Stochastic Lagrangian models for turbulent flows

Application to a downscaling method for wind forecast at small scales

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Application: Near-Surface Wind Speed at fine scale

- Prospective: evaluate the wind potential
  - 2006: French wind turbines produced 1 GW / 10 GW in 2010.
- Prediction: forecasting wind at small scale
  - Integration of wind power inside the French Electric Network
  - High variability of the near-surface wind

- Numerical Weather Prediction (NWP)
  - Météo France: 25 km to *few* km (mesoscale) for short time prediction (24h, 36h), European Centre for Medium-Range Weather Forecasts (ECMWF) > 10 km

$\Longrightarrow$ Needs a Downscaling Method.

The SDM Project: Stochastic Downscaling Method

Aim: propose a new numerical method to improve the wind forecasting at small scale.

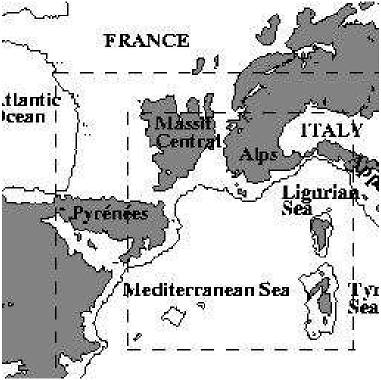
Joint work with

- J.F. Jabir (INRIA): **mathematical analysis of Lagrangian models and their confined version.**
- F. Bernardin (CETE), A. Rousseau, C. Chauvin (INRIA & LJK): **development of the numerical method.**
- P. Drobinski, T. Salameh (LMD): **application to meteorology and first validations of SDM.**

SDM is funded by the French Agency for the Environment and Energy Management (ADEME).

Geographical framework

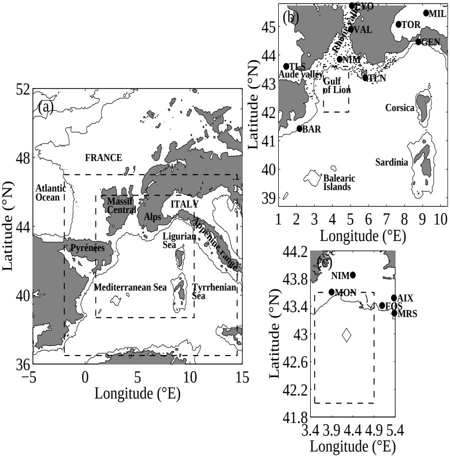
The French part of the Mediterranean basin:
(Languedoc-Roussillon, Provence-Alpes-Côtes d'Azur, Rhône-Alpes)



- First region in terms of production, with a high potential to develop.
- Mediterranean climate, mainly forced by the large scale climatic conditions, during the winter (November to Marsh).
- Complex association between large scale and regional scale (10 – 100 km). Important role of orography, the ground and sea contrast.

The Numerical Weather Prediction for large scales

We used the numerical model MM5 (mesoscale meteorological solver developed at NCAR, USA)



MM5 is run over 3 nested domains with respective horizontal resolutions of 27 km, 9 km and 3 km.

Coarse, medium and fine domain are centered at 43.7°N, 4.6°E and cover an area of 1350×1350 km^2, 738×738 km^2 and 120×174 km^2, respectively.

The $\diamond$ on Fig.(c) indicates the position of the buoy ASIS.

Our downscaling approach

- Our domain: one or several meshes of MM5 in the fine domain.
  - The boundary condition: the MM5 velocity
- Fluid dynamics represented in a Lagrangian approach
  - No more stability condition (CFL)
  - Nonlinear McKean Stochastic Differential Equations simulated by a particle method
  - Computational complexity very attractive.
- Stochastic Downscaling Method
  - Integrate more and more physics inside this model
  - A totally new kind of simulation: a lot of open problems (on the model and on its discretization)
  - Introduction of the splitting scheme to integrate the pressure effects.

Outline

- Modeling turbulent flows
  - The RANS equations
  - The Lagrangian approach
- SDM: the model
  - The basic Lagrangian model
  - The meteorologic closure
  - The guidance
- SDM: the numerical framework
  - The particle in cell method
  - The general algorithm
  - The splitting scheme
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Modeling turbulent flows: Statistical approach of turbulent flows

The Reynolds averages (or ensemble averages) are expectations:

$$\langle \mathscr{U} \rangle(t,x) := \int_\Omega \mathscr{U}(t,x,\omega) d\mathbb{P}(\omega).$$

The corresponding Reynolds decomposition of the velocity is

$$\mathscr{U}(t,x,\omega) = \langle \mathscr{U} \rangle(t,x) + \mathbf{u}(t,x,\omega),$$
$$\mathscr{P}(t,x,\omega) = \langle \mathscr{P} \rangle(t,x) + \mathbf{p}(t,x,\omega)$$

The random field $\mathbf{u}(t,x,\omega)$ is the turbulent part of the velocity.

Incompressible Navier Stokes equation in $\mathbb{R}^3$, for the velocity field $(\mathscr{U}^{(1)}, \mathscr{U}^{(2)}, \mathscr{U}^{(3)})$ and the pressure $\mathscr{P}$, with constant mass density ρ

$$\partial_t \mathscr{U} + (\mathscr{U} \cdot \nabla)\mathscr{U} = \nu \Delta \mathscr{U} - \frac{1}{\rho}\nabla \mathscr{P}, \quad t > 0,\ x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$
$$\nabla \cdot \mathscr{U} = 0, \quad t \geq 0,\ x \in \mathbb{R}^3,$$
$$\mathscr{U}(0,x) = \mathscr{U}_0(x),\ x \in \mathbb{R}^3.$$

The Reynolds averaged equation for the mean velocity

Assuming Reynolds decomposition, we obtain the unclosed equation with constant mass density ρ

$$\partial_t \langle \mathscr{U}^{(i)} \rangle + \sum_{j=1}^{3} \langle \mathscr{U}^{(j)} \rangle \partial_{x_j} \langle \mathscr{U}^{(i)} \rangle + \sum_{j=1}^{3} \partial_{x_j} \langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \nu \Delta \langle \mathscr{U}^{(i)} \rangle - \frac{1}{\rho} \partial_{x_i} \langle \mathscr{P} \rangle,$$

$$\nabla . \langle \mathscr{U} \rangle = 0, \ t \geq 0, \ x \in \mathbb{R}^3,$$

$$\langle \mathscr{U} \rangle (0, x) = \langle \mathscr{U}_0 \rangle (x), \ x \in \mathbb{R}^3,$$

where $\langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle - \langle \mathscr{U}^{(i)} \rangle \langle \mathscr{U}^{(j)} \rangle$.

Direct modeling of the Reynolds stress by a turbulent viscosity model:

$$\text{kinetic turbulent energy } k(t,x) := \sum_{i=1}^{3} \frac{1}{2} \langle \mathbf{u}^{(i)} \mathbf{u}^{(i)} \rangle (t,x)$$

and

$$\text{pseudo-dissipation } \varepsilon(t,x) := \nu \sum_{i=1}^{3} \sum_{j=1}^{3} \langle \partial_{x_j} \mathbf{u}^{(i)} \partial_{x_j} \mathbf{u}^{(i)} \rangle (t,x).$$

The equation for the Reynolds stress $(\langle \mathbf{u}^{(i)}\mathbf{u}^{(j)} \rangle, i, j)$

$$
\partial_t \langle \mathbf{u}^{(i)}\mathbf{u}^{(j)} \rangle + \Big(\langle \mathscr{U} \rangle \cdot \nabla_x \langle \mathbf{u}^{(i)}\mathbf{u}^{(j)} \rangle \Big) + \sum_{k=1}^{3} \partial_{x_k} \langle \mathbf{u}^{(i)}\mathbf{u}^{(j)}\mathbf{u}^{(k)} \rangle
$$

$$
= -\frac{1}{\rho} \langle \mathbf{u}^{(j)} \partial_{x_i} \mathbf{p} + \mathbf{u}^{(i)} \partial_{x_j} \mathbf{p} \rangle + \nu \sum_{k=1}^{3} \partial^2_{x_k} \langle \mathbf{u}^{(i)}\mathbf{u}^{(j)} \rangle
$$

$$
+ \nu \sum_{k=1}^{3} \langle \partial_{x_k} \mathbf{u}^{(i)} \partial_{x_k} \mathbf{u}^{(j)} \rangle - \sum_{k=1}^{3} \Big(\langle \mathbf{u}^{(i)}\mathbf{u}^{(k)} \rangle \partial_{x_k} \langle \mathscr{U}^{(j)} \rangle + \langle \mathbf{u}^{(j)}\mathbf{u}^{(k)} \rangle \partial_{x_k} \langle \mathscr{U}^{(i)} \rangle \Big) .
$$

Higher order closure : model equation for the Reynolds stress.

An alternative approach to compute the Reynolds stress

Let $f_E(t,x;V)$ be the probability density function (PDF) of the random field $\mathscr{U}(t,x)$, then

$$\langle \mathscr{U}^{(i)} \rangle (t,x) = \int_{\mathbb{R}^3} V^{(i)} f_E(t,x;V) dV,$$
$$\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle (t,x) = \int_{\mathbb{R}^3} V^{(i)} V^{(j)} f_E(t,x;V) dV.$$

The closure problem is reported on the PDE satisfied by the probability density function f_E.

In a series of papers (see e.g. Pope 85, ...,Dreben Pope 03), Stephen B. Pope propose to model the PDF f_E with a Lagrangian description of the flow.

Fluid particle model family

On a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the state vector $(\mathrm{X}_t, \mathrm{U}_t, \psi_t)$ satisfying

$$
\begin{aligned}
d\mathrm{X}_t =& \mathrm{U}_t dt, \\
d\mathrm{U}_t =& \left[-\frac{1}{\rho} \nabla_x \langle \mathscr{P} \rangle (t, \mathrm{X}_t) + \nu \triangle_x \langle \mathscr{U} \rangle (t, \mathrm{X}_t) \right] dt \\
& - G(t, X_t) \left(\mathrm{U}_t - \langle \mathscr{U} \rangle (t, \mathrm{X}_t) \right) dt + \sqrt{C(t, \mathrm{X}_t) \varepsilon(t, \mathrm{X}_t)} dW_t, \\
d\psi_t =& D_1(t, \mathrm{X}_t, \psi_t) dt + D_2(t, \mathrm{X}_t, \psi_t) d\widetilde{W}_t.
\end{aligned}
$$

$\left(W, \widetilde{W}\right)$ is a $4D$-Brownian motion.

- Compute de Eulerian fields $\langle \mathscr{U}^{(i)} \rangle (t, x)$, $\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle (t, x)$.
- Determine ε, C, G, D_1, D_2 by the RANS closure.

Compute the Reynolds averages $\langle \mathscr{U}^{(i)} \rangle$ and $\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle$

We call $f_L(t; x, V, \psi)$ the probability density function of $(\mathrm{X}_t, \mathrm{U}_t, \psi_t)$.

f_L satisfies a closed PDE: the Fokker-Planck equation a associated to the particle fluid SDE.

Case of incompressible flow with a constant mass density:

$$f_E(t, x; V, \phi) = \frac{f_L(t; x, V, \phi)}{\displaystyle\int_{\mathbb{R}^4} f_L(t; x, V, \psi) dV d\psi},$$

and for any bounded measurable function $F(v)$,

$$\langle F(\mathscr{U}) \rangle (t, x) = \mathbb{E}\left(F(\mathrm{U}_t) / \mathrm{X}_t = x \right).$$

In particular,

$$\left\langle \mathscr{U}^{(i)} \right\rangle (t, x) = \int_{\mathbb{R}^4} V^{(i)} \frac{f_L(t; x, V, \phi)}{\displaystyle\int_{\mathbb{R}^4} f_L(t; x, U, \psi) dU d\psi} dV d\phi = \mathbb{E}\left(\mathrm{U}_t^{(i)} / \mathrm{X}_t = x \right).$$

The underlying RANS Equations

$\mathbb{P}\left((\mathrm{X}_t, \mathrm{U}_t, \psi_t) \in dxdvd\phi\right) := f_L(t; x, v, \phi) dxdvd\phi$.

Fokker Planck Equation

$$
\begin{aligned}
\partial_t f_L + (v \cdot \nabla_x f_L) &= \frac{1}{\rho}\left(\nabla_x \langle \mathscr{P} \rangle(t,x) \cdot \nabla_v f_L\right) \\
&- \nu \left(\triangle_x \langle \mathscr{U} \rangle(t,x) \cdot \nabla_v f_L\right) - \nabla_v \cdot \left(G(t,x)(\langle \mathscr{U} \rangle(t,x) - v) f_L\right) \\
&+ \frac{C(t,x)\varepsilon(t,x)}{2} \triangle_v f_L - \nabla_\phi \cdot \left(D^1(t,x,\phi) f_L\right) \\
&+ \frac{1}{2} \triangle_\phi \left(D^2(t,x,\phi) f_L\right).
\end{aligned}
$$

- Integrating w.r.t. $dvd\phi$: conservation of mass Equation for $\rho(t,x) = \int f_L(t; x, v, \phi) dvd\phi$

$$
\partial_t \int f_L dvd\phi + \nabla_x \cdot \left(\frac{\int v f_L dvd\phi}{\int f_L dvd\phi} \int f_L dvd\phi \right) = 0
$$

$$
\partial_t \rho + \nabla_x \cdot (\rho \langle \mathscr{U} \rangle) = 0.
$$

The underlying RANS Equations

- Multiplying by v_i, integrating w.r.t. $dvd\phi$: RANS Equation

$$\partial_t \int v_i f_L dvd\phi + \int v_i v_j \partial_{x_j} f_L dvd\phi$$
$$= -\frac{1}{\rho}\nabla_x \langle \mathscr{P}^{(i)} \rangle \int f_L dvd\phi + \nu \triangle_x \langle \mathscr{U}^{(i)} \rangle \int f_L dvd\phi$$

$\Leftrightarrow$

$$\partial_t \left(\rho \langle \mathscr{U}^{(i)} \rangle \right) + \sum_j \partial_{x_j} \left(\rho \langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle \right) = -\nabla_x \langle \mathscr{P}^{(i)} \rangle + \nu \triangle_x \langle \mathscr{U}^{(i)} \rangle \rho.$$

- Multiplying by $v_i v_j$, integrating w.r.t. $dvd\phi$: model equation on the Reynolds stress.

$\Rightarrow$ Identification of the Lagrangian model coefficients ε, C, G, D_1, D_2.

The Simplified Langevin model (Pope 94)

$$
\begin{cases}
d\mathrm{X}_t = & \mathrm{U}_t dt, \\
d\mathrm{U}_t^{(i)} = & \Big[-\frac{1}{\rho} \frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \mathrm{X}_t) \\
& \quad - \left(\frac{1}{2} + \frac{3}{4} C_0\right) \frac{\varepsilon(t, \mathrm{X}_t)}{k(t, \mathrm{X}_t)} \Big(\mathrm{U}_t^{(i)} - \left\langle \mathscr{U}^{(i)} \right\rangle (t, \mathrm{X}_t) \Big) \Big] dt \\
& + \sqrt{C_0 \varepsilon(t, \mathrm{X}_t)} dW_t^{(i)}, \ \forall \ i \in \{1, 2, 3\}
\end{cases}
$$

\+ boundary conditions + wall boundary functions.

where $\varepsilon(t, x)$ and $k(t, x)$ are supposed to be known. $\langle \mathscr{P} \rangle (t, x)$ must be recovered by the Poisson equation

$$
\nabla^2 \langle \mathscr{P} \rangle = - \frac{\partial^2 \left\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \right\rangle}{\partial x_i \partial x_j}
$$

which guarantees that the averaged Eulerian velocity is divergence free.

The Basic model (Dreeben Pope 98)

Include the instantaneous turbulence frequency ω, satisfying

$$\begin{cases} d\mathrm{X}_t = & \mathrm{U}_t dt, \\ d\mathrm{U}_t^{(i)} = & \Big[-\frac{1}{\rho}\frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \mathrm{X}_t) \\ & \quad - \left(\frac{1}{2} + \frac{3}{4}C_0\right)\langle\omega\rangle(t,\mathrm{X}_t)\left(\mathrm{U}_t^{(i)} - \left\langle \mathscr{U}^{(i)} \right\rangle (t, \mathrm{X}_t)\right)\Big] dt \\ & + \sqrt{C_0 k(t,\mathrm{X}_t)\langle\omega\rangle(t,\mathrm{X}_t)}dW_t^{(i)}, \ \forall\ i \in \{1,2,3\} \\ & \\ d\omega_t = & -C_3\langle\omega\rangle(t,X_t)\left(\omega_t - \langle\omega\rangle(t,X_t)\right)dt - S_\omega\langle\omega\rangle(t,X_t)\omega_t dt \\ & + \sqrt{2C_3C_4\langle\omega\rangle^2(t,X_t)\omega_t}dW_t^{(4)}. \end{cases}$$

where

$$S_\omega = C_{\omega 2} + C_{\omega 1}\frac{\langle \mathbf{u}^{(i)}\mathbf{u}^{(j)}\rangle(t,x)}{\varepsilon(t,x)}\frac{\partial\langle\mathscr{U}^{(i)}\rangle}{\partial x_j}(t,x).$$

$\varepsilon(t,x)$ is recovered by the closure relation $\langle\omega\rangle(t,x) = \dfrac{\varepsilon(t,x)}{k(t,x)}$.

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The SDM model in $\mathcal{D}$

$$
\begin{cases}
d\mathrm{X}_t = & \mathrm{U}_t dt, \\
d\mathrm{U}_t^{(i)} = & \Big[-\frac{1}{\rho} \frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \mathrm{X}_t) \\
& \quad - \left(\frac{1}{2} + \frac{3}{4} C_0\right) \frac{\varepsilon(t, \mathrm{X}_t)}{k(t, \mathrm{X}_t)} \left(\mathrm{U}_t^{(i)} - \left\langle \mathscr{U}^{(i)} \right\rangle (t, \mathrm{X}_t) \right) \Big] dt \\
& + \sqrt{C_0 \varepsilon(t, \mathrm{X}_t)} dW_t^{(i)}, \ \forall \ i \in \{1, 2, 3\} \\
& \\
& + \text{ boundary conditions on } \partial\mathcal{D}.
\end{cases}
$$

- $k(t, x)$ is computed inside the model.
- $\langle \mathscr{P} \rangle (t, x)$ must be recovered by the Poisson equation

$$
\nabla^2 \langle \mathscr{P} \rangle = - \frac{\partial^2 \left\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \right\rangle}{\partial x_i \partial x_j}
$$

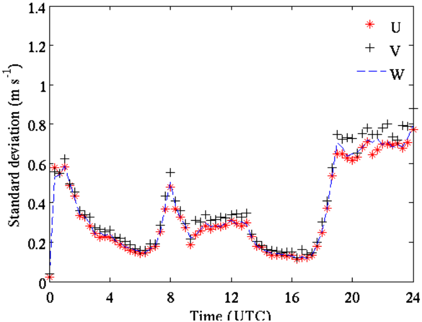
which guarantees that the averaged Eulerian velocity is divergence free.

The turbulent-kinetic-energy model
Meteorologic Closure in SDM

The components:

- The mixing length $\ell_m = \ell_m(z)$
- the turbulent viscosity $\nu_T = \frac{C_k}{\ell_m} k^{1/2}$
- A model for the dissipation rate: $\varepsilon(t,x,y,z) = \frac{C_\varepsilon}{\ell_m(z)} k^{3/2}(t,x,y,z)$

- Calibrate the coefficients to include more and more fine physics
- Link with the similarity theory
  Figure: Square root of $\langle \mathbf{u}^{(i)} \mathbf{u}^{(i)} \rangle$, for $i = 1, 2, 3$ in one cell of $\mathcal{D}$.
- Initial condition should satisfy the guessed physical behavior. ($\Longrightarrow k$)



The Guidance with an external velocity field (0.1)

The Downscaling method

Let $\mathcal{D}$ be an open set of $\mathbb{R}^3$, and a velocity V_{ext} given at $\partial\mathcal{D}$:

$$\begin{cases} d\mathrm{X}_t = & \mathrm{U}_t dt, \\ d\mathrm{U}_t = & \Big[-\frac{1}{\rho}\nabla \langle \mathscr{P} \rangle (t, \mathrm{X}_t) \\ & \quad - \left(\frac{1}{2} + \frac{3}{4}C_0\right) \frac{\varepsilon(t,\mathrm{X}_t)}{k(t,\mathrm{X}_t)} \left(\mathrm{U}_t - \langle \mathscr{U} \rangle (t, \mathrm{X}_t)\right)\Big] dt \\ & + \sqrt{C_0 \varepsilon(t,\mathrm{X}_t)} dW_t \\ & - \sum_{0 \le s \le t} 2\mathrm{U}_{s^-} \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathcal{D}\}} + \sum_{0 \le s \le t} V_{ext}(s, \mathrm{X}_s) \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathcal{D}\}}. \end{cases}$$

The two last terms should guarantee that

$$\langle \mathscr{U} \rangle (t,x) := \mathbb{E}\left[\mathrm{U}_t / \mathrm{X}_t = x\right] = V_{ext}(t,x), \forall x \in \partial\mathcal{D}.$$

The Guidance with an external velocity field (0.2)

The Downscaling method

Let $\mathcal{D}$ be an open set of $\mathbb{R}^3$, and a velocity V_{ext} given at $\partial\mathcal{D}$:

$$\begin{cases} d\mathrm{X}_t = & \mathrm{U}_t dt - V_{ext}(t,\mathrm{X}_t)\,\mathbb{1}_{\{\mathrm{X}_s\in\partial\mathcal{D}\}}dt, \\ d\mathrm{U}_t = & \left[-\frac{1}{\rho}\nabla\langle\mathscr{P}\rangle(t,\mathrm{X}_t) \right. \\ & \left. \quad -\left(\frac{1}{2}+\frac{3}{4}C_0\right)\frac{\varepsilon(t,\mathrm{X}_t)}{k(t,\mathrm{X}_t)}\left(\mathrm{U}_t-\langle\mathscr{U}\rangle(t,\mathrm{X}_t)\right)\right]dt \\ & +\sqrt{C_0\varepsilon(t,\mathrm{X}_t)}dW_t \\ & -\sum_{0\le s\le t} 2\mathrm{U}_{s^-}\,\mathbb{1}_{\{\mathrm{X}_s\in\partial\mathcal{D}\}} + \sum_{0\le s\le t} 2V_{ext}(s,\mathrm{X}_s)\,\mathbb{1}_{\{\mathrm{X}_s\in\partial\mathcal{D}\}}. \end{cases}$$

The two last terms should guarantee that

$$\langle\mathscr{U}\rangle(t,x) := \mathbb{E}\left[\mathrm{U}_t/\mathrm{X}_t = x\right] = V_{ext}(t,x), \forall x\in\partial\mathcal{D}.$$

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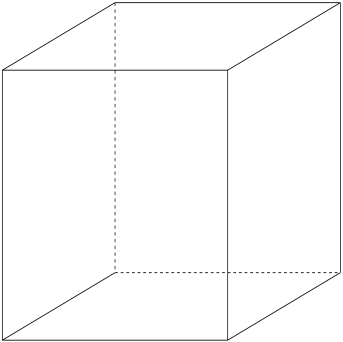
The numerical framework: particle method



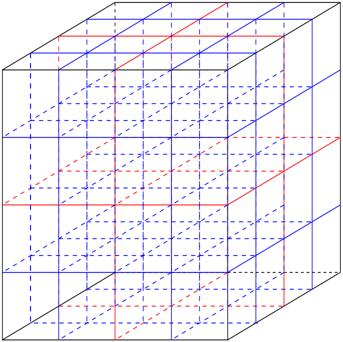
Our computational domain $\mathcal{D}$,
for example, a given cell of the MM5 solver.

Boundary condition:

$$\forall x \in \partial\mathcal{D},\ \langle \mathscr{U} \rangle(t,x) = V_{MM5}(t,x)$$

($MM5$ guideline.)

The numerical framework: particle method

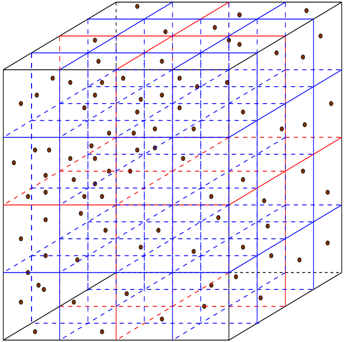


The computational space is divided in cells of given size.
Particle in cell (P.I.C.) technique to approximate the Eulerian fields like $\left\langle \mathscr{U}^{(i)} \right\rangle (t, x)$.

We compute the Eulerian fields (mean fields) at the center of each sub-cell only.

The numerical framework

The Particle in Cell method



- We introduce N_p particles in $\mathcal{D}$.
- $G(y,x) = \mathbb{1}\{y \in \mathcal{C}(x)\}$.
- Each cell $\mathcal{C}$ contains N_{pc} particles (constant mass density constraint).

$$\langle F(\mathscr{U})\rangle(t,x) \simeq \sum_{k=1}^{N_p} F\left(\mathrm{U}_t^{k,N_p}\right) \frac{G(\mathrm{X}_t^{k,N_p},x)}{\sum_{j=1}^{N_p} G(\mathrm{X}_t^{k,N_p},\mathrm{X}_t^{j,N_p})}.$$

Convergence: Propagation of chaos result.

The external velocity V_{ext} is imposed at the boundaries of $\mathcal{D}$.

The numerical algorithm

The N_p-Particles dynamic: for $j = 1, \ldots, N_p$

$$\begin{cases} d\mathrm{X}_t^{j,N_p} = \mathrm{U}_t^{j,N_p} dt, \\ d\mathrm{U}_t^{(i),j,N_p} = -\dfrac{1}{\rho}\dfrac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \mathrm{X}_t^{j,N_p})dt \\ \qquad\qquad +D_{\mathrm{U}}(t, \mathrm{X}_t^{j,N_p})dt + B_{\mathrm{U}}(t, \mathrm{X}_t^{j,N_p})dW_t^{(i),j,N_p} \\ \qquad\qquad +MM5 \text{ guideline terms at the boundary, } \forall\ i \in \{1,2,3\}. \end{cases}$$

- The coefficients D_{U}, B_{U} depend on the particles approximations of $\langle \mathscr{U} \rangle$, $\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle$ and its derivatives.

- $-\dfrac{1}{\rho}\dfrac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \mathrm{X}_t^{j,N_p})$ ensures that $\nabla \cdot \langle \mathscr{U} \rangle = 0$ and maintains the mass density constant.

$$\nabla^2 \langle \mathscr{P} \rangle = -\frac{\partial^2 \langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle}{\partial x_i \partial x_j}$$

The algorithm

A fractional step method: $n\Delta t \longrightarrow (n+1)\Delta t$ (Pope 85)

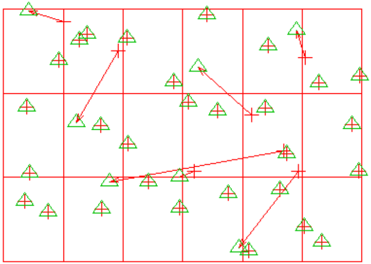
The N_p-Particles dynamic: for $j = 1, \dots, N_p$, for $n\Delta t \leq t \leq (n+1)\Delta t$,

$$\begin{cases} d\tilde{\mathrm{X}}_t^{j,N_p} = \tilde{\mathrm{U}}_t^{j,N_p} dt, \\ d\tilde{\mathrm{U}}_t^{(i),j,N_p} = -\frac{1}{\rho}\frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \tilde{\mathrm{X}}_t^{j,N_p})dt \\ \qquad\qquad + D_{\tilde{\mathrm{U}}}(t, X_t^{j,N_p})dt + B_{\tilde{\mathrm{U}}}(t, X_t^{j,N_p})dW_t^{(i),j,N_p} \\ \qquad\qquad + MM5 \text{ guideline terms at the boundary, } \forall\, i \in \{1,2,3\} \\ \\ \mathrm{X}_{n\Delta t}^{j,N_p}, \mathrm{U}_{n\Delta t}^{(i),j,N_p} \text{ given.} \end{cases}$$

- Correction of the particles positions $\tilde{\mathrm{X}}_{(n+1)\Delta t}^{j,N_p} \longrightarrow \mathrm{X}_{(n+1)\Delta t}^{j,N_p}$, in order to maintain the (discrete) uniform distribution.
- Correction of the particles velocities $\tilde{\mathrm{U}}_{(n+1)\Delta t}^{j,N_p} \longrightarrow \mathrm{U}_{(n+1)\Delta t}^{j,N_p}$ such that $\nabla.\langle \mathscr{U}^{(n+1)} \rangle = 0$.

The constant mass density constraint

- The density of particles has to be constant in each cell
  - Acts on $\{X_n^k\}_{1\leq k\leq N_p}$ ($+$)
  - An optimal transport problem



$N_{pc} = 2$.

$+$: particles after advancement

Δ: particles after density uniformization.

- Aim: Minimize the number of crossed cells
- Impacts on the statistics

Solve the optimal transportation problem

- Move the particles, such that the corresponding distribution becomes uniform.
- Minimize the global amount of displacement.

The density $\rho(x)$ is an Eulerian quantity approximated thanks to the nearest grid point formula

$$\rho(x_i) = \frac{\#\{\text{particles in } C_i\}}{N_{pc}}, \quad N_{pc} = \frac{N_p}{\#\{\text{cells}\}}$$

Can be viewed as a discretization of an optimal continuous transport problem (see e.g. Brenier 03):

Find a transport map $\phi : \mathcal{D} \to \mathcal{D}$, satisfying $\forall A \subset \mathcal{D}$

$$\int_{\phi^{-1}(A)} \rho(x) dx = \int_A \rho_0(x) dx$$

minimizing the L^2-cost

$$K(\phi) = \int_{\mathcal{D}} |x - \phi(x)|^2 dx.$$

Solve the optimal transportation problem

Well-known problem, having a well-known solution (see Benamou Brenier 2000 and ref. herein): ϕ is unique and given by

$$\phi = \mathbb{1}_{\mathcal{D}} - \nabla\gamma$$

with γ satisfying the Monge Ampère equation

$$\rho(x) = \det \begin{pmatrix} 1 - \frac{\partial^2\gamma}{\partial x_1^2} & -\frac{\partial^2\gamma}{\partial x_1 \partial x_2} & -\frac{\partial^2\gamma}{\partial x_1 \partial x_3} \\ -\frac{\partial^2\gamma}{\partial x_1 \partial x_2} & 1 - \frac{\partial^2\gamma}{\partial x_2^2} & -\frac{\partial^2\gamma}{\partial x_2 \partial x_3} \\ -\frac{\partial^2\gamma}{\partial x \partial x_3} & -\frac{\partial^2\gamma}{\partial x_2 \partial x_3} & 1 - \frac{\partial^2\gamma}{\partial x_3^2} \end{pmatrix}$$

- Numerical discretization: difficult
- Explicite solution in dimension one.

Solve the discrete optimal transportation problem

Classical assignment problem in network optimization.

The auction algorithm and its ε-scaling improvement.
(Bertsekas 98)

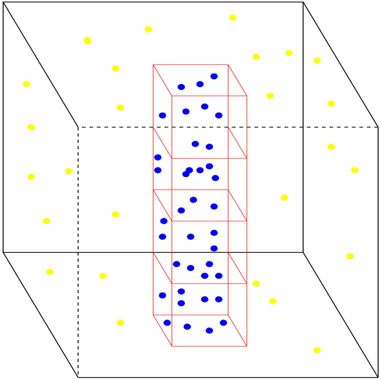
Worse case complexity: $\mathcal{O}(N^3 log(N))$
Averaged complexity in SDM: $\mathcal{O}(N^2)$.

The triangular 1D optimal transport

Suppose $\mathcal{D} = (0,1)$. The optimal transport is then entirely determined by the transfer condition:

$$\forall\, x \in \mathcal{D}, \quad \phi(x) = \int_0^x \rho(y)dy,$$

- The 1D discrete optimal transport problem is easy to solve.
- Solve the 3D case as a collection of 1D cases in the three directions.
- See Chauvin etal 08, for a comparision of the various methods for solving the OTP



Correction of the particles velocities

The divergence-free constraint

- $\tilde{\mathrm{U}}^{j,N_p}_{n\Delta t} \longrightarrow \mathrm{U}^{j,N_p}_{n\Delta t}$. The new velocity field $\langle \mathscr{U} \rangle_n$ must be divergence free.
- Classically obtained by solving a Poisson equation:

$$\begin{cases} \Delta P = -\frac{1}{\Delta t} \nabla \cdot \langle \tilde{\mathscr{U}} \rangle_n, \ x \in \mathcal{D}, \\ \left. \frac{\partial P}{\partial n} \right|_{\partial \mathcal{D}} = 0, \end{cases}$$

  and update the velocity field thanks to:

$$\langle \mathscr{U} \rangle_{n\Delta t} = \langle \tilde{\mathscr{U}} \rangle_{n\Delta t} + \Delta t \nabla P.$$

$$\mathrm{U}^{j,N_p}_{n\Delta t} = \tilde{\mathrm{U}}^{j,N_p}_{n\Delta t} + \Delta t \nabla P(\mathrm{X}^{j,N_p}_{n\Delta t})$$

  This insures the free divergence of $\langle \mathscr{U} \rangle_{n\Delta t}$.
- BUT the velocity field has to fulfill:

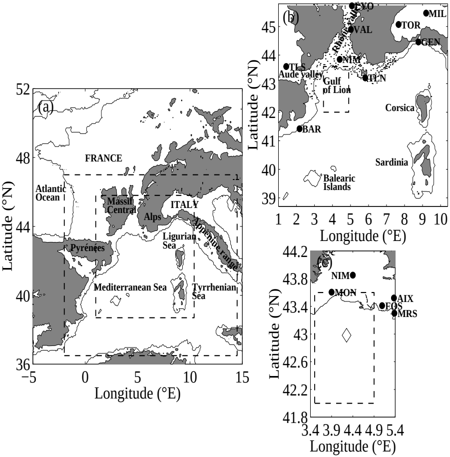
$$\forall x \in \partial\mathcal{D}, \ \langle \mathscr{U} \rangle_{n\Delta t}(t,x) = V_{MM5}(t,x)$$

$\Longrightarrow$ A specific projection procedure?

Outline

- Modeling turbulent flows
  - The RANS equations
  - The Lagrangian approach
- SDM: the model
  - The basic Lagrangian model
  - The meteorologic closure
  - The guidance
- SDM: the numerical framework
  - The particle in cell method
  - The general algorithm
  - The splitting scheme
- Numerical results (with Bernardin, Chauvin, Drobinski, Rousseau, Salameh)
  - Numerical convergence
  - Meteorologic validation
- Mathematical study
  - A simplified Lagrangian model
  - Spatially confined Lagrangian model
  - The Vlasov-Fokker-Planck Equation with specular boundary condition

Numerical Experiments: Application to Wind Refinement in a Realistic Case



The MM5 model is run for 3 days between March 23rd and 25th, 1998 over the 3 nested domains with 3 with respective horizontal resolutions of 27, 9 and 3 km.

The initial and boundary conditions are taken from the ECMWF (European Centre for Medium Range Weather Forecast) reanalyses.

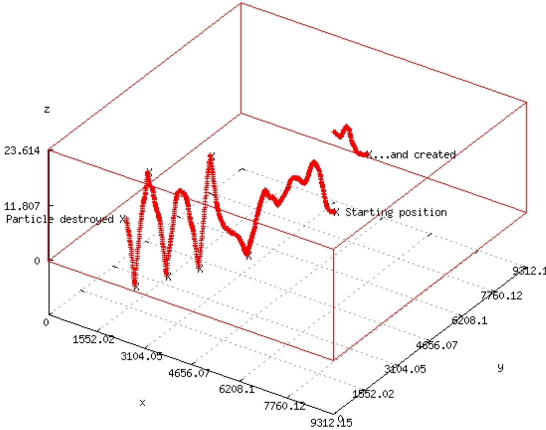
Diamond represents the location of the buoy ASIS.

Numerical results

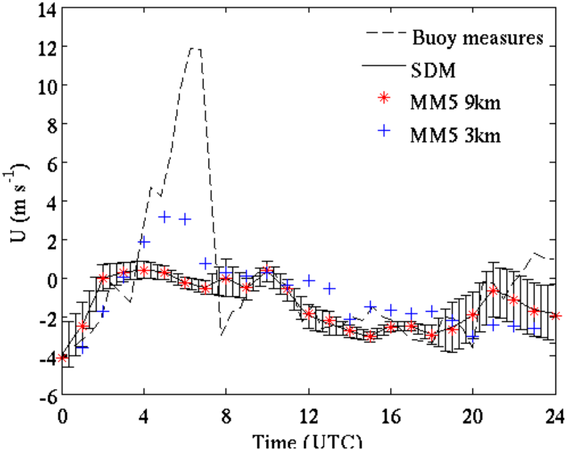
SDM Validation

- Calibrate the coefficients C_ε, ℓ_m on a simple model.
  - No terrain elevation.
- Test made on $6 \times 6 \times 6$ cells, with $T = 25\,h$.

  - $V^{(1)}_{MM5} \sim -1 m/s$,
    $V^{(2)}_{MM5} \sim -8 m/s$,
    $V^{(3)}_{MM5} \sim 0.0005 m/s$.
  - $\Delta t = 1s$
  - Run $\sim 8\,h$ for $N_{pc} = 800$.
  - Standard deviation σ independent of N_{pc}.
  - Small spin-up.

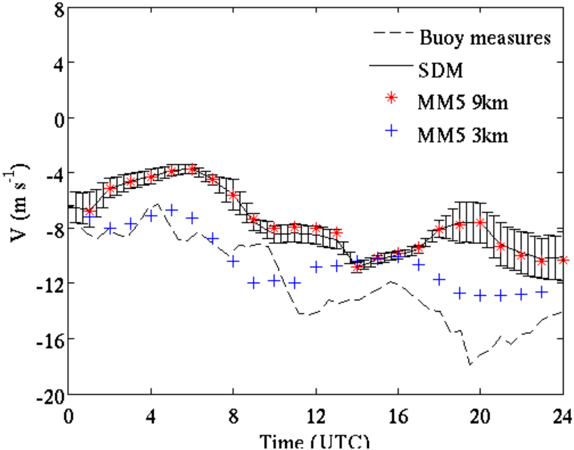


Numerical results



Comparison of MM5 at several scales, the buoy, and SDM in the cell containing the buoy.

Numerical results



Comparison of MM5 at several scales, the buoy, and SDM in the cell containing the buoy.

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Mathematical study of a simplified Langevin model

$2d$ dimensional SDE in the phase space (position, velocity):

$$\begin{cases} d\mathrm{X}_t = \mathrm{U}_t dt, \\ d\mathrm{U}_t = \mathbb{E}\left[b(u, \mathrm{U}_t)/\mathrm{X}_t\right]\big|_{u=\mathrm{U}_t} dt + \sigma(t, \mathrm{X}_t, \mathrm{U}_t) dW_t, \;\; t \in [0, T]. \end{cases}$$

Nonlinear drift term in the sense of McKean.

Related works:

Sznitman (86): Propagation of chaos for the Burgers Equation:

$$X_t = X_0 + W_t + 2\int_0^t u(s, X_s) ds$$

$$u(t,x)dx \text{ is the law of } X_t.$$

Dermoune (03): Conditional propag. of chaos for pressurless gas Eq.

$$X_t = X_0 + W_t + \int_0^t \mathbb{E}[v(X_0)/X_s] ds.$$

Here: local interaction in the d first variables $(x_1, \ldots, x_d)$. Hypoelliptic Fokker-Plank equation. We need a Propagation of chaos result.

Mathematical study of a simplified Langevin model

If $b : \mathbb{R}^{2d} \to \mathbb{R}^d$ is bounded, by the Girsanov theorem, any weak solution $(\mathrm{X}_t, \mathrm{U}_t, t \in [0, T])$ has a strictly positive density $(\rho_t(x, u), t \in [0, T])$,

$$\text{and} \qquad \mathbb{E}\left[b(u, \mathrm{U}_t)/\mathrm{X}_t = x\right] = B[x, u; \rho_t].$$

$$\text{where} \qquad B[x, u; \gamma] = \begin{cases} \dfrac{\int_{\mathbb{R}^d} b(u, v)\gamma(x, v)dv}{\int_{\mathbb{R}^d} \gamma(x, v)dv}, \text{ if } \displaystyle\int_{\mathbb{R}^d} \gamma(x, v)dv \neq 0, \\ 0, \text{ elsewhere} \end{cases}$$

$$\begin{cases} \mathrm{X}_t = \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = \mathrm{U}_0 + \int_0^t B\left[\mathrm{X}_s, \mathrm{U}_s; \rho_s\right] ds + \int_0^t \sigma(s, \mathrm{X}_s, \mathrm{U}_s)\, dW_s, \\ \mathbb{P}((\mathrm{X}_t, \mathrm{U}_t) \in dxdu) = \rho_t(x, u)dxdu, \ t \in [0, T], \end{cases}$$

Mathematical study of a simplified Langevin model

- σ is bounded and strongly elliptic: for $a := \sigma\sigma^*$, there exists $\lambda > 0$ s.t. for all $t \in (0, T]$, x, u, $v \in \mathbb{R}^d$,

$$\frac{|v|^2}{\lambda} \leq \sum_{i,j=1}^{d} a^{(i,j)}(t,x,u) v_i v_j \leq \lambda |v|^2 .$$

- For all $1 \leq i$, $j \leq d$, $\sigma^{(i,j)}(t,x,u)$ is B-Hölder continuous.

Theorem

Let $b \in C_b(\mathbb{R}^{2d}, \mathbb{R}^d)$, let $(\mathrm{X}_0, \mathrm{U}_0)$ s.t. $\mathbb{E}_{\mathbb{P}}\left[\|\mathrm{X}_0\|_{\mathbb{R}^d} + \|\mathrm{U}_0\|^2_{\mathbb{R}^d}\right] < +\infty$. On the previous hypotheses on the velocity diffusion coefficient σ, the system has a unique weak solution.

The smoothed system in the space variables

$$\begin{cases} \mathrm{X}_t^\varepsilon = \mathrm{X}_0 + \int_0^t \mathrm{U}_s^\varepsilon \, ds, \\ \mathrm{U}_t^\varepsilon = \mathrm{U}_0 + \int_0^t B_\varepsilon\left[\mathrm{X}_s^\varepsilon, \mathrm{U}_s^\varepsilon; \rho_s^\varepsilon\right] ds + \int_0^t \sigma(s, \mathrm{X}_s, \mathrm{U}_s)\, dW_s, \end{cases}$$

where $\mathcal{L}aw(\mathrm{X}_t^\varepsilon, \mathrm{U}_t^\varepsilon) = \rho_t^\varepsilon(x,u)dxdu$, and for every non–negative γ in $L^1(\mathbb{R}^{2d})$,

$$B_\varepsilon\left[x, u; \gamma\right] = \frac{\int_{\mathbb{R}^{2d}} b(v,u)\phi_\varepsilon(x-y)\gamma(y,v)\, dy\, dv}{\int_{\mathbb{R}^{2d}} \phi_\varepsilon(x-y)\gamma(y,v)\, dy\, dv + \varepsilon},$$

for a given regularization ϕ_ε of the Dirac mass in $\mathbb{R}^d$ in $C_c^1(\mathbb{R}^d)$.

$$\begin{cases} d\mathrm{X}_t^\varepsilon = \mathrm{U}_t^\varepsilon dt, \\ d\mathrm{U}_t^\varepsilon = \dfrac{\mathbb{E}\left[b(v, \mathrm{U}_t^\varepsilon)\phi_\varepsilon(x - \mathrm{X}_t^\varepsilon)\right]\Big|_{x=\mathrm{X}_t^\varepsilon, v=\mathrm{U}_t^\varepsilon}}{\mathbb{E}\left[\phi_\varepsilon(x - \mathrm{X}_t^\varepsilon)\right]\Big|_{x=\mathrm{X}_t^\varepsilon} + \varepsilon} dt + \displaystyle\int_0^t \sigma(s, \mathrm{X}_s^\varepsilon, \mathrm{U}_s^\varepsilon)\, dW_s, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [0, T]. \end{cases}$$

Existence of the smoothed system

On $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \in [0,T]), \mathbb{Q})$,
$\left(W_t^i;\ t \in [0,T];\ i \in \mathbb{N}\right)$ independent Brownian motions in $\mathbb{R}^d$,
$\{(\mathrm{X}_0^i, \mathrm{U}_0^i);\ i \in \mathbb{N}\}$ i.i.d, independent of the Brownian family, such that

$$\mathbb{Q}\left((\mathrm{X}_0^1, \mathrm{U}_0^1) \in dx\, du\right) = \rho_0(x,u)\, dx\, du.$$

$$\begin{cases} \mathrm{X}_t^{i,\varepsilon,N} = \mathrm{X}_0^i + \int_0^t \mathrm{U}_s^{i,\varepsilon,N} ds, \\ \mathrm{U}_t^{i,\varepsilon,N} = \mathrm{U}_0^i + \displaystyle\int_0^t \frac{\sum_{j=1,\ j\neq i}^N b(\mathrm{U}_s^{i,\varepsilon,N}, \mathrm{U}_s^{j,\varepsilon,N})\phi_\varepsilon(\mathrm{X}_s^{i,\varepsilon,N} - \mathrm{X}_s^{j,\varepsilon,N})}{\sum_{j=1,\ j\neq i}^N \left(\phi_\varepsilon(\mathrm{X}_s^{i,\varepsilon,N} - \mathrm{X}_s^{j,\varepsilon,N}) + \varepsilon\right)} ds + W_t^i, \\ \hfill i = 1, \ldots, N. \end{cases}$$

The sequence $\{\pi^N = \mathcal{L}aw\left(\frac{1}{N}\sum_{i=1}^N \delta_{\{\mathrm{X}^{i,\varepsilon,N}, \mathrm{U}^{i,\varepsilon,N}, W^i\}}\right), N \in \mathbb{N}\}$ is tight on $\mathcal{P}(C([0,T];\mathbb{R}^{3d}))$.

Spatially Confined Langevin model in $\mathcal{D} \subset \mathbb{R}^d$

Impact problem with stochastic forcing.
(Deterministic motions, see e.g. Schatzman 98 , Ballard 01).

Homogeneous Dirichlet condition for the impact problem:

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = & \mathrm{U}_0 + \int_0^t \langle b(\mathrm{X}_s, \mathrm{U}_s) \rangle \, ds + W_t \\ & - \sum_{0 \leq s \leq t} 2 \left(\mathrm{U}_{s-} \cdot n(\mathrm{X}_s) \right) n(\mathrm{X}_s) \, \mathbb{1}_{\{\mathrm{X}_s \in \partial \mathcal{D}\}} \end{cases}$$

must satisfy the averaged no-permeability condition: $\forall x \in \partial \mathcal{D}$,

$$\langle \mathscr{U} \cdot n \rangle (t, x) = \mathbb{E} \left[\mathrm{U}_t \cdot n(\mathrm{X}_t)) \big/ \mathrm{X}_t = x \right] = 0.$$

Sufficient condition for the averaged no-permeability

Lemma

Assume that $\gamma(\rho)$ satisfies the following properties:

$$i)\ \gamma(\rho)(t,x,u) = \gamma(\rho)(t,x,u - 2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)),$$
$$dt \otimes d\sigma_{\mathcal{D}} \otimes du,\ a.e.$$

$$ii)\ \int_{\mathbb{R}^d} |(v\cdot n_{\mathcal{D}}(x))|\gamma(\rho)(t,x,v)\,dv < +\infty,\ \ dt\otimes d\sigma_{\mathcal{D}},\ \ a.e.$$

$$iii)\ \int_{\mathbb{R}^d} \gamma(\rho)(t,x,v)\,dv > 0,\quad dt\otimes d\sigma_{\mathcal{D}},\ \ a.e.$$

Then the averaged no-permeability holds.

Confined Langevin model in $\mathcal{D} = \mathbb{R}^{d-1} \times \mathbb{R}^+$

Theorem

$b : \mathbb{R}^{2d} \to \mathbb{R}^d$ bounded Lipschitz, $(\mathrm{X}_0, \mathrm{U}_0)$ s.t. $\mathbb{E}[\|\mathrm{X}_0\|^2 + \|\mathrm{U}_0\|^4] < +\infty$. ρ_0 has its support in $\mathcal{D}$.

There exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \in [0, T]), \mathbb{P}, (W_t))$ and $\mathcal{F}_t$–adapted process, (X, U) valued in $C([0, T]; \mathbb{R}^{d-1} \times \mathbb{R}^+) \times \mathcal{D}([0, T]; \mathbb{R}^d)$ s.t.

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = & \mathrm{U}_0 + \int_0^t \mathbb{E}\left[b(u, \mathrm{U}_s)/\mathrm{X}_s\right]\big|_{u=\mathrm{U}_s} ds + W_t \\ & - \sum_{0<s\leq t} 2\left(\mathrm{U}_{s-} \cdot n_{\mathcal{D}}(\mathrm{X}_s)\right) n_{\mathcal{D}}(\mathrm{X}_s)\, \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathcal{D}\}} \end{cases}$$

and the sequence $\{\tau_n;\ n \in \mathbb{N}\}$ defined by

$$\tau_0 = \inf\{t \geq 0; \mathrm{X}_t^d = 0\}, \qquad \tau_n = \inf\{t > \tau_{n-1}; \mathrm{X}_t^d = 0\}$$

is well defined and grows to infinity.

The Vlasov-Fokker-Planck Equation

Theorem

The joint law of $(\mathrm{X}_t, \mathrm{U}_t)$ *has a density* $\rho(t,x,v)$*, which is the unique weak solution of the following Vlasov-Fokker-Planck Equation with specular boundary condition:*

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + v \cdot \nabla_x \rho + \left(\left[\dfrac{\int_{\mathbb{R}^2} b(v,u)\rho(t,x,u)du}{\int_{\mathbb{R}^2} \rho(t,x,u)du} \right] \cdot \nabla_v \rho \right) = \dfrac{1}{2}\Delta_v \rho, \\ \qquad\qquad\qquad\qquad\qquad (t,x,v) \in (0,T) \times \mathcal{D} \times \mathbb{R}^d, \\ \rho(0,x,v) = \rho_0(x,v) \text{ given}, (x,v) \in \mathcal{D} \times \mathbb{R}^d, \\ \rho(t,x,v) = \rho(t,x,v - 2(v \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \\ \qquad\qquad\qquad\qquad\qquad (t,v) \in (0,T) \times \mathbb{R}^d, x \in \partial\mathcal{D}. \end{cases}$$

The sufficient condition for the averaged no-permeability is fulfilled.

The confined Brownian motion primitive in the half line

Starting from $(\mathrm{X}_0, \mathrm{U}_0)$ with $\mathrm{X}_0 > 0$, and a (B_t) Brownian motion in $\mathbb{R}$,

$$\mathscr{Y}_t = \mathrm{X}_0 + \int_0^t \mathscr{V}_s ds, \qquad \mathscr{V}_t = \mathrm{U}_0 + B_t.$$

$$\text{Set } \mathrm{X}_t = |\mathscr{Y}_t|,$$
$$\mathrm{U}_t = \mathscr{V}_t \mathscr{S}_{t^+}, \text{ with } \mathscr{S}_t := sign(\mathscr{Y}_t).$$

Lemma

If ρ_0 has its support in $\mathbb{R} \times (0, +\infty) \times \mathbb{R}$, then $\mathscr{S}_t$ jumps a countable number of times and U_t solves

$$\mathrm{U}_t = \mathrm{U}_0 + W_t - 2 \sum_{0<s\leq t} \mathrm{U}_{s-} \mathbb{1}_{\{\mathrm{X}_s = 0\}}, \ \mathbb{P}.a.s.$$

where W_t is a Brownian motion.

(Lachal 97: Passage time of the Brownian motion primitive at 0)

And for other domains ?

Theorem

$\mathcal{D}$ a smooth bounded domain in $\mathbb{R}^d$.
$b : \mathbb{R}^{2d} \to \mathbb{R}^d$ bounded.
Weak existence (in $L^2\left((0,T)\times\mathcal{D}; H^1(\pi,\mathbb{R}^d)\right)$) and uniqueness of the Vlasov-Fokker-Planck Equation with specular boundary condition:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + v \cdot \nabla_x \rho + \left(\left[\dfrac{\int_{\mathbb{R}^d} b(v,u)\rho(t,x,u)du}{\int_{\mathbb{R}^d} \rho(t,x,u)du} \right] \cdot \nabla_v \rho \right) = \dfrac{1}{2}\Delta_v \rho, \\ \qquad\qquad\qquad\qquad (t,x,v) \in (0,T)\times\mathcal{D}\times\mathbb{R}^d, \\ \rho(0,x,v) = \rho_0(x,v),\ (x,v)\in\mathcal{D}\times\mathbb{R}^d, \\ \rho(t,x,v) = \rho(t,x,v - 2(v\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \\ \qquad\qquad\qquad\qquad (t,v)\in(0,T)\times\mathbb{R}^d, x\in\partial\mathcal{D}. \end{cases}$$

Propagation of initial Maxwellian bounds for the sub- and super- solutions.

Euler scheme for linear confined models, $\mathcal{D} = \mathbb{R}^+$

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = & \mathrm{U}_0 + \int_0^t b(\mathrm{U}_s) ds + W_t - \sum_{0<s\leq t} 2\mathrm{U}_{s-} \mathbb{1}_{\{X_s=0\}}. \end{cases}$$

Euler scheme: $\Delta t > 0$ and $K \in \mathbb{N}$ s.t. $T = K\Delta t$; $t_k := k\Delta t$, $1 \leq k \leq K$, $(\bar{\mathrm{X}}_{t_k}, \bar{\mathrm{U}}_{t_k})$ given, compute $(\bar{\mathrm{X}}_{t_{k+1}}, \bar{\mathrm{U}}_{t_{k+1}})$:

if $\bar{\mathrm{X}}_{t_k} + \Delta t \bar{\mathrm{U}}_{t_k} \geq 0$ then

$$\begin{aligned} \bar{\mathrm{X}}_{t_{k+1}} &= \bar{\mathrm{X}}_{t_k} + \Delta t \bar{\mathrm{U}}_{t_k} \\ \bar{\mathrm{U}}_{t_{k+1}} &= \bar{\mathrm{U}}_{t_k} + \Delta t b(\bar{\mathrm{U}}_{t_k}) + (W_{t_{k+1}} - W_{t_k}). \end{aligned}$$

else

$$\begin{aligned} \tau_k &= t_k + \bar{\mathrm{X}}_{t_k} / \bar{\mathrm{U}}_{t_k}. \\ \bar{\mathrm{X}}_{t_{k+1}} &= -(t_{k+1} - \tau_k)\bar{\mathrm{U}}_{t_k} \\ \bar{\mathrm{U}}_{t_{k+1}} &= \underbrace{-\bar{\mathrm{U}}_{t_k} - (\tau_k - t_k) b(\bar{\mathrm{U}}_{t_k})}_{-\bar{\mathrm{U}}_{\tau_k}} + (t_{k+1} - \tau_k) b(-\bar{\mathrm{U}}_{t_k}) + (W_{t_{k+1}} - W_{t_k}). \end{aligned}$$

Weak convergence of the Euler scheme

Lemma

If $b(u) = -cu$ then $h(t, x, u)$ have bounded spatial derivatives up to the order 4 and

$$\left|\mathbb{E}f(\mathrm{X}_T) - \mathbb{E}f(\bar{\mathrm{X}}_T)\right| \leq C\Delta t$$

for f in $C_b(\mathbb{R})$.

Where

$$h(t,x,u) = \mathbb{E}\left(f(X_T^{t,x,u})\right) \text{ solves the following PDE in } [0,T] \times \mathbb{R}^+ \times \mathbb{R}:$$

$$\begin{cases} \dfrac{\partial h}{\partial t} + u\nabla_x h + b(u)\nabla_u h + \dfrac{1}{2}\Delta_u h = 0, \\ h(t,0,u) = h(t,0,-u), \\ h(T,x,u) = f(x). \end{cases}$$

Concluding remarks

Stochastic Downscaling Method: next step

- More physics! (terrain elevation, temperature...)

Numerical analysis

- Validation of the splitting algorithm for Lagrangian models.
- Numerical analysis of the PIC method.

On Lagrangian models

- Confined Brownian primitive in a domain with a forcing
- Divergence free Lagrangian model (work in progress with J. Fontbona).
- Regularity and upper-bounds for the solution of a linear backward PDE with specular boundary condition.