REGULARITY OF THE FREE BOUNDARY IN A TWO-PHASE SEMILINEAR PROBLEM IN TWO DIMENSIONS

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ABSTRACT. We study minimizers of the energy functional

$$\int_D \left(|\nabla u|^2 + 2(\lambda_+(u^+)^p + \lambda_-(u^-)^p)\right)\,\mathrm{d}x$$

for $p \in (0,1)$ without any sign restriction on the function u. The main result states that in dimension two the free boundaries $\Gamma^+ = \partial\{u > 0\} \cap D$ and $\Gamma^- = \partial\{u < 0\} \cap D$ are C^1-regular.

1. Introduction and Main Result

1.1. **Problem.** Given a bounded domain $D \subset \mathbb{R}^n$ with smooth boundary and $u_0 \in W^{1,2}(D) \cap L^\infty(D)$ consider the variational problem

$$J(u) = \int_D \left(|\nabla u|^2 + 2F(u)\right)\,\mathrm{d}x \to \min, \qquad u - u_0 \in W_0^{1,2}(D), \tag{1.1}$$

where $F(u)$ is a Hölder continuous function of the type

$$F(u) = \lambda_+(u^+)^p + \lambda_-(u^-)^p, \tag{1.2}$$

$$\lambda_\pm > 0, \quad 0 < p < 1, \quad u^\pm = \max\{\pm u, 0\}. \tag{1.3}$$

The existence of minimizers is straightforward and is obtained by the direct methods of calculus of variations. Note that generally there may exist more than one minimizer with given boundary values u_0, since the functional J is not convex, see e.g. [Phi83b].

We are not imposing any sign constraints on u_0, so the minimizers u may take both positive and negative values. We regard the regions

$$\Omega^+(u) = \{u > 0\}, \quad \Omega^-(u) = \{u < 0\}$$

as two different phases of u and the main objective in this paper is to study their interfaces

$$\Gamma^\pm(u) = \partial\Omega^\pm(u) \cap D,$$

which we also call *free boundaries*, as they are apriori unknown. Note that since F is nonsmooth, a fattening of the zero set $\{u = 0\}$ may occur so that Γ^+ and Γ^- split from each other, see Figure 1.

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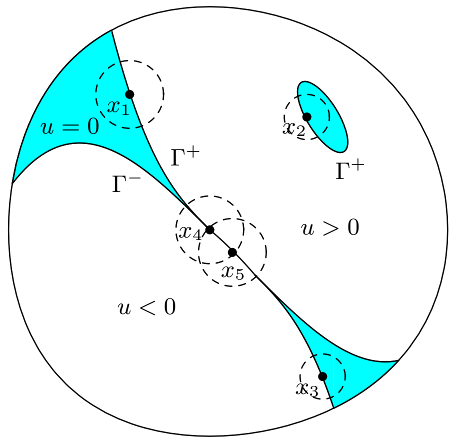

FIGURE 1. $\Gamma^+ = \partial\{u > 0\} \cap D$, $\Gamma^- = \partial\{u > 0\} \cap D$; x_1, x_2, x_3 one-phase points; x_4 branching two-phase point; x_5 non-branching two-phase point.

The Euler-Lagrange equation associated with the variational problem (1.1) is

$$\Delta u = p\left(\lambda_+ (u^+)^{p-1}\chi_{\{u>0\}} - \lambda_-(u^-)^{p-1}\chi_{\{u<0\}}\right) \quad \text{in } D. \tag{1.4}$$

The minimizers will satisfy this equation, however, since even the local integrability of the right-hand side is apriori unknown, the equation should be understood is the sense of the first domain variation

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} J(u(x+\epsilon\phi(x))) = 2\int_D [\nabla u \cdot D\phi\nabla u - (\operatorname{div}\phi)F(u)]\,\mathrm{d}x = 0, \quad \phi \in C^1_0(D).$$

This very weak notion of solution is sufficient to derive Weiss's monotonicity formula (see Lemma 3.3 below), which is one of the key tools in the study of this equation, both in elliptic case [Wei98] and its parabolic counterpart [Wei99].

Note that because of the regularity of minimizers (see below), the sets $\Omega^\pm(u)$ are open, and therefore u is real analytic in $\Omega^-(u) \cup \Omega^+(u)$, and the equation (1.4) is satisfied in the classical sense there.

1.2. **Known results.** The problem has been well studied in the one-phase setting, i.e. when the minimizers are assumed to be nonnegative. It has been established by Phillips [Phi83a] that nonnegative minimizers u of J satisfy

$$u \in C^{1,\beta-1}_{\text{loc}}(D), \qquad \beta = \frac{2}{2-p}. \tag{1.5}$$

This is the best regularity possible, as there is a one-dimensional example $u(x_1) = C_+(x_1^+)^\beta$, for a suitably chosen $C_+ > 0$. Concerning the regularity of the free boundary, Phillips [Phi83b] and Alt and Phillips [AP86] have proved that that there exists a singular set $\Sigma \subset \Gamma^+$ of $(n-1)$-Hausdorff measure zero such that $\Gamma^+ \setminus \Sigma$ is C^∞ (actually real analytic). Moreover, they have shown that in dimension $n = 2$ the singular set $\Sigma = \emptyset$, i.e. the free boundary is fully regular.

Furthermore, Nagano [Nag03] proved the differentiability of the free boundary in two dimensions for a class of solutions containing non-minimizers by using an

Alexandrov-type reflection-comparison argument, somewhat similar to the one in [SW06] and Section 5 below.

The two-phase case, i.e. when there are no sign assumptions on u, has also been studied in the literature. Thus, Giaquinta and Giusti [GG84] have proved that the optimal regularity (1.5) holds also for minimizers of a general class of energy functionals that includes J. Recently, a direct proof of this optimal $C^{1,\beta-1}_{\mathrm{loc}}$ regularity, specifically for the minimizers of J, has been given in a note by Lindgren and Silvestre [LS05] by using a blowup argument combined with a Liouville-type theorem.

Despite the fact that the optimal regularity for the minimizers of J is known, there are virtually no results available concerning the regularity of the free boundaries $\Gamma^\pm$ in the two-phase case. The present paper contributes in this direction, albeit only in dimension $n = 2$ (see Theorem 1.1 below). An important tool that we use is a monotonicity formula due to Weiss [Wei98] that allows to study the so-called blowups at free boundary points.

Versions of the problem above for the values $p = 0$ and $p = 1$ have also been studied in the literature.

For $p = 0$, one takes $F(u) = \lambda_+\chi_{\{u>0\}} + \lambda_-\chi_{\{u<0\}}$ for $u \neq 0$ and $F(0) = \min\{\lambda_+, \lambda_-\}$. The corresponding Euler-Lagrange equation becomes

$$\Delta u = 0 \quad \text{in } \Omega^+(u) \cup \Omega^-(u)$$
$$|\nabla u^+|^2 - |\nabla u^-|^2 = \lambda_+ - \lambda_- \quad \text{on } \{u = 0\},$$

which should be understood in a suitable weak sense. This problem has been studied by Alt, Caffarelli and Friedman in [ACF84], where they introduced their celebrated monotonicity formula and proved the optimal $C^{0,1}_{\mathrm{loc}}$ regularity of the minimizers. Later, in the seminal series of papers [Caf87, Caf89, Caf88], Caffarelli has proved that a Reifenberg-type flatness condition on the free boundary implies its Lipschitz continuity, $C^{1,\alpha}$ regularity, and real-analyticity, similar to the regularity theory of minimal surfaces.

In the case $p = 1$, the equation (1.4) becomes

$$\Delta u = \lambda_+\chi_{\{u>0\}} - \lambda_-\chi_{\{u<0\}} \quad \text{in } D,$$

which is known as the two-phase obstacle problem. The optimal $C^{1,1}_{\mathrm{loc}}$ regularity has been proved in this case by Ural'tseva [Ura01] and Weiss [Wei01]. The regularity of the free boundary for this problem is also well understood. Thus, Shahgholian, Ural'tseva and Weiss [SUW07] (Shahgholian and Weiss [SW06] in dimension $n = 2$) have shown that near so-called branching points the free boundaries Γ^+ and Γ^- are C^1 regular and tangent to each other (with an example showing that $\Gamma^\pm$ are not generally of class $C^{1,\mathrm{Dini}}$). Thus, the singularity in this case may occur only at one-phase points.

1.3. **Main result.** The main result of this paper concerns the the regularity of the free boundary in dimension $n = 2$.

Theorem 1.1 (C^1 regularity of $\Gamma^\pm$ in $n = 2$)**.** *Let u be a minimizer of* (1.1) *with $0 < p < 1$ in dimension $n = 2$. Then the free boundaries $\Gamma^\pm(u)$ are C^1 regular.*

This result is akin to that of Shahgholian and Weiss [SW06] for the two-phase obstacle problem ($p = 1$) and Alt and Phillips [AP86] for the full regularity of the free boundary in the one-phase case ($0 < p < 1$) in dimension $n = 2$. The key difficulty lies in analyzing the behavior of Γ^+ and Γ^- at the points of their contact (so called branching points).

The outline of the paper is as follows.

- In Section 2 we give a general description of the structure of the free boundary, as can be seen on Figure 1.
- Section 3 contains preliminary material such as rescalings and blowups, nondegeneracy. Here, we also introduce the monotonicity formula of Weiss [Wei98], which implies that blowups at branching points are homogeneous of order β.
- In Section 4 we give a classification of homogeneous global minimizers of the functional J (that include all blowups) in dimension $n = 2$, see Theorem 4.1.
- In Section 5 we prove a key uniqueness theorem for blowups at branching free boundary points (Theorem 5.1), thus establishing the differentiability of the free boundaries $\Gamma^\pm$ (Corollary 5.5).
- Finally, in Section 6 we give the proof of the C^1 regularity of $\Gamma^\pm$ (Theorem 1.1) by a careful application of the methods of Section 5.

2. The structure of the free boundary

We start with a brief discussion of various types of free boundary points, as illustrated on Figure 1.

Let u be a minimizer of (1.1) and $x_0 \in \Gamma = \Gamma^+ \cup \Gamma^-$.

1) We say that x_0 is a *one-phase* free boundary point if there exists a ball $B_\delta(x_0) \subset D$ such that $u \geq 0$ (or $u \leq 0$) in $B_\delta(x_0)$. Equivalently, x_0 is one-phase if

$$x_0 \in (\Gamma^+ \setminus \Gamma^-) \cup (\Gamma^- \setminus \Gamma^+).$$

Note the regularity of the free boundary points near one-phase points is reduced is to the case already studied by Alt and Phillips [AP86]. In particular, in dimension $n = 2$, δ can be chosen so small that $B_\delta(x_0) \cap \Gamma^+$ (or $B_\delta(x_0) \cap \Gamma^-$) will be a real-analytic surface.

2) We say that x_0 is a *two-phase* free boundary point, if

$$x_0 \in \Gamma^+ \cap \Gamma^-.$$

We distinguish two types of two-phase points. The first kind is so-called *branching points*, where the condition

$$|\nabla u(x_0)| = 0$$

is satisfied. The name comes from the fact that at the points x_0 where Γ branches out to $\Gamma^\pm$, i.e., $x_0 \in \Gamma^+ \cap \Gamma^- \cap \overline{\{u = 0\}^\circ}$, this condition holds automatically (but not necessarily vice versa).

The second kind of two-phase points are the *non-branching points* is where

$$|\nabla u(x_0)| > 0.$$

Since $u \in C^{1,\beta-1}_{\mathrm{loc}}(D)$, the implicit function theorem implies that for such points there exists a small δ such that $B_\delta(x_0) \cap \Gamma_+ = B_\delta(x_0) \cap \Gamma_-$ is a graph of a $C^{1,\alpha}$ function.

3. Rescalings and Blowups

One of the key ideas in studying the infinitesimal properties of the free boundary is to make an infinite "zoom-in" (or "blowup") at a free boundary point.

More specifically, given a minimizer u of (1.1), $x_0 \in \Gamma$ and $r > 0$ define the *rescaling*

$$u_{x_0,r}(x) = \frac{u(x_0 + rx)}{r^\beta}, \quad \beta = \frac{2}{2-p}$$

for $x \in D_{x_0,r} = \frac{1}{r}(D - x_0)$. We will use the notation u_r for $u_{x_0,r}$ if $x_0 = 0$. If $x_0 \in \Gamma \cap K$ for $K \subset\subset D$ and is such that $|\nabla u(x_0)| = 0$, we will have the uniform estimates

$$|u_{x_0,r}(x)| \le C_K |x|^\beta, \quad \text{for } |x| \le \frac{\delta}{r},$$

where $\delta = \frac{1}{2}\operatorname{dist}(K, \partial\Omega)$. This will follow from the optimal $C^{1,\beta-1}_{\mathrm{loc}}$-regularity of u. Hence, for a fixed x_0, we may extract a sequence $r_j \to 0$ such that

$$u_{x_0,r_j} \to u_0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n),$$

where $u_0 \in C^{1,\beta-1}(\mathbb{R}^n)$. We will call u_0 a *blowup* of u at x_0. It is a simple exercise to show that u_0 is a global minimizer of functional J, i.e. it minimizes J on every subdomain $U \subset \mathbb{R}^n$ among the functions in $W^{1,2}(U)$ with the same trace on ∂U as u. Note that the blowup is not defined at free boundary points x_0 where $|\nabla u(x_0)| > 0$, i.e. at non-branching points. Moreover, at points where blowups exist, it is not clear apriori if the blowup is unique. Namely, taking a different subsequence $r'_j \to 0$ may result in convergence of u_{x_0,r'_j} to a different blowup u'_0. One of our key results is that such blowup is actually unique, if the space dimension $n = 2$, see Theorem 5.1.

3.1. Nondegeneracy.

Another possibility that needs to be ruled out is that u_0 vanishes identically in $\mathbb{R}^n$. This is accomplished with the help of the following nondegeneracy lemma.

Lemma 3.1 (Nondegeneracy)**.** *Let u be a solution of* (1.1) *and $x_0 \in \Gamma^+$. Then*

$$\sup_{\partial B_r(x_0)\cap\Omega^+} u \ge c_+ r^\beta$$

for any $r > 0$ such that $B_r(x_0) \subset\subset D$. Similarly, if $x_0 \in \Gamma^-$, then

$$\inf_{\partial B_r(x_0)\cap\Omega^-} u \le -c_- r^\beta,$$

provided $B_r(x_0) \subset\subset D$. Here $c_\pm = c(p, n, \lambda_\pm) > 0$.

Proof. The proof is similar to the one in [Phi83a] which in turn follows the same idea as in [CR76]. Let $y \in \Omega^+$ and let $r_0 > 0$ be so small that $B_{r_0}(y) \subset D$. Set

$$w(x) = |u(x)|^{2/\beta} - c\,|x - y|^2$$

for some constant $c > 0$ to be specified later. Then

$$\Delta w = \frac{2p\lambda_+}{\beta} + \frac{2}{\beta}\left(\frac{2}{\beta} - 1\right)\frac{|\nabla u|^2}{|u|^p} - 2nc \quad \text{in } \Omega^+ \cap B_r(y)$$

for any $r < r_0$. Thus, taking $c = (p\lambda_+)/(\beta n)$, we obtain that $\Delta w \geq 0$. Since $w(y) > 0$ and w is subharmonic, there must exist $x_y \in \partial(B_r(y) \cap \Omega^+)$ such that $w(x_y) > 0$. Since $w \leq 0$ on Γ^+, necessarily $x_y \in \partial B_r(y) \cap \Omega^+$, which gives that

$$\sup_{\partial B_r(y)\cap\Omega^+} w > 0,$$

or, equivalently,

$$\sup_{\partial B_r(y)\cap\Omega^+} u^{2/\beta} > c\, r^2.$$

If now $x_0 \in \Gamma^+$, we can find a sequence of points $y = y_n \in \Omega^+$ converging to x_0. Then passing to the limit, by continuity, we obtain

$$\sup_{\partial B_r(x)\cap\Omega^+} u^{2/\beta} \geq c\, r^2.$$

This proves the lemma for $x_0 \in \Gamma^+$.

On Γ^- we can argue similarly. □

A simple corollary from the nondegeneracy is that if $x_0 \in \Gamma^{\pm}(u) \cap \{|\nabla u| = 0\}$, then $0 \in \Gamma^{\pm}(u_0)$. In particular, no blowup is identically zero.

3.2. **Homogeneity of blowups.** The next proposition characterizes the blowups of solutions.

Proposition 3.2 (Blowups are homogeneous). *Let u be a minimizer of* (1.1) *and $x_0 \in \Gamma \cap \{|\nabla u| = 0\}$. Then any blowup u_0 of u at x_0 is a homogeneous function of degree β with respect to the origin, i.e.,*

$$u_0(rx) = r^\beta u_0(x), \quad x \in \mathbb{R}^n,\ r > 0.$$

The proof of this proposition is based on an important monotonicity formula due to Weiss [Wei98].

Lemma 3.3 (Weiss's monotonicity formula). *Let u be a minimizer of* (1.1) *and*

$$W(r, x_0) = \frac{1}{r^{n+2\beta-2}} \int_{B_r(x_0)} \big(|\nabla u|^2 + 2F(u)\big)\, \mathrm{d}x - \frac{\beta}{r^{n+2\beta-1}} \int_{\partial B_r(x_0)} u^2(x)\, \mathrm{d}\sigma,$$

for $r > 0$ such that $B_r(x_0) \subset\subset D$. Then W is monotonically increasing with respect to r. Moreover, $W(r, x_0) = 0$ for $0 < r < r_0$ iff u is a homogeneous function of degree β with respect to x_0 in $B_{r_0}(x_0)$.

Sometimes we will use the abbreviated notation $W(r)$ for $W(r, x_0)$ if the point x_0 is clear from the context, and more expanded notation $W(r, x_0, u)$, if we want to specify the function u.

Proof. For the complete proof we refer to the original paper of Weiss [Wei98]. Here we just indicate that using the identity

$$W(r, x_0, u) = W(1, 0, u_{x_0,r}),$$

where $u_{x_0,r}(x) = u(x_0 + rx)/r^\beta$, one can derive that

$$W'(r) = \frac{2}{r^{n+2\beta-1}} \int_{\partial B_r(x_0)} ((x - x_0) \cdot \nabla u - \beta u)^2\, \mathrm{d}\sigma.$$

The last part of the lemma follows from the fact that $(x - x_0) \cdot \nabla u - \beta u = 0$ in $B_{r_0}(x_0)$ is equivalent to the homogeneity of u. □

By using this monotonicity formula, one can give a quick proof of Proposition 3.2

Proof of Proposition 3.2. Let $u_{x_0,r_j} \to u_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then for any $\rho > 0$, we have

$$W(\rho, 0, u_0) = \lim_{j\to\infty} W(\rho, 0, u_{x_0,r_j}) = \lim \lim_{j\to\infty} W(r_j\rho, x_0, u) = W(0+, x_0, u).$$

Hence $W(\rho, 0, u_0)$ is constant in ρ, which implies that u_0 is homogeneous of degree β. □

4. Classification of homogeneous global minimizers

As the blowups of minimizers u are homogeneous of degree β, it would be desirable to obtain classification of such global minimizers. This poses a challenging open problem even in one-phase case in higher dimensions. In dimension $n = 2$, the problem is much simpler and, loosely speaking, reduces to identifying the solutions of an ODE with period 2π.

From now on, unless stated otherwise, we will be working in dimension $n = 2$.

Theorem 4.1 (Homogeneous global minimizers)**.** *Let u_0 be a homogeneous global minimizer of J in dimension $n = 2$. Then after a suitable rotation of coordinate axes we have the following possibilities:*

1) (One-phase nonnegative) $u_0(x) = C_+(x_1^+)^\beta$.
2) (One phase nonpositive) $u_0(x) = -C_-(x_1^-)^\beta$
3) (Two-phase) $u_0(x) = C_+(x_1^+)^\beta - C_-(x_1^-)^\beta$

The constants $C_\pm = C(p, n, \lambda_\pm) > 0$ are chosen so that u_0 solves (1.4).

The proof is based on the following lemma. It will be more convenient to use polar coordinates (r, θ) in the statement and the proof of this lemma.

Lemma 4.2. *Let $u(r, \theta) = r^\beta f(\theta)$ be a* positive *solution of* (1.4) *in the cone $\mathcal{C}_\gamma = \{(r, \theta) : r > 0,\ \theta \in (0, \gamma)\}$, vanishing continuously on $\partial\mathcal{C}_\gamma$: $u(r, 0) = u(r, \gamma) = 0$. Suppose also that $u \in C^1\left(\overline{\mathcal{C}_\gamma}\right)$. Then*

$$\frac{\pi}{\beta} \leq \gamma \leq \pi.$$

Furthermore, if $f'(0) = 0$ or $f'(\gamma) = 0$ then $\gamma = \pi$. Conversely, if $\gamma = \pi$ then necessarily $f'(0) = f'(\pi) = 0$.

The same result holds also for negative *u.*

Proof. Let v be a homogeneous harmonic function in the cone $\theta \in (0, \gamma)$ such that $v(r, 0) = v(r, \gamma) = 0$. Then v must be of the form $v(r, \theta) = r^{\pi/\gamma} g(\theta)$. Now, for C large enough $u \leq Cv$ on $\partial B_1 \cap \mathcal{C}_\gamma$. Since u is subharmonic in $\mathcal{C}_\gamma$ by the maximum principle we will have $u \leq Cv$ in this cone. In particular, this means

$$r^{\beta - (\pi/\gamma)} \leq Cg/f \quad \text{for } r \leq 1.$$

Letting $r \to 0$ we obtain that $\beta - (\pi/\gamma) \geq 0$ and thus $\gamma \geq \pi/\beta$.

To prove the upper bound on γ, we claim that $u^{1/\beta}$ is superharmonic. Indeed, inserting $u(r, \theta) = r^\beta f(\theta)$ into (1.4), we obtain the following equation for f:

$$\beta^2 f + f'' = p\lambda_+ f^{p-1}$$

Multiplying by f' and integrating from 0 to θ we arrive at

$$\beta^2 f^2(\theta) + (f')^2(\theta) = (f')^2(0) + 2\lambda_+ f^p(\theta).$$

Writing out $|\nabla u|^2$ in polar coordinates, we see that this equality is equivalent to

$$|\nabla u|^2 = c_0 r^{\beta p} + 2\lambda_+ u^p, \tag{4.1}$$

where $c_0 = (f')^2(0) \geq 0$. Using (1.4) and (4.1), we now obtain

$$\begin{aligned}\Delta\left(u^{1/\beta}\right) &= \frac{1}{\beta} u^{(1/\beta)-1}\Delta u + \frac{1}{\beta}\left(\frac{1}{\beta}-1\right) u^{(1/\beta)-2}|\nabla u|^2 \\ &= \frac{1}{\beta} u^{(1/\beta)-1} p\lambda_+ u^{p-1} + \frac{1}{\beta}\left(\frac{1}{\beta}-1\right) u^{(1/\beta)-2}(c_0 r^{\beta p} + 2\lambda_+ u^p) \\ &\leq \lambda_+ u^{(1/\beta)+p-2}\left(\frac{p}{\beta} + \frac{2}{\beta}\left(\frac{1}{\beta}-1\right)\right) \\ &= 0.\end{aligned}$$

Hence, if we take $c > 0$ small enough so that $u \geq cv$ on $\partial B_1 \cap \mathcal{C}_\gamma$ we will have

$$u^{1/\beta} \geq cv$$

in B_1. This implies that $r^{(\pi/\gamma)-1}$ is bounded for $r \leq 1$. Taking $r \to 0$ we obtain $\gamma \leq \pi$.

In the case $f'(0) = 0$ the calculations above imply that $\Delta\left(u^{1/\beta}\right) = 0$. From representation $u^{1/\beta} = r(f(\theta))^{1/\beta}$ one easily obtains that $u^{1/\beta} = Cr\sin\theta$ for some constant $C > 0$ and therefore $\gamma = \pi$. If $f'(\gamma) = 0$ we can argue the same way just by integrating from $\theta = \gamma$ instead of $\theta = 0$.

Suppose now $\gamma = \pi$. Then we claim that necessarily $f(\theta) = C\sin^\beta\theta$ for a certain constant $C > 0$. Note that this would follow from the argument in the previous paragraph if we knew that $f'(0) = 0$ or $f'(\pi) = 0$. So assume that both $|f'(0)| > 0$, $|f'(\pi)| > 0$. Then there exists a constant $C > 0$ such that the graph of $C\sin^\beta(\theta)$ touches the graph of $f(\theta)$ at an interior point $\theta_0 \in (0, \pi)$:

$$f(\theta) \geq C\sin^\beta(\theta), \quad f(\theta_0) = C\sin^\beta(\theta_0).$$

Indeed, this follows from the fact that $\sin^\beta(\theta)$ has a vanishing derivative at $\theta = 0, \pi$. Consequently, we obtain that a superharmonic function $u^{1/\beta}$ touches the harmonic function $C^{1/\beta}x_1$ at an interior point in $\mathcal{C}_\gamma$. Therefore, by the strong maximum principle both functions must coincide. Hence, $f(\theta) = C\sin^\beta\theta$.

The non-positive case can be handled in the same manner. □

Proof of Theorem 4.1. Consider three cases:

1) 0 is a positive one-phase point, i.e., $0 \in \Gamma^+(u_0) \setminus \Gamma^-(u_0)$. In this case $u_0 \geq 0$. Consider then the positivity set $\Omega^+(u_0)$. From the homogeneity, the connected components of Ω^+ are cones. Lemma 4.2 implies that the cones have opening between π/β and π. In fact, since $|\nabla u_0| = 0$ on Γ^+ for nonnegative solutions, the opening of the components of Ω^+ is exactly π. Hence, there are either two, or just one components of Ω^+ of opening π, which after a rotation, correspond to

$$u_0(x) = C_+|x_1|^\beta$$

and

$$u_0(x) = C_+(x_1^+)^\beta,$$

respectively. The former case is actually impossible, since for nonnegative minimizers the zero set $\{u_0 = 0\}$ must have nonzero Lebesgue density at free boundary points, see [Phi83b].

2) 0 is a negative one-phase point, i.e., $0 \in \Gamma^-(u_0) \setminus \Gamma^+(u_0)$. This case is treated similarly to the previous one.

3) 0 is a two-phase point, i.e., $0 \in \Gamma^+(u_0) \cap \Gamma^-(u_0)$. In this case both Ω^+ and Ω^- are nonempty. By Lemma 4.2 each component of $\Omega^\pm$ is a cone of opening between π/β and π. Since $\beta < 2$ there could be no more than 3 different components in $\Omega^\pm$.

If there are three components, then we have two possibilities: either there are two components of the same sign sharing a common side, or the set $\{u = 0\}$ has a nonempty interior. In both cases, $|\nabla u| = 0$ on one side of at least two of the components, which implies that they both must have opening π. This doesn't leave space for the third component.

Hence, there are precisely two components, one in Ω^+, the other in Ω^-. We claim that both have opening π. Indeed, otherwise the set $\{u = 0\}$ will have nonempty interior and therefore $|\nabla u| = 0$ on one side of both components, implying that their opening must be π. Thus, using Lemma 4.2 one more time, we obtain that after a suitable rotation

$$u_0(x) = C_+(x_1^+)^\beta - C_-(x_1^-)^\beta. \quad \square$$

5. Uniqueness of the blowup

Let $x_0 \in \Gamma^+ \cap \Gamma^-$ be a branching point for a minimizer u of the functional J and consider its rescalings

$$u_r(x) = u_{x_0,r}(x) = \frac{u(x_0 + rx)}{r^\beta}, \quad r > 0.$$

By Proposition 3.2, Lemma 3.1, and Theorem 4.1, the only possible subsequential limits of u_r as $r = r_j \to 0$ are the rotations of

$$u_0(x) = C_+(x_1^+)^\beta - C_-(x_1^-)^\beta,$$

i.e. one of the functions

$$u_0^\omega = u_0 \circ U^{-\omega}, \qquad \omega \in [0, 2\pi)$$

where U^ω is a counterclockwise rotation by angle ω. In this section we show that the limit is unique.

Theorem 5.1 (Uniqueness of the blowup at branch points)**.** *Let u be a minimizer of J and $x_0 \in \Gamma^+ \cap \Gamma^- \cap \{|\nabla u| = 0\}$. Then there exists a unique $\omega \in [0, 2\pi)$ such that*

$$\lim_{r \to 0} u_r(x) = u_0^\omega.$$

The proof is based on Alexandrov-type reflection-comparison arguments, that we adopted from [SW06].

We start by analyzing the reflection-comparison properties of the homogeneous solution.

Lemma 5.2. *For the global solution u_0 define a 2π-periodic function*

$$\phi_0(\theta) = u_0(\cos\theta, \sin\theta).$$

and consider the differences

$$\xi_0^\omega(\theta) = \phi_0(\omega + \theta) - \phi_0(\omega - \theta), \quad \omega \in [0, 2\pi).$$

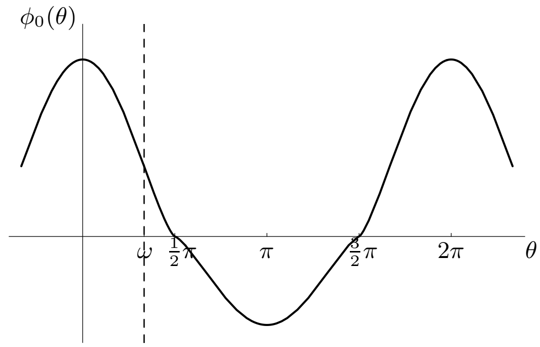

FIGURE 2. The graph of $\phi_0(\theta)$

Then $\xi_0^\omega(\theta) = 0$ *if* $\theta = 0,\ \pi$ *or* $\omega = 0,\ \pi$ *and*

$$\xi_0^\omega(\theta) < 0 \quad \textit{for all} \quad \theta \in (0,\pi), \quad \textit{if} \quad \omega \in (0,\pi)$$
$$\xi_0^\omega(\theta) > 0 \quad \textit{for all} \quad \theta \in (0,\pi), \quad \textit{if} \quad \omega \in (\pi,2\pi).$$

Proof. This follows easily from the fact that $\phi_0(\theta)$ is strictly decreasing for $\theta \in [0,\pi]$ and strictly increasing for $\theta \in [\pi, 2\pi]$, see Figure 2. □

Lemma 5.3. *For a minimizer u in B_R and $0 < r < R$ define the 2π-periodic function*

$$\phi_r(\theta) = u_r(\cos\theta, \sin\theta) = \frac{u(r\cos\theta, r\sin\theta)}{r^\beta}$$

and consider the differences

$$\xi_r^\omega(\theta) = \phi_r(\omega+\theta) - \phi_r(\omega-\theta), \quad \omega \in [0,2\pi).$$

Then for any $\delta > 0$ there exists $\epsilon > 0$ with the following property: if

$$\|\phi_r - \phi_0\|_{C^1([0,2\pi])} \le \epsilon$$

then

$$\xi_r^\omega(\theta) < 0 \quad \textit{for all} \quad \theta \in (0,\pi), \quad \textit{if} \quad \omega \in [\delta, \pi/2-\delta] \cup [\pi/2+\delta, \pi-\delta]$$
$$\xi_r^\omega(\theta) > 0 \quad \textit{for all} \quad \theta \in (0,\pi), \quad \textit{if} \quad \omega \in [\pi+\delta, 3\pi/2-\delta] \cup [3\pi/2+\delta, 2\pi-\delta].$$

Proof. We will only consider

$$\omega \in [\delta, \pi/2-\delta] \cup [\pi/2+\delta, \pi-\delta],$$

the other case being analogous.

1) For $\theta \in [\delta/2, \pi-\delta/2]$, by Lemma 5.2, we will have

$$\xi_0^\omega(\theta) \le -c_1(\delta) < 0.$$

2) For $\theta \in [0, \delta/2]$, we will have

$$\partial_\theta \xi_0^\omega(\theta) = \partial_\theta\phi_0(\omega+\theta) + \partial_\theta\phi_0(\omega-\theta) \le -c_2(\delta) < 0,$$

since

$$\omega \pm \theta \in [\delta/2, \pi/2-\delta/2] \cup [\pi/2+\delta/2, \pi-\delta/2].$$

3) For $\theta \in [\pi-\delta/2, \pi]$, similarly to case 2), we will have

$$\partial_\theta \xi_0^\omega(\theta) \ge c_3(\delta) > 0,$$

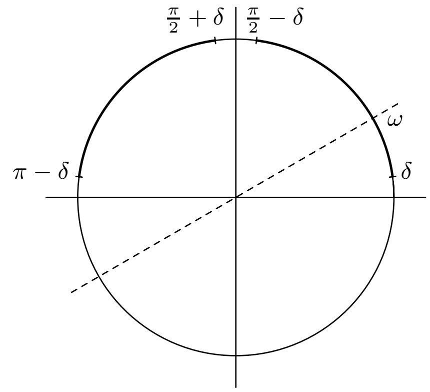

FIGURE 3. The range of ω for which $\xi_r^\omega(\theta)<0$ for all $\theta\in(0,\pi)$

since

$$\omega\pm\theta\in[\pi+\delta/2,3\pi/2-\delta/2]\cup[3\pi/2+\delta/2,2\pi-\delta/2].$$

Now, collecting the estimates in 1)–3), we will have that if

$$\|\phi_r-\phi_0\|_{C^1([0,2\pi])}<\min\{c_1,c_2,c_3\}$$

then

$$\begin{aligned}\partial_\theta\xi_r^\omega(\theta)&<0,\quad\text{for}\quad\theta\in[0,\delta/2]\\ \xi_r^\omega(\theta)&<0,\quad\text{for}\quad\theta\in[\delta/2,\pi-\delta/2]\\ \partial_\theta\xi_r^\omega(\theta)&>0,\quad\text{for}\quad\theta\in[\pi-\delta/2,\pi].\end{aligned}$$

This immediately implies that $\xi_r^\omega(\theta)<0$ for $\theta\in(0,\pi)$, since $\xi_r^\omega(0)=\xi_r^\omega(\pi)=0$. The proof is complete. □

To state the next lemma we use the following notation: for $x=(x_1,x_2)$ denote by x^* the reflection of x with respect to the x_1-axis, i.e.

$$x^*=(x_1,-x_2).$$

Respectively, for any set $E\subset\mathbb{R}^2$ we denote

$$E^*=\{x^*:x\in E\}.$$

We also denote

$$E^+=E\cap\{x_2>0\}.$$

Lemma 5.4 (Strict reflection-comparison)**.** *Let u be a minimizer of* (1.1) *in B_r, and suppose that*

$$u(x)<u(x^*)\quad\textit{for all}\quad x\in(\partial B_r)^+.$$

Then

$$u(x)\le u(x^*)\quad\textit{for all}\quad x\in B_r^+.$$

Moreover,

$$u(x)<u(x^*)\quad\textit{for all}\quad x\in B_r^+\setminus\{u(x)=u(x^*)=0\}.$$

Even though we are not going to use the latter strict inequality in this paper, we actually establish it first and then obtain the unstrict inequality as a corollary.

Proof. 1) Define

$$u^*(x) = u(x^*), \quad x \in B_r^+.$$

Then clearly it is a minimizer of J in B_r^+ and

$$u < u^* \quad \text{on} \quad (\partial B_r)^+.$$

Define also

$$\overline{v} = \max\{u, u^*\}, \quad \underline{v} = \min\{u, u^*\}$$

and note that

$$\overline{v} = u^*, \quad \underline{v} = u \qquad \text{on} \quad \partial(B_r^+).$$

From the minimality properties of u and u^* we therefore have

$$J(\overline{v}) \geq J(u^*), \quad J(\underline{v}) \geq J(u).$$

On the other hand, the structure of J implies that

$$J(u^*) + J(u) = J(\max\{u^*, u\}) + J(\min\{u^*, u\}) = J(\overline{v}) + J(\underline{v}).$$

As a consequence,

$$J(\overline{v}) = J(u^*), \quad J(\underline{v}) = J(u)$$

and therefore $\overline{v}$ and $\underline{v}$ are minimizers of J themselves.

2) Suppose now that at some point $x_0 \in B_r^+$ we have $u(x_0) = u^*(x_0) \neq 0$. Then we also have $\overline{v}(x_0) = \underline{v}(x_0)$, i.e. $\overline{v}$ touches $\underline{v}$ at x_0 by above. Consider then the difference

$$w = \overline{v} - \underline{v}.$$

By construction, we have

$$w \geq 0, \qquad w(x_0) = 0.$$

Assume for a moment that $\overline{v}(x_0) = \underline{v}(x_0) > 0$. Then for a small $\delta > 0$,

$$\Delta w = p\lambda_+(\overline{v}^{p-1} - \underline{v}^{p-1}) \leq 0 \quad \text{in} \quad B_\delta(x_0),$$

i.e. w is superharmonic in $B_\delta(x_0)$. We arrive at exactly same conclusion also when $\overline{v}(x_0) = \underline{v}(x_0) < 0$. Then the strong maximum principle for harmonic functions now implies that $w = 0$ in $B_\delta(x_0)$, which is equivalent to

$$u = u^* \quad \text{in} \quad B_\delta(x_0).$$

3) Consider now the set

$$E = \{u \geq u^*\} \cap B_1^+ \setminus \{u = u^* = 0\},$$

which is open by the arguments above. Suppose that $E \neq \emptyset$ and take $x_0 \in E$. Let E_0 be the component of E that contains x_0 (see Figure 4). Consider then the boundary of E_0. Clearly, $\partial E_0 \cap (\partial B_r)^+ = \emptyset$. Next, we claim that

$$u = u^* = 0 \quad \text{on} \quad (\partial E_0)^+.$$

Indeed, we readily have that $u = u^*$ on $(\partial E_0)^+$ and if $u(x) = u^*(x) \neq 0$ for some $x \in (\partial E_0)^+$ then by 2) above $u = u^*$ in a neighborhood of x, which is a contradiction with definition of E. Consider now the set

$$\widetilde{E}_0 = E_0 \cup E_0^*.$$

Then by the arguments above, it is easy to see that

$$u = 0 \quad \text{on} \quad \partial\widetilde{E}_0.$$

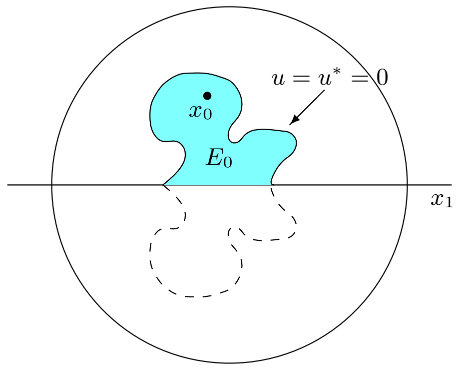

FIGURE 4. Step 3) in the proof of the strict reflection-comparison Lemma 5.4

Then from minimality of u we infer that $u = 0$ in $\widetilde{E}_0$. But this implies $u = u^* = 0$ in E_0, which is a contradiction with definition of E. Therefore $E = \emptyset$, which is equivalent to

$$u < u^* \quad \text{on} \quad B_r^+ \setminus \{u = u^* = 0\}.$$

This proves the second assertion in the lemma. The first assertion

$$u \le u^* \quad \text{on} \quad B_r^+$$

is a simple corollary. The proof is complete. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Pick a subsequence u_{r_j}, $r_j \to 0$, converging to $u_0^{\omega_0}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$. Without loss of generality we may assume that $\omega_0 = 0$. Then we will have

$$\phi_{r_j} \to \phi_0 \quad \text{in} \quad C^1([0, 2\pi])$$

and therefore by Lemma 5.3

$$\xi^\omega_{r_j}(\theta) = \phi_{r_j}(\omega + \theta) - \phi_{r_j}(\omega - \theta) < 0$$

for all $\theta \in (0, \pi)$ and

$$\omega \in [\delta, \pi/2 - \delta] \cup [\pi/2 + \delta, \pi - \delta]$$

for sufficiently large $j \ge j_0$. Applying now the strict reflection-comparison (Lemma 5.4) to functions $u \circ U^\omega$ in $B_{r_{j_0}}$, we obtain that

$$\xi^\omega_r(\theta) \le 0$$

for all $\theta \in (0, \pi)$, $0 < r \le r_{j_0}$, and the same range of ω as above.

Suppose now that for another subsequence $r'_j \to 0$, the rescalings $u_{r'_j}$ converge to $u_0^{\omega_1}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$. Starting from some large j, $r'_j \le r_{j_0}$ and, using that $\xi^\omega_{r'_j}(\theta) \le 0$ for $\theta \in (0, \pi)$, we obtain in the limit that

$$\xi_0^{\omega - \omega_1}(\theta) \le 0, \quad \text{for all} \quad \theta \in (0, \pi)$$

and all $\omega \in [\delta, \pi/2 - \delta] \cup [\pi/2 + \delta, \pi - \delta]$. Hence, by Lemma 5.2, we must have

$$|\omega_1| \le \delta,$$

and since we can take δ arbitrarily small,

$$\omega_1 = 0.$$

This implies that every converging subsequence of u_r converges to u_0 and this completes the proof of the theorem. □

As a consequence, the free boundary is differentiable at branch points.

Corollary 5.5 (Differentiability of $\Gamma^\pm$ at branching points)**.** *Let $x_0 \in \Gamma^+ \cap \Gamma^- \cap \{\nabla u = 0\}$. Then both Γ^+ and Γ^- are differentiable at x_0 and have a common tangent.*

Proof. Without loss of generality, let $x_0 = 0$ and suppose that the rescalings u_r converge to

$$u_0(x) = C_+(x_1^+)^\beta - C_-(x_1^-)^\beta.$$

Hence, for r small enough we have $|u_r(x) - u_0(x)| < \epsilon$ in $B_1(0)$, which means $|u - u_0| < r^\beta \epsilon$ in $B_r(x_0)$. This means that $\Gamma^+ \cap \Gamma^- \cap B_r$ lies inside the strip $\{-cr\epsilon \le x_1 \le cr\epsilon\}$ for some $c = c(p, \lambda_+, \lambda_-)$. Hence the free boundary has a cone at each side with angle $\arcsin c\epsilon$, one in Ω^+, the other in Ω^-. Here ϵ can be made as small as we wish by taking r small. Hence, the line $\{x_2 = 0\}$ will be tangent to both Γ^+ and Γ^- at x_0. □

6. C^1 regularity of the free boundaries

Proof of Theorem 1.1. We start by pointing out that we know the C^1 regularity of $\Gamma^\pm$ near one-phase points by the result of Alt and Phillips [AP86] (recall that we work in dimension $n = 2$). We also know the C^1 regularity near non-branching two-phase points ($|\nabla u| > 0$) by the implicit function theorem. Therefore we will focus our attention to the proof of the C^1 regularity near branching points.

At branching points we know the existence of normals by Corollary 5.5. Thus, normals exist at every free boundary point. So let us denote the unit normal on Γ^+ pointing inward Ω^+ by ν^+ and the one on Γ^- pointing outward Ω^- by ν^-.

We next show that $\nu^\pm$ are continuous at branching points. To this end, fix a branching point $x_0 \in \Gamma^+ \cap \Gamma^- \cap \{|\nabla u| = 0\}$. Without loss of generality we may assume that $x_0 = 0$ at that the blowup of u at x_0 is

$$u_0(x) = C_+(x_1^+)^p - C_-(x_1^-)^p.$$

For $x \in \Gamma = \Gamma^+ \cup \Gamma^-$ near 0 define

$$\phi_{x,r}(\theta) = u_{x,r}(\cos\theta, \sin\theta).$$

Fix a sequence $r_j \to 0$. Since u is $C^{1,\beta-1}$ regular, it is clear that there exists $\kappa_j \to 0$ with the property that

$$\phi_{x_j,r_j} \to \phi_0 \quad \text{in } C^1([0, 2\pi]),$$

whenever $x_j \in \Gamma \cap B_{\kappa_j}$. Then we claim that for any given $\delta > 0$ exists $j = j_\delta$ such that

$$\xi^\omega_{x_j,r_j}(\theta) = \phi_{x_j,r_j}(\omega + \theta) - \phi_{x_j,r_j}(\omega - \theta)$$

will satisfy

$$\xi^\omega_{x_j,r_j}(\theta) < 0 \quad \text{for all} \quad \theta \in (0, \pi), \quad \text{if} \quad \omega \in [\delta, \pi/2 - \delta] \cup [\pi/2 + \delta, \pi - \delta]$$
$$\xi^\omega_{x_j,r_j}(\theta) > 0 \quad \text{for all} \quad \theta \in (0, \pi), \quad \text{if} \quad \omega \in [\pi + \delta, 3\pi/2 - \delta] \cup [3\pi/2 + \delta, 2\pi - \delta],$$

whenever $x_j \in B_{\kappa_j} \cap \Gamma$ and $j \ge j_\delta$. Indeed, if not, we would obtain a sequence $x_j \in B_{\kappa_j}$ such that $\phi_{x_j,r_j} \to \phi_0$, but the above property is violated. On the other hand, Lemma 5.3 would imply that the above inequalities for $\xi^\omega_{x_j,r_j}$ are in fact satisfied for large j, which is a contradiction.

Next we apply the strong reflection-comparison Lemma 5.4. It implies that if $\xi^\omega_{x_j,r_j} < 0$ on $(0,\pi)$ for some fixed ω, then $\xi^\omega_{x_j,r} \le 0$ on $(0,\pi)$ also for all $r \le r_j$. Hence, we obtain that

$$\begin{aligned} &\xi^\omega_{x,r}(\theta) \le 0 \quad \text{for all} \quad \theta\in(0,\pi), \quad \text{if} \quad \omega\in[\delta,\pi/2-\delta]\cup[\pi/2+\delta,\pi-\delta]\\ &\xi^\omega_{x,r}(\theta) \ge 0 \quad \text{for all} \quad \theta\in(0,\pi), \quad \text{if} \quad \omega\in[\pi+\delta,3\pi/2-\delta]\cup[3\pi2+\delta,2\pi-\delta]\end{aligned}$$

for any $x\in B_{\kappa_{j_\delta}}\cap\Gamma$ and $0<r<r_{j_\delta}$. Letting $r\to 0$, it is now easy to realize, independent of the type of the free boundary point x,

$$|\nu^\pm(x)-e_1|\le\delta, \quad \text{for any } x\in B_{\kappa_{j_\delta}}\cap\Gamma^\pm. \tag{6.1}$$

Finally, let us show that $\Gamma^\pm$ can be represented as graphs $x_1 = f_\pm(x_2)$ near the origin. This will follow, once we show that the horizontal lines $x_2=\eta$ intersect $\Gamma^\pm$ near the origin exactly once. Consider therefore the sets

$$\Lambda_\eta = \{x_1\in[-a,a] : u(x_1,\eta)=0\}$$

for some small $a>0$ and $|\eta|\le a$. Note that arguing as in Corollary 5.5, we will have that $u(-a,\eta)<0$ and $u(a,\eta)>0$ (if a is small enough) so the sets Λ_η are nonempty. Let

$$f_+(\eta)=\sup\Lambda_\eta, \quad f_-(\eta)=\inf\Lambda_\eta.$$

We claim that

$$\Lambda_\eta=[f_-(\eta),f_+(\eta)]. \tag{6.2}$$

Assuming that $U=[f_-(\eta),f_+(\eta)]\setminus\Lambda_\eta$ is nonempty, let U_0 be one of its connected components. Since U is an open set in $\mathbb{R}^1$, U_0 is an open interval (a_0,b_0). Since $u\ne 0$ on $U\times\{\eta\}$, u will not change sign in $U_0\times\{\eta\}$.

1) Assume that $u>0$ on $U_0\times\{\eta\}$. Then clearly $(b_0,\eta)\in\Gamma^+$. Moreover, it is easy to realize that one must have $\nu^+(b_0,\eta)\cdot e_1\le 0$. However, that contradicts (6.1).

2) Similarly, if $u<0$ in $U_0\times\{\eta\}$, then $(a_0,\eta)\in\Gamma^-$ and $\nu^-(a_0,\eta)\cdot e_1\le 0$, again contradicting (6.1).

Thus, we must necessarily have (6.2) for $|\eta|\le a$. As a direct corollary, we obtain that

$$\begin{aligned}\Omega^+\cap K_a&=\{x\in K_a : x_1>f_+(x_2)\}\\ \Omega^-\cap K_a&=\{x\in K_a : x_1<f_-(x_2)\},\end{aligned}$$

where $K_a=(-a,a)\times(-a,a)$. Since $\Omega^\pm$ are open, the functions f_+ must be upper semicontinuous and f_- lower semicontinuous. Besides, using (6.1), we can easily conclude that $f_\pm$ are continuous and in fact differentiable at every point and that $f'_\pm(\eta)\to 0$ as $\eta\to 0$.

Finally, considering three possibilities for the points $(f_\pm(\eta),\eta)\in\Gamma^\pm$ (one-phase, non-branching two-phase, branching two-phase) we will have that $f'_\pm$ are continuous and therefore $\Gamma^\pm\cap K_a$ are C^1 graphs.

This completes the proof of the theorem. □

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