_{ij}. \end{cases}$$

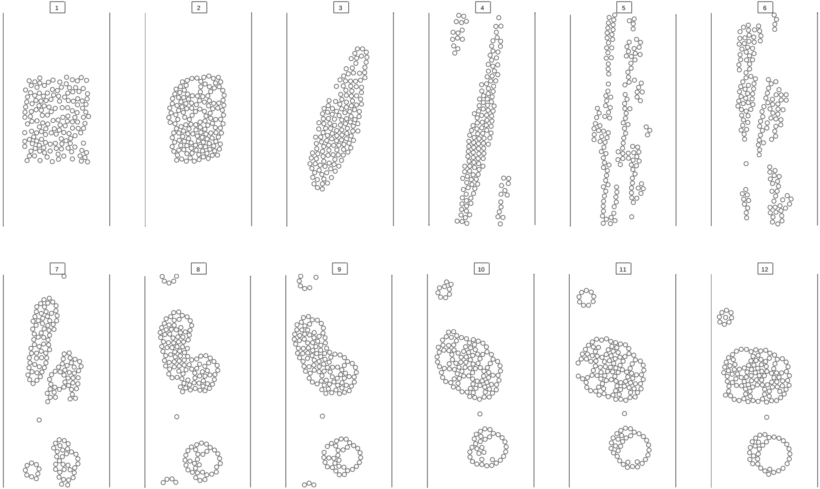
$$\mathcal{C}_{\mathbf{q}} = \{\mathbf{v} \in \mathrm{T}_Q \ , \ \ D_{ij}(\mathbf{q}) = 0 \Rightarrow \mathbf{G}_{ij} \cdot \mathbf{v} \geq 0 \ , \ \ \gamma_{ij} < 0 \Rightarrow \mathbf{G}_{ij} \cdot \mathbf{v} = 0\} \, ,$$

Remark : tangential forces can be included.

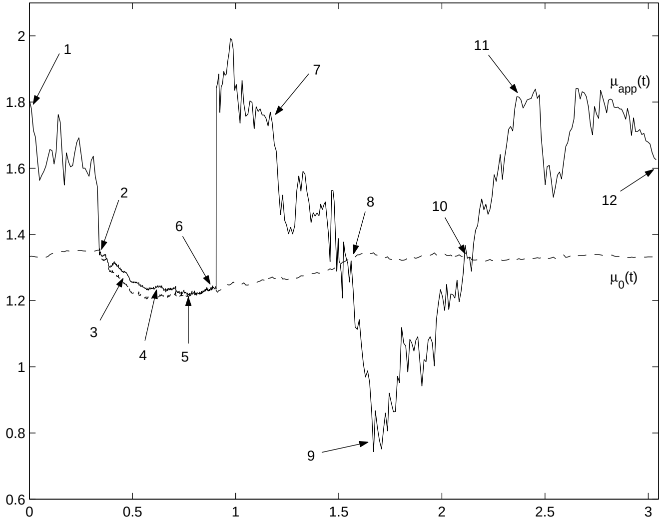
$\rightarrow$30-40-42-50-52

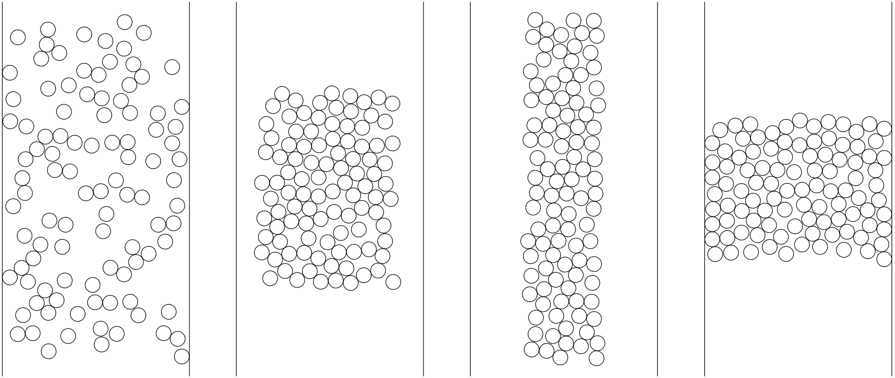
Toward macroscopic ?

$$
\left|
\begin{array}{rcr}
\partial_t \rho + \partial_x(\rho u) & = & 0 \\
\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p & = & f \\
\partial_t \gamma + \partial_x(\gamma u) & = & -p \\
\gamma \leq 0\ ,\ \ \rho \leq 1\ ,\ \ \gamma(1-\rho) & = & 0
\end{array}
\right.
$$


1
2
3
4
5
6
7
8
9
10
11
12

History of the apparent viscosity





Apparent viscosities : 2.0, 2.45, 1.62, and 6.54.

Some animations to download

`http ://www.math.u-psud.fr/~maury/`

`http ://www.cmap.polytechnique.fr/~lefebvre/`