Long-time asymptotic behaviour of a multiscale rod-like model of polymeric fluids

Hui Zhang

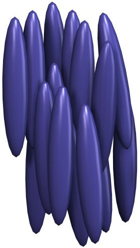
School of Mathematical Sciences, Beijing Normal University

Outline

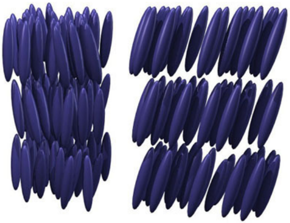
Polymeric fluids

- Polymeric fluids: liquid crystal, egg white, etc....
- Special properties : shear thinning, kayaking, tumbling, phase transition, defects ...

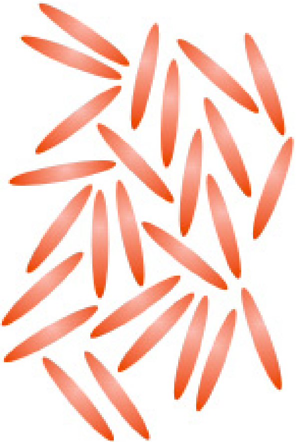
liquid crystals-phases



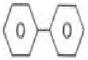
Nematic Phase



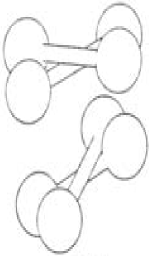
Smectic Phase



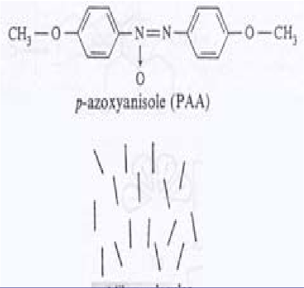
isotropic Phase

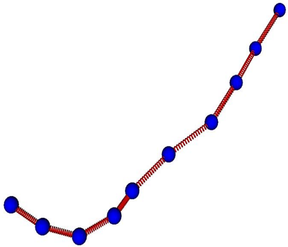


(biphenyl)



(dumbbells)





Kinetic model

$\mathbf{m}$—the orientation of a rodlike particle
$\psi(\mathbf{x}, \mathbf{m}, t)$—the distribution function

$$\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla)\psi = \frac{1}{k_B T}\nabla \cdot \{[D_{\|}\mathbf{mm} + D_{\perp}(\mathbf{I} - \mathbf{mm})] \cdot (\psi \nabla \mu)\}$$
$$+\frac{D_r}{k_B T}\mathcal{R} \cdot (\psi \mathcal{R}\mu) - \mathcal{R} \cdot (\mathbf{m} \times \kappa \cdot \mathbf{m}\psi), \quad \mathbf{m} \in \mathbb{S}^2,$$

$\mathcal{R} = \mathbf{m} \times \frac{\partial}{\partial \mathbf{m}}$: rotational operator $\quad D_r = \frac{\xi_r}{k_B T}$: rotary diffusivity
$\mu = \ln \psi + \bar{U}$: the chemical potential
$\bar{U}$: the excluded-volume potential

$$\bar{U}(\mathbf{x}, \mathbf{m}, t) = k_B T \alpha \int_{\Omega} \int_{|\mathbf{m}'|=1} B(\mathbf{x}, \mathbf{x}', \mathbf{m}, \mathbf{m}')\psi(\mathbf{x}', \mathbf{m}', t) d\mathbf{m}' d\mathbf{x}'.$$

$B(\mathbf{x}, \mathbf{x}', \mathbf{m}, \mathbf{m}') = \frac{1}{\varepsilon^3}\chi(\frac{\mathbf{x}-\mathbf{x}'}{\varepsilon})|\mathbf{m} \times \mathbf{m}'|^2$

Macroscopic model[E & Zhang , Meth. Appl. Anal., 06]

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nabla \cdot \tau + \mathbf{F},$$
$$\nabla \cdot \mathbf{u} = 0.$$

$$\tau = \underbrace{\tau^s}_{\text{viscous stress}} + \underbrace{\tau^e}_{\text{elastic stress}}$$

$\tau^s = 2\eta_s \mathbf{D} + \frac{1}{2}\xi_r \mathbf{D} : \langle \mathbf{mmmm} \rangle$

η_s : solvent viscosity

$\mathbf{D} := \frac{1}{2}(\kappa + \kappa^T) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ strain tensor

$\tau^e = -\langle (\mathbf{m} \times \mathcal{R}\mu)\mathbf{m} \rangle \longleftarrow$ the virtual work principle

$\mathbf{F} = -\langle \nabla \mu \rangle$

$\langle \cdot \rangle$ denotes averaging with respect to the distribution ψ, i.e.,

$\langle g \rangle = \int_{|\mathbf{m}|=1} g\psi d\mathbf{m}.$

Dimensionless rodlike model

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \frac{\gamma}{Re}\triangle\mathbf{u} + \frac{1-\gamma}{2Re}\nabla\cdot(\mathbf{D} : \langle\mathbf{mmmm}\rangle)$$

$$+\frac{1-\gamma}{ReDe}(\nabla\cdot\tau^e + \mathbf{F}) \text{ for } \mathbf{x}\in\Omega$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \text{for } \mathbf{x}\in\Omega.$$

$$\frac{\partial\psi}{\partial t} + \nabla\cdot(\mathbf{u}\psi) = \frac{\varepsilon^2}{De}\nabla\cdot[(\mathbf{I}+\mathbf{mm})(\psi\nabla\mu)]$$

$$+\frac{1}{De}\mathcal{R}\cdot(\psi\mathcal{R}\mu) - \mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}\psi), \quad \mathbf{m}\in\mathbb{S}^2.$$

$$\varepsilon = \frac{L}{L_0} = \frac{\text{the charateristic length of the rods}}{\text{the typical size of the flow region}}$$

energy law

the energy law($\lambda = \frac{1-\gamma}{ReDe}$):

$$\frac{d}{dt}\left[\frac{1}{2}\int_\Omega |\mathbf{u}|^2 d\mathbf{x} + \lambda E(\psi)\right] = -\int_\Omega \left[\frac{\gamma}{De}|\nabla \mathbf{u}|^2 + \frac{1-\gamma}{2Re}\langle(\mathbf{mm}:\mathbf{D})^2\rangle\right] d\mathbf{x}$$
$$-\lambda \int_\Omega \left[\frac{\varepsilon^2}{De}\langle \nabla\mu \cdot [(\mathbf{I}+\mathbf{mm})\nabla\mu\rangle + \frac{1}{De}\langle \mathcal{R}\mu \cdot \mathcal{R}\mu\rangle\right] d\mathbf{x},$$

where $E(\psi)$ is a nonlocal intermolecular potential. Here it is

$$E(\psi) = \int_\Omega \int_{|\mathbf{m}|=1} \psi(\mathbf{x},\mathbf{m},t)\ln\psi(\mathbf{x},\mathbf{m},t) + \frac{1}{2}U(\mathbf{x},\mathbf{m},t)\psi(\mathbf{x},\mathbf{m},t) d\mathbf{m} d\mathbf{x}.$$

questions

- Wellposed analysis [H. Zhang & P.W. Zhang, SIAM J. Math. Anal. 08]
- Numerical simulation [H.J. Yu & P.W. Zhang, J. Non-Newtonian Fluid Mech. 07]
- Steady states analysis [H.L. Liu, H. Zhang, P.W. Zhang, G. Warnecke, P. Constantin, I. Kevrekidis, E.S. Titi, I. Fatkullin, V. Slastikov, Q. Wang]
- Long time behavior?

$$\varepsilon = 0, B = |\mathbf{m} \times \mathbf{m}|^2.$$

stationary system

$$
\begin{aligned}
&(\mathbf{u}_\infty \cdot \nabla)\mathbf{u}_\infty + \nabla p_\infty = \frac{\gamma}{Re}\triangle \mathbf{u}_\infty + \frac{1-\gamma}{ReDe}\nabla \cdot \tau_\infty, \qquad \text{for } \mathbf{x} \in \Omega,\\
&\nabla \cdot \mathbf{u}_\infty = 0, \ \text{for } \mathbf{x} \in \Omega,\\
&(\mathbf{u}_\infty \cdot \nabla)\psi_\infty = \frac{1}{De}\mathcal{R}\cdot\mathcal{R}\psi_\infty + \frac{1}{De}\mathcal{R}\cdot(\psi_\infty \mathcal{R}U_\infty)\\
&\qquad\qquad\qquad -\mathcal{R}\cdot(\mathbf{m}\times\kappa_\infty\cdot\mathbf{m}\psi_\infty),\\
&U_\infty = \alpha\int_{|\mathbf{m}'|=1}|\mathbf{m}\times\mathbf{m}'|^2\psi_\infty(\mathbf{x},\mathbf{m}')d\mathbf{m}',\\
&\tau_\infty = \tau^s_\infty + \tau^e_\infty, \quad \kappa_\infty = (\nabla\mathbf{u}_\infty)^T,\\
&(\tau^s)_\infty = \frac{De}{2}\kappa_\infty : \langle\mathbf{mmmm}\rangle_\infty, \ (\tau^e)_\infty = 3S_\infty - \langle(\mathbf{m}\times\mathcal{R}U_\infty)_\infty\mathbf{m}\rangle_\infty.
\end{aligned}
$$

Potential

$$U(\mathbf{m}) \triangleq U(\mathbf{m},[\psi]) = \int_{\mathbb{S}^2} K(\mathbf{m},\mathbf{m}')\psi(\mathbf{m}',\mathbf{x},t)d\mathbf{m}',$$

$K(\mathbf{m},\mathbf{m}')$ is a smooth, real valued, symmetric kernel.

- the dipolar potential: $K(\mathbf{m},\mathbf{m}') = -\alpha\mathbf{m}\cdot\mathbf{m}'$
- Onsager potential: $K(\mathbf{m},\mathbf{m}') = \alpha|\mathbf{m}\times\mathbf{m}'|$
- Maier-Saupe potential: $K(\mathbf{m},\mathbf{m}') = \alpha|\mathbf{m}\times\mathbf{m}'|^2$
  where α is a parameter that measures the potential intensity.

Here we can see that the potential depend on the PDF from the appearances.

Potential relation

1D Onsager potential

$$K(\mathbf{m}, \mathbf{m}') = \alpha |\sin(\theta - \theta')|$$

1D Maier-Saupe potential

$$K(\mathbf{m}, \mathbf{m}') = \alpha |\sin(\theta - \theta')|^2$$

1D Maier-Saupe potential is an approximation of the 1D Onsager potential since $\sin^2(\theta - \theta') = \frac{1}{2}(1 - \cos 2(\theta - \theta'))$ and

$$|\sin(\theta - \theta')| = \frac{2}{\pi}\left[1 - \sum_{k=1}^{\infty} \frac{1}{2k-1} \cos 2k(\theta - \theta')\right].$$

Potential

The intrinsical potential forms are exactly some well-known functions.

- Example 1: dipolar potential:

$$U_{\theta\theta} + U = 0$$

$$U = \eta \cos(\theta - \theta_0).$$

- Example 2: Onsager potential:

$$U_{\theta\theta} + U = 4\alpha \frac{e^{-U}}{\int_0^{2\pi} e^{-U} d\theta}.$$

$$U_{\theta\theta\theta} + U_\theta U_{\theta\theta} + U U_\theta + U_\theta = 0.$$

Potential

- Example 3: Maier-Saupe potential:

$$U_{\theta\theta} + 4U = 2\alpha$$

$$U = \frac{\alpha}{2} + \eta \cos 2(\theta - \theta_0)$$

- Example 4: Maier-Saupe potential:

$$\mathcal{R} \cdot \mathcal{R} U + 6U = 4\alpha$$

$$U = \frac{2\alpha}{3} - \eta \left(|\mathbf{m} \times \mathbf{d}|^2 - \frac{2}{3} \right)$$

Entropy

- A. Arnold et al, Comm. Partial Diff. Equs. 01
- B. Jourdain et al, Arch. Rational Mech. Anal. 06

Denote $f(t,v)(v \in \mathbb{R}^n)$: the distribution function, *The physical entropy* (Boltzamann's H-functional) is

$$H(f) = \int_{\mathbb{R}^n} f \ln f dv.$$

$M^f(v)$: the Maxwellian distribution function, *the relative to the Maxwellian entropy* is

$$e(f|M^f) = \int_{\mathbb{R}^n} f \ln(\frac{f}{M^f}) dv. \tag{1}$$

Entropy

an admissible relative entropy: Let J be either $\mathbb{R}$ or $\mathbb{R}^+ := (0, \infty)$. Let $\psi \in C(\bar{J}) \cap C^4(J)$ satisfying the conditions

$$\begin{aligned}
&\psi(1) = 0, \\
&\psi'' \geq 0, \quad \psi'' \not\equiv 0 \quad \text{on} \quad J, \\
&(\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV} \quad \text{on} \quad J.
\end{aligned}$$

Let $\rho_1 \in L^1(\mathbb{R}^n)$, $\rho_2 \in L^1_+(\mathbb{R}^n)$ with $\int \rho_1 dx = \int \rho_2 dx = 1$ and $\rho_1/\rho_2 \in \bar{J}\rho_2(dx)$-a.e. Then

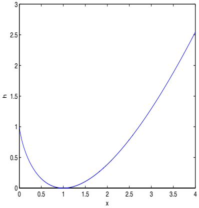
$$e_\psi(\rho_1|\rho_2) = \int_{\mathbb{R}^n} \psi\left(\frac{\rho_1}{\rho_2}\right)\rho_2(dx)$$

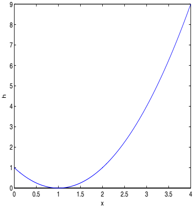
is called *an admissible relative entropy (of ρ_1 with respect to ρ_2) with generating function ψ.*

Entropy

- Admissible relative entropies $\longleftarrow$strictly convex function ψ.

$$h(x) = x\ln x - (x-1) \quad h(x) = x^p - 1 - p(x-1), p = 2$$





- The typical example: the physical relative entropy (1) generated by $\chi_{ph}(\sigma) = \sigma\ln\sigma - \sigma + 1$ not by $\psi = \sigma\ln\sigma$.

Entropy

- The physical relative entropy $e = e_{\chi_{ph}}$ can be written as

$$e(\rho|\rho_\infty) = F(\rho|A) - F(\rho_\infty|A); \quad F(\rho|A) = \int_{\mathbb{R}^n} (\rho \ln \rho + A(x)\rho) dx.$$

A potential

- The relative entropy is continuous: $\rho_j \to \rho(\text{as } j \to \infty)$ in $L^2_+(\mathbb{R}^n, \rho_\infty^{-1}(dx))$ with the normalization $\int \rho_j dx = \int \rho_\infty dx = 1$.

$$e_\psi(\rho_j|\rho_\infty) \to e_\psi(\rho|\rho_\infty) \quad as \quad j \to \infty.$$

0+1 Model and results

$$\psi_t = \frac{1}{De}[\psi_{\theta\theta} + (\psi U_\theta)_\theta], \quad \int_0^{2\pi} \psi(\theta, t)d\theta = 1$$

$$U = \alpha \int_0^{2\pi} \sin^2(\theta - \theta')\psi(\theta', t)d\theta'.$$

Theorem

$$\frac{1}{2}\left(\int_0^{2\pi} |\psi - \psi_\infty| d\theta\right)^2 \le H(t) := \int_0^{2\pi} \psi \ln\left(\frac{\psi}{\psi_\infty}\right)d\theta \le H(0)e^{-2\beta t}$$

provided that $\alpha^2 \le \frac{\lambda_1}{De}(1 - \frac{1}{2De}) - \beta.$

0+1 Model and results(continuous)

Here $\psi_\infty = \frac{e^{-U_\infty}}{\int_0^{2\pi} e^{-U_\infty} d\theta}$ is a formal expression, which satisfies the steady state equation $0 = \psi_{\theta\theta} + (\psi U_\theta)_\theta$.

Theorem

(i) $\alpha \leq 4$, *the only stationary solution* $\psi_\infty = 1/2\pi$.

(ii) $\alpha > 4$, $\psi_\infty = 1/2\pi$ *and* $\psi_\infty(\theta) = \frac{e^{-\eta^* \cos 2(\theta-\theta_0)}}{\int_0^{2\pi} e^{-\eta^* \cos 2\theta} d\theta}$, θ_0 *depends on the initial data,* η^* *is uniquely determined by*

$$\frac{\int_0^{2\pi} \cos 2\theta \, e^{-\eta^* \cos 2\theta} d\theta}{\int_0^{2\pi} e^{-\eta^* \cos 2\theta} d\theta} + \frac{2\eta^*}{\alpha} = 0.$$

[P. Constantin et al 05, I. Fatkullin et al 05, C. Luo et al 05, H.L. Liu et al 05]

Proof of results

$$\psi_t = \frac{1}{De}\partial_\theta\left[\psi\partial_\theta \ln(\frac{\psi}{\psi_\infty}) + \psi(U-U_\infty)_\theta\right]$$

Multiplication by $\mu = \ln\psi + U_\infty = \ln(\frac{\psi}{\psi_\infty})$ and integration

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}\psi\ln(\frac{\psi}{\psi_\infty})d\theta + \frac{1}{De}\int_0^{2\pi}\psi\left|(\ln\frac{\psi}{\psi_\infty})_\theta\right|^2 d\theta$$

$$\leq \frac{1}{2De^2}\int_0^{2\pi}\psi\left|(\ln\frac{\psi}{\psi_\infty})_\theta\right|^2 d\theta + \frac{1}{2}\int_0^{2\pi}\psi|(U-U_\infty)_\theta|^2 d\theta.$$

Proof of results(continuous)

- $\int_0^{2\pi} \psi|(U-U_\infty)_\theta|^2 d\theta \leq 2\alpha^2 \int_0^{2\pi} \psi \ln(\frac{\psi}{\psi_\infty}) d\theta,$
- The well-known Csiszár-Kullback inequality

$$\left(\int_0^{2\pi} |\psi-\psi_\infty| d\theta\right)^2 \leq 2\int_0^{2\pi} \psi \ln(\frac{\psi}{\psi_\infty}) d\theta.$$

- There exists a constant $\lambda_1 > 0$ such that

$$\int_0^{2\pi} \psi \ln(\frac{\psi}{\psi_\infty}) d\theta \leq \frac{1}{\lambda_1} \int_0^{2\pi} \psi \left|(\ln \frac{\psi}{\psi_\infty})_\theta\right|^2 d\theta$$

from Theorem 3.4 in [A. Arnold et al , 01].

Proof of results(continuous)

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}\psi\ln(\frac{\psi}{\psi_\infty})d\theta \le \left[\alpha^2+\lambda_1(-\frac{1}{De}+\frac{1}{2De^2})\right]\int_0^{2\pi}\psi\ln(\frac{\psi}{\psi_\infty})d\theta.$$

Here we can see that

$$H(t):=\int_0^{2\pi}\psi\ln(\frac{\psi}{\psi_\infty})d\theta \le H(0)e^{-2\beta t}$$

provided that

$$\alpha^2+\lambda_1(-\frac{1}{De}+\frac{1}{2De^2})\le -\beta<0,$$

where β is a arbitrary small positive constant.

1+1 Model and result

$$\psi_t = \frac{1}{De}[\psi_{\theta\theta} + (\psi U_\theta)_\theta] + \gamma(\psi \sin^2\theta)_\theta$$

Theorem

$$\frac{1}{2}\left(\int_0^{2\pi} |\psi - \bar{\psi}_\infty| d\theta\right)^2 \leq G(t) := \int_0^{2\pi} \psi \ln\left(\frac{\psi}{\bar{\psi}_\infty}\right) d\theta \leq G(0)e^{-2\beta t}$$

provided that $\alpha^2 \leq \frac{\lambda_1}{De}(1 - \frac{1}{2De}) - \beta.$

1+1 Model and results(continuous)

Here $\bar{\psi}_\infty = \frac{e^{-V_\infty}}{\int_0^{2\pi} e^{-V_\infty} d\theta}$, $V_\infty(\theta) = U_\infty + De\,\gamma(\frac{1}{2} - \frac{1}{4}\sin 2\theta)$ is a formal expression, which satisfies the steady state equation
$0 = \psi_{\theta\theta} + (\psi U_\theta)_\theta + De\gamma(\psi \sin^2\theta)_\theta$

Theorem (G. Warnecke & H. Zhang, 09)

$\bar{\psi}_\infty(\theta) = \frac{1}{Z}\left[1 + b(\theta)\right] e^{-a(\theta)}$
$a(\theta) = \frac{\alpha}{2} + \eta\cos 2(\theta - \theta_0) + \frac{\gamma\theta}{2}$, $b(\theta) = (e^{\gamma\pi} - 1)\frac{\int_0^\theta e^{a(\tau)}d\tau}{\int_0^{2\pi} e^{a(\tau)}d\tau}$,

$$\frac{1}{Z}\int_0^{2\pi}\cos 2(\theta - \theta_0)[1 + b(\theta)]\, e^{-a(\theta)}d\theta + \frac{2\eta}{\alpha} + \frac{\gamma}{2\alpha}\sin 2\theta_0 = 0,$$
$$\frac{1}{Z}\int_0^{2\pi}\sin 2(\theta - \theta_0)[1 + b(\theta)]\, e^{-a(\theta)}d\theta + \frac{\gamma}{2\alpha}\cos 2\theta_0 = 0.$$

1+1 Model and results(continuous)

- $\alpha < \alpha_1(\alpha_1 \approx 4.083)$, there is only one pair of solutions(η, θ_0)
- $\alpha > \alpha_2(\alpha_2 \approx 5.125)$, there is only a pair of solutions (η, θ_0).
- $\alpha_1 < \alpha < \alpha_2$, there are possible many pairs of solutions (η, θ_0), one/ two/ three.
  $(\eta, \theta_0) = (0.1333, 0.8374), (0.967, 1.728), (1.0596, 1.935)$ are solutions for $\gamma De = 0.01$ and $\alpha = 4.5$

Model and results for the decoupled case in 3D

Set $\mathbf{x}(t, \mathbf{x}_0)$ to be the flow map satisfying

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{u}(t, \mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

$\tilde{\psi}(t, \mathbf{m}) = \psi(t, \mathbf{x}(t, \mathbf{x}_0), \mathbf{m})$. Then

$$\frac{\partial \tilde{\psi}}{\partial t}(t, \mathbf{m}) = \frac{1}{De}\mathcal{R} \cdot \mathcal{R}\tilde{\psi} + \mathcal{R} \cdot [(\frac{1}{De}\mathcal{R}U - \mathbf{m} \times \kappa \cdot \mathbf{m})\tilde{\psi}],$$

If find a scalar function $A(\mathbf{m})$ and $B(\mathbf{m})$ such that

$$\mathcal{R}A(\mathbf{m}) = \mathbf{m} \times \kappa_\infty \cdot \mathbf{m}, \quad \mathcal{R}B(\mathbf{m}) = \mathbf{m} \times (\kappa - \kappa_\infty) \cdot \mathbf{m} \qquad (2)$$

When $\|\kappa - \kappa_\infty\|_{L^\infty} \to 0$ as $t \to \infty$,

$$\tilde{G}(t) := \int_{\mathbb{S}^2} \tilde{\psi} \ln(\frac{\tilde{\psi}}{\tilde{\psi}_\infty}) d\mathbf{m} \quad \leq \quad \tilde{G}(0)e^{-2\beta t}$$

Model and results for the decoupled case in 3D(continuous)

Here formally $\tilde{\psi}_\infty = e^{-(U_\infty + De\ A)} / \int_{\mathbb{S}^2} e^{-(U_\infty + De\ A)} d\mathbf{m}$. For example, when κ_∞ is symmetric $(\kappa_\infty^T = \kappa_\infty)$(elongational flows), $A = \frac{1}{2}\mathbf{m} \cdot \kappa_\infty \cdot \mathbf{m}$. Thus

$$\tilde{\psi}_\infty = e^{-(U_\infty + \frac{De}{2}\mathbf{m}\cdot\kappa_\infty\cdot\mathbf{m})} / \int_{\mathbb{S}^2} e^{-(U_\infty + \frac{De}{2}\mathbf{m}\cdot\kappa_\infty\cdot\mathbf{m})} d\mathbf{m}.$$

But for some cases we can prove such $\tilde{\psi}_\infty$ does not exist. e.g.

$$\kappa_\infty = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

solutions at the weak shear flow

[H. Zhang & P.W. Zhang, Physica D, 07]

- **Tumbling**
- **Logrolling**
- **Kayaking**

0+2 Model and results

$$\psi_t = \frac{1}{De}\mathcal{R}\cdot(\mathcal{R}\psi + \psi\mathcal{R}U),$$

$$U = \alpha\int_{\mathbb{S}^2}|\mathbf{m}\times\mathbf{m}'|^2\psi(\mathbf{m}',t)d\mathbf{m}',$$

$$\int_{\mathbb{S}^2}\psi(\mathbf{m},t)d\mathbf{m} = 1.$$

Similar result

$$\frac{1}{2}\left(\int_0^{2\pi}|\psi-\psi_\infty|d\theta\right)^2 \le N(t) := \int_0^{2\pi}\psi\ln\left(\frac{\psi}{\psi_\infty}\right)d\theta \le N(0)e^{-2\beta t}$$

where $\psi_\infty = \frac{e^{-U_\infty}}{\int_0^{2\pi}e^{-U_\infty}d\theta}$ satisfies

$$0 = \mathcal{R}\cdot(\mathcal{R}\psi + \psi\mathcal{R}U).$$

0+2 Model and results(continuous)

Theorem

$$\alpha^* = \min_{\eta} \frac{\int_0^1 e^{-\eta z^2} dz}{\int_0^1 (z^2 - z^4) e^{-\eta z^2} dz} \approx 6.731393. \tag{3}$$

All solutions are given explicitly by

$$\psi = k\, e^{-\eta(\mathbf{m}\cdot\mathbf{d})^2},$$

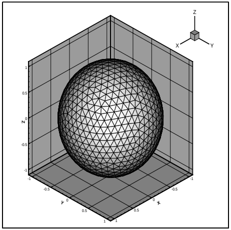
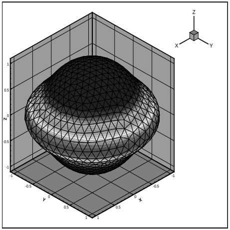
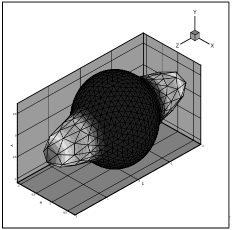
where $\mathbf{d} \in \mathbb{S}^2$ *is a parameter,* $\eta = \eta(\alpha)$ *and* $k = [4\pi \int_0^1 e^{-\eta z^2} dz]^{-1}$

$$\frac{3e^{-\eta}}{\int_0^1 e^{-\eta z^2} dz} - \left(3 - 2\eta + \frac{4\eta^2}{\alpha}\right) = 0. \tag{4}$$

0+2 Model and results(continuous)

More precisely,

- $0 < \alpha < \alpha^*$, $\psi_0 = 1/4\pi$.
- $\alpha = \alpha^*$, $\psi_0 = 1/4\pi$ and $\psi_1 = k_1\, e^{-\eta_1 (\mathbf{m}\cdot\mathbf{d})^2}$, $\eta_1 < 0$.
- $\alpha^* < \alpha < 7.5$, $\psi_0 = 1/4\pi$ and $\psi_i = k_i\, e^{-\eta_i (\mathbf{m}\cdot\mathbf{d})^2}$, $\eta_i < 0$ (i=1,2).
- $\alpha = 7.5$, $\psi_0 = 1/4\pi$ and $\psi_1 = k_1\, e^{-\eta_1 (\mathbf{m}\cdot\mathbf{d})^2}$, $\eta_1 < 0$.
- $\alpha > 7.5$, $\psi_0 = 1/4\pi$,
  $\psi_i = k_i\, e^{-\eta_i (\mathbf{m}\cdot\mathbf{d})^2} (i = 1, 2)$, $\eta_1 < 0$, $\eta_2 > 0$.

Entropy

Theorem

The unique stationary solution to the coupled problem with homogeneous Dirichlet boundary conditions on the velocity is

$$\mathbf{u}_\infty = 0 \quad \textit{and} \quad \psi_\infty \propto exp(-U_\infty).$$

Theorem

Set $(\mathbf{u}, \psi)$ to the coupled problem in the case homogeneous Dirichlet boundary conditions on the velocity. Then $\mathbf{u}$ converges exponentially fast in the $L^2_{\mathbf{x}}$ norm to $\mathbf{u}_\infty = 0$ and the entropy $H(t)$, where $\psi_\infty \propto exp(-U_\infty)$, converges exponential fast to 0. Therefore, ψ converges exponentially fast in the $L^2_{\mathbf{x}}(L^1_{\mathbf{m}})$ norm to ψ_∞.

Stress

Theorem

Consider a solution $(\mathbf{u}, \psi)$ to the coupled problem in the case homogeneous Dirichlet boundary conditions on the velocity. Then we have

$$\|\tau^e - \tau^e_\infty\|_{L^1_{\mathbf{x}}} \approx O(e^{-Ct}), \|\tau^s - \tau^s_\infty\|_{L^1_{\mathbf{x}}} < \infty, \text{ for a.e. } t > 0.$$

Generalization of the entropy method

Let $(\mathbf{u}, \psi)$ be a solution of time evolution system with the boundary condition $\mathbf{u} = \mathbf{g}(t)$ on $\partial\Omega$. And let $(\mathbf{u}_\infty, \psi_\infty)$ be a solution to the system with the same initial boundary conditions.
Set

$$\bar{\mathbf{u}}(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{x}) - \mathbf{u}_\infty(\mathbf{x}), \quad \bar{\psi}(t, \mathbf{x}, \mathbf{m}) = \psi(t, \mathbf{x}, \mathbf{m}) - \psi_\infty(\mathbf{x}, \mathbf{m}).$$

introduce the following quantities:

$$E = \frac{1}{2}\int_\Omega |\bar{\mathbf{u}}|^2 d\mathbf{x},$$

$$H = \int_\Omega \int_{\mathbb{S}^2} \psi \ln\left(\frac{\psi}{\psi_\infty}\right) d\mathbf{m} d\mathbf{x},$$

$$F = E + \lambda H, \quad \lambda = \frac{1-\gamma}{ReDe}.$$

Generalization of the entropy method

$$
\begin{aligned}
& \frac{dF}{dt} + \frac{\gamma}{Re}\int_\Omega |\nabla \bar{\mathbf{u}}|^2 d\mathbf{x} + \frac{\lambda}{De}\int_\Omega\int_{\mathbb{S}^2} \psi \left| \mathcal{R}\ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 d\mathbf{m}d\mathbf{x} \\
& + \lambda\frac{De}{2}\int_\Omega \langle(\mathbf{mm}:\nabla\bar{\mathbf{u}})^2\rangle\, d\mathbf{x} \\
= \;& -I_1 - \lambda I_2 - \lambda I_3 + \lambda I_4 + \lambda I_5 + 3\lambda I_6 - \lambda I_7 + \frac{\lambda}{De}(I_8 + I_9)
\end{aligned}
$$

$$
I_1 = \int_\Omega \bar{\mathbf{u}}\cdot\nabla\mathbf{u}_\infty\bar{\mathbf{u}}d\mathbf{x}, \quad I_2 = \int_\Omega\int_{\mathbb{S}^2} \bar{\mathbf{u}}\psi\cdot\nabla(\ln\psi_\infty)\, d\mathbf{m}d\mathbf{x},
$$

$$
I_3 = \int_\Omega \kappa_\infty : (\langle\mathbf{mmmm}\rangle - \langle\mathbf{mmmm}\rangle_\infty) : \nabla\bar{\mathbf{u}}\, d\mathbf{x},
$$

Generalization of the entropy method

$$I_4 = \int_\Omega \langle (\mathbf{m} \times \mathcal{R}(U - U_\infty))\mathbf{m} \rangle : \nabla \bar{\mathbf{u}} \, d\mathbf{x},$$

$$I_5 = \int_\Omega \int_{\mathbb{S}^2} (\mathbf{m} \times \mathcal{R}U_\infty)\mathbf{m}\bar{\psi} : \nabla \bar{\mathbf{u}} \, d\mathbf{x},$$

$$I_6 = \int_\Omega \langle \mathbf{m}\mathbf{m} \rangle_\infty : \nabla \bar{\mathbf{u}} \, d\mathbf{x},$$

$$I_7 = \int_\Omega \int_{\mathbb{S}^2} (\mathbf{m} \times (\kappa - \kappa_\infty) \cdot \mathbf{m})\psi \cdot \mathcal{R}(\ln \psi_\infty) d\mathbf{m} d\mathbf{x},$$

$$I_8 = \int_\Omega \int_{\mathbb{S}^2} \psi \left[\mathcal{R} \cdot \mathcal{R}(U - U_\infty)\right] d\mathbf{m} d\mathbf{x},$$

$$I_9 = \int_\Omega \int_{\mathbb{S}^2} \psi \left[+\mathcal{R}(U - U_\infty) \cdot \mathcal{R}(\ln \psi_\infty)\right] d\mathbf{m} d\mathbf{x}.$$

Generalization of the entropy method

When $\mathbf{u}_\infty$ is homogeneous flow, i.e., with a constant $\nabla \mathbf{u}_\infty$. Precisely, we assume that the boundary conditions on $\mathbf{u}$ are such that a homogeneous flow $\mathbf{u}_\infty(\mathbf{x}) = M\mathbf{x}$.

Generalization of the entropy method

M is antisymmetric

$$\begin{aligned} \mathbf{u} \to \mathbf{u}_\infty \quad & \text{in} \quad L^2_{\mathbf{x}}, \\ \psi \to \psi_\infty \quad & \text{in} \quad L^2_{\mathbf{x}}(L^1_{\mathbf{m}}) \end{aligned}$$

provided that

$$\begin{aligned} & \frac{\gamma}{Re} - (2\alpha + 1) - \|(\int_{\mathbb{S}^2} \psi_0^2 d\mathbf{m})^{\frac{1}{2}}\|_{L^\infty} > a_1 > 0, \\ & \frac{1}{C_{SLI}} \frac{\lambda}{De} - \left[3\alpha + 2\alpha \|(\int_{\mathbb{S}^2} \psi_0^2 d\mathbf{m})^{\frac{1}{2}}\|_{L^\infty} \right] > a_2 > 0 \end{aligned}$$

where C_{SLI} is from the Sobolev logarithmic inequality:

$$\int_\Omega \int_{\mathbb{S}^2} \phi \ln(\frac{\phi}{\psi_\infty}) d\mathbf{m} d\mathbf{x} \le C_{SLI} \int_\Omega \int_{\mathbb{S}^2} \phi \left| \mathcal{R} \ln(\frac{\phi}{\psi_\infty}) \right|^2 d\mathbf{m} d\mathbf{x}.$$

Generalization of the entropy method

M is symmetric(e.g. elongational flow)

$$
\begin{aligned}
\mathbf{u} &\to \mathbf{u}_\infty \quad \text{in} \quad L^2_{\mathbf{x}}, \\
\psi &\to \psi_\infty \quad \text{in} \quad L^2_{\mathbf{x}}(L^1_{\mathbf{m}})
\end{aligned}
$$

provided that

$$
\begin{aligned}
&\frac{\gamma}{Re} - (2\alpha+1) - \|(\int_{\mathbb{S}^2} \psi_0^2 d\mathbf{m})^{\frac{1}{2}}\|_{L^\infty} - \|M\|_{L^\infty} > a_3 > 0, \\
&\frac{1}{C_{SLI}} \frac{\lambda}{De} - \left[3\alpha + 2\alpha\|(\int_{\mathbb{S}^2} \psi_0^2 d\mathbf{m})^{\frac{1}{2}}\|_{L^\infty} - \|M\|_{L^\infty}\right] > a_4 > 0.
\end{aligned}
$$

Conclusion

- long time asymptotic behavior of the rodlike model in various cases $0+1$, $1+1$, $0+2$ and the given flow case.
- long time asymptotic behavior of entropy and stress for homogenous Dirichlet boundary condition.
- long time asymptotic behavior of the solution for non-homogenous Dirichlet boundary condition.