Optimal regularity of a parabolic free boundary problem of two-phase type with coefficients worse than Lipschitz

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Abstract

We study the regularity of a parabolic free boundary problem of two-phase type with coefficients below the Lipschitz threshold. For the Lipschitz coefficient case one can apply a monotonicity formula to prove the optimal $C_x^{1,1} \cap C_t^{0,1}$-regularity of the solution and that the free boundary is, near the so-called branching points, the union of two graphs that are Lipschitz in time and C^1 in space. In our case, the same monotonicity formula does not apply in the same way. Instead we use scaling arguments similar to the ones used for the elliptic case in [4] to prove the optimal regularity. However, whenever the spatial gradient does not vanish on the free boundary, we are in the parabolic setting faced with some extra difficulties, that forces us to strain our assumptions slightly.

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1 Introduction and main result

In this paper, we study the interior regularity of solutions to

$$Hu = \Delta u - \partial_t u = \lambda_1(x,t)\chi_{\{u>0\}} - \lambda_2(x,t)\chi_{\{u<0\}} \quad \text{in } Q_1^-, \tag{1.1}$$

where $Q_1^- = (-1,0] \times B_1$ is a finite cylinder with B_1 the unit ball in $\mathbb{R}^n$ and λ_1, λ_2 are strictly positive functions. We denote by $\Gamma(u)$ the whole free boundary $\partial\{u>0\} \cup \partial\{u<0\} \cap Q_1^-$, with its two parts $\Gamma^\pm(u) = \partial\{\pm u > 0\} \cap Q_1^-$. The points in the free boundary where u changes sign and has zero spatial gradient,

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i.e. the set $\Gamma^+(u) \cap \Gamma^-(u) \cap \{|\nabla u| = 0\}$, are usually referred to as "branching points". The name is a bit misleading since we do not necessarily have that $\{u > 0\}$ and $\{u < 0\}$ branches out at "branching points". The free boundary points in $\Gamma^+(u) \setminus \Gamma^-(u)$ (resp. $\Gamma^-(u) \setminus \Gamma^+(u)$) are usually referred to as positive (respectively, negative) one-phase points, or simply one-phase points.

Since the right hand side of (1.1) is bounded, the parabolic estimates (for instance, Theorem 7.22 on page 175 in [8] combined with the Sobolev embedding) implies that u is locally in $C_x^{1,\alpha} \cap C_t^{\alpha}(Q_1^-)$ for any $\alpha < 1$. In the set $Q_1^- \setminus \Gamma(u)$, the regularity of u is completely determined by the λ_is. In our case, it means that u is at least in $C_x^{2,\beta} \cap C_t^{1,\beta}$ for some $\beta \in (0,1)$ by standard parabolic estimates (again, using Theorem 7.30 in [8] combined with the Sobolev embedding). Thus, we focus on the regularity in a neighbourhood of $\Gamma(u)$. We observe that, if u changes sign, it will in general, due to the structure of the right hand side, have a jump discontinuity across $\Gamma(u)$. Therefore, we cannot expect any better regularity than $C_x^{1,1} \cap C_t^{0,1}$. The main result of the paper shows that indeed this is the case for solutions belonging to a certain class denoted by $P_R(M, M_0, M_1)$ defined as follows:

Definition 1.1. *We say that* $u \in P_R(M, M_0, M_1)$*, for positive constants* M*,* M_0 *and* M_1 *if,*

1. $Hu = \lambda_1(x,t)\chi_{\{u>0\}} - \lambda_2(x,t)\chi_{\{u<0\}}$ *in* $Q_R^- = B_R \times (-R^2, 0]$*, in the sense of distributions.*
2. $\sup_{Q_R^-} |u| \leq M_0$.
3. $\sup_{Q_R^-} \max_i(\frac{1}{\lambda_i}, \lambda_i) \leq 1/M$.
4. λ_1 *and* λ_2 *are uniformly continuous satisfying*
$$\sup_{i=1,2} \sup_{|x-y|\leq r, |t-s|\leq r^2} |\lambda_i(x,t) - \lambda_i(y,s)| \leq M_1 r^\alpha$$
for $r < 1$ *and for some* $0 < \alpha < 1$.
5. $\partial_t \lambda_i \in L^p(Q_R^-)$ *for* $p > \frac{2+n}{2}$.
6. $(0,0) \in \Gamma^+(u) \cap \Gamma^-(u) \cap \{|\nabla u| = 0\}$.

The main theorem is

Theorem 1.2. *Assume that* $u \in P_1(M, M_0, M_1)$*. Then, there exists* C*,* $r_0 > 0$ *depending on positive constants* M*,* M_0*,* M_1 *and the dimension* n *such that*

$$||u||_{C_x^{1,1} \cap C_t^{0,1}(Q_{r_0}^-)} \leq C.$$

In the definition above, and also in the sequel, ∇u denotes the spatial gradient of u. Sometimes we will also abuse the word gradient, when we in fact refer to the spatial gradient. In general, we will use the notation

$$Q_r^-(x_0, t_0) = (t_0 - r_0^2, t_0] \times B_r(x_0, t_0),$$

and also $Q_r^- = Q_r^-(0,0)$ and $B_r = B_r(0,0)$.

Our result is a parabolic version of the one proved in [4]. As explained earlier, this regularity is optimal, since the right hand side might in general have a jump across $\Gamma(u)$. In the absence of any useful monotonicity formula, Theorem 1.2 is proved using scaling arguments (see [4]) combined with a classification of global solutions to a slightly perturbed version of (1.1).

Remark 1.3. *Condition (5) in the Definition 1.2 above might seem less natural than the others for the reader. This is also the authors feeling. However, it is needed in order to prove that $\partial_t u$ is continuous over the free boundary whenever the (spatial) gradient is not zero, where we use Hölder estimates for equation of the type*

$$Hu \in L^p(Q_1^-).$$

Recently, a lot of effort has been made to prove regularity theory for free boundary problems similar to (1.1), elliptic as well as parabolic, under minimal assumptions on the functions λ_i, $i = 1, 2$. For the case, when the λ_is are supposed to be Lipschitz in both space and time, it was recently proved in [11] that $u \in C_x^{1,1} \cap C_t^{0,1}$ and that close to branching points, $\Gamma(u)$ is the union of two graphs that are Lipschitz in time and C^1 in space. Another very similar problem, but with a slightly different right hand side, has been studied in [5], where the authors prove that if the ingredients (those corresponding to λ_i) are Lipschitz in space and Dini continuous in time, then the solutions enjoy the optimal $C_x^{1,1} \cap C_t^{0,1}$-regularity. The very same problem, with weaker assumptions on the ingredients (Hölder continuity) together with some geometric and energetic assumptions, has been treated recently in [3], where the local $C_x^{1,1} \cap C_t^{0,1}$ regularity is proved.

In the stationary case, i.e., the corresponding elliptic problem, several similar studies has been done. In [10], the authors prove that if the λ_is are Lipschitz, then the solutions are $C^{1,1}$ and the free boundary is, close to branching points, locally the union of two C^1-graphs. This result was partially generalized in [4] to the case when the λ_is are Hölder continuous, where it is also proved that also $u \in C^{1,1}$.

Remark 1.4. *We would like to point out that in the proof of Theorem 1.2 in [4] there is a mistake. In order to localize the problem with the scaling*

$$v(x) = \frac{u(\rho x)}{\rho^2},$$

we need to invoke result like Proposition 12 in [13] but for Hölder continuous coefficients instead of constants. This result can be proved exactly as Proposition 12 in [13], but we need to use a monotonicity formula adapted to our case, see for instance [9].

As a motivation for the study of the problem (1.1), we mention here a possible application.

Temperature control of a liquid: Suppose that we put a liquid in a container K, which has a given temperature g at its boundary ∂K. Assume further that we would like to keep the temperature u of the liquid as close to zero as possible with the aid of some limited equipment, and that the boundary temperature g attains both positive and negative values, which means that we will never reach exactly zero. If the equipment is limited so that we have one machine that can cool down the liquid with an effect λ_1 and another that can heat it up with an effect of λ_2, then the simplest model for the heat diffusion in the liquid will be

$$\begin{cases} Hu = \lambda_1\chi_{\{u>0\}} - \lambda_2\chi_{\{u<0\}} \text{ in } K, \\ u = g \text{ on } \partial K, \\ u = u_0 \text{ when } t = 0, \end{cases}$$

where u_0 is assumed to be the initial temperature distribution of the liquid.

2 Organization of the paper

The organization of the paper is as follows. In Section 3, we briefly introduce the notion of blow-ups, which is used in frequently in this paper, and state some more or less known results. This is followed by Section 4 where we classify solutions to a certain two-phase type free boundary problem using blow-up type arguments.

This classification is later used in Section 5 to prove quadratic estimates in a neighbourhood of the free boundary which depends on the modulus of the spatial gradient of the solution. The fact that solutions enjoy quadratic growth at branching points seems to be fairly accepted in the community of free boundary problems. In the elliptic case, the proof is given (in fact only for the case λ_i constant, but it applies in the Hölder continuous case as well), but in the parabolic case it cannot be found in the literature. Therefore we describe it somewhat briefly in the appendix.

The estimates near free boundary points which are not in the aforementioned neighbourhood is accomplished in Section 6 using the method of Von Mises transform combined with some known results for the corresponding elliptic problem. The essential idea is that in this case, the gradient is sufficiently big. Following this, in Section 7 we combine the earlier results with the result from the appendix and prove Theorem 1.2. Finally, we give a brief description of the proof of quadratic growth at branching points, where we mimic of the proof of Proposition 4.1 in [13].

3 Preliminaries

One important notion that will be used extensively in this paper, which has been successfully used before in many papers on free boundary problems, is the notion of rescalings. The main idea is, given a solution u to

$$Hu = f \text{ in } Q_1^-,$$

we define a rescaled function

$$v_r(x,t) := \frac{u(rx, r^2t)}{r^2}$$

which satisfies

$$Hv_r = f(r\cdot) \text{ in } Q^-_{\frac{1}{r}}.$$

Now, if we moreover have estimates of u that implies that, for instance, $|v_r| \leq C$ in Q_1^- we can, thanks to the estimates known for parabolic equation (for instance Theorem 7.30 in [8] combined with the Sobolev embedding), use compactness and say that there is a subsequence v_{r_j} converging to a solution in $\mathbb{R}^n \times \mathbb{R}^-$ as $r_j \to 0$. If we instead start with a solution in the whole space $\mathbb{R}^n \times \mathbb{R}^-$, it also makes sense to let r tend to ∞. By studying the possible limits we can sometimes get useful information about the original function u.

We thus begin by collecting some of the the well known results that will be required for proof of Theorem 1.2. The following proposition is exactly the same as Lemma 4.1 in [11] which shows that the solutions of (1.1) does not grow too slow around free boundary points.

Proposition 3.1. (Nondegeneracy) *Let $u \in P_1(M, M_0, M_1)$. Then there is a constant c depending on the dimension and M such that for all $(y,t) \in \Gamma^\pm(u) \cap Q_1^-$*

$$\sup_{Q_r^-(y,t)\cap\{\pm u>0\}} \pm u \geq cr^2,$$

for r small enough.

In particular, Proposition 3.1 implies the stability of free boundary points under the blow-up procedure, which is essential in many situations in this subject.

Next, we recall a monotonicity formula that we will be using in the sequel, whose proof can be found in [2].

Theorem 3.2. (Monotonicity formula) *Let u, v be two disjoint subcaloric functions in $\mathbb{R}^n \times (-1, 0]$ such that $u(0) = v(0) = 0$. Define*

$$\phi(r,u,v) = \int_{-r^2}^0 \int_{\mathbb{R}^n} |\nabla u|^2 G(x,-s)\,\mathrm{d}x\,\mathrm{d}s \int_{-r^2}^0 \int_{\mathbb{R}^n} |\nabla v|^2 G(x,-s)\,\mathrm{d}x\,\mathrm{d}s,$$

with $G(x,t)$ being the usual heat kernel. Then ϕ is nondecreasing in r.

4 Global solutions of a related problem

In order to go through with our scaling argument we will need to classify global solutions to the equation

$$Hu = \lambda_1\chi_{\{u>-ax_1\}} - \lambda_2\chi_{\{u<-ax_1\}}, \tag{4.1}$$

where λ_1 and λ_2 are constants. One major difference here from the global solutions studied in [11] is that one cannot apply the monotonicity formula from [2] to the pair of functions $(\partial_e u)^\pm$ for all spatial directions e, but only for those (spatial) directions orthogonal to e_1. Of course, translating the function u linearly with ax_1 yields a global solution to the two-phase obstacle problem. However, the gradient will not vanish at the origin, so again we cannot apply the classification from [11] directly. Therefore, an approach in the flavour of Lemma 2.2 in [4] is required. Nevertheless, here we choose to prove that solutions of (4.1) must be time independent and then apply the Lemma 2.2 from [4].

Lemma 4.1. *Let u be a solution of the following problem*

$$\begin{cases} Hu = \lambda_1 \chi_{\{u>-ax_1\}} - \lambda_2 \chi_{\{u<-ax_1\}} & \text{in } \mathbb{R}^n \times \mathbb{R}^-, \\ u(0,0) = |\nabla u(0,0)| = 0, \\ (0,0) \in \Gamma(u), \end{cases} \tag{4.2}$$

where $a > 0$, λ_1 and λ_2 are positive constants. Assume also that there exists $C_0 > 0$

$$\sup_{Q_r^-} |u| \le C_0 r^2, \tag{4.3}$$

whenever $r > 1$. Then, one of the following holds

1. *$u(x,t) = \frac{\lambda_1}{2}(x_1^+)^2 - \frac{\lambda_2}{2}(x_1^-)^2$;*
2. *$u(x,t) = \frac{\lambda_1}{2}(x_1^+)^2$;*
3. *$u(x,t) = -\frac{\lambda_2}{2}(x_1^-)^2$*

where $x_1^+ = x_1^+(x) := \max\{x_1, 0\}$ and $x_1^- = x_1^-(x) := \min\{x_1, 0\}$ for any $x \in \mathbb{R}^n$. In particular,

$$\sup_{Q_1^-} |u| \le \frac{1}{2} \max(\lambda_1, \lambda_2).$$

Remark 4.2. *A more general form of this lemma with the right hand side equal to $\lambda_1 \chi_{\{u>-L(x)\}} - \lambda_2 \chi_{\{u<-L(x)\}}$ with L being a linear function can easily be obtained by a change of coordinates.*

Proof. Since λ_1 and λ_2 are both constants, Theorem 4.1 in [5] applies and due to (4.3) we then have uniform $C_x^{1,1} \cap C_t^{0,1}$-estimates for the solutions of (4.2). For $R > 0$, consider the rescaled functions

$$u_R(x) = \frac{u(Rx, R^2 t) + ax_1 \cdot R}{R^2}.$$

It can be easily verified that

1. $Hu_R = \lambda_1 \chi_{\{u_R>0\}} - \lambda_2 \chi_{\{u_R<0\}}$ in $\mathbb{R}^n \times \mathbb{R}^-$,
2. $(0,0) \in \Gamma^+(u_R) \cap \Gamma^-(u_R) \cap \{|\nabla u_R| = 0\}$, and
3. $\sup_{Q_\rho^-} |u_R| \le C\rho^2$ for some $C > 0$ and for $\rho > 1/R$.

Applying the classical estimates for parabolic equations, we obtain a subsequence u_{R_j} converging locally in $C_x^{1,\alpha} \cap C_t^\alpha$, $0 < \alpha < 1$, to a function u_∞. In addition, u_∞ satisfies

1. $Hu_\infty = \lambda_1 \chi_{\{u_\infty>0\}} - \lambda_2 \chi_{\{u_\infty<0\}}$ in $\mathbb{R}^n \times \mathbb{R}^-$,
2. $(0,0) \in \Gamma^+(u_\infty) \cap \Gamma^-(u_\infty) \cap \{|\nabla u_\infty| = 0\}$, and
3. $\sup^-_{Q_\rho^-} |u_\infty| \le C\rho^2$ for some $C > 0$ and for $\rho > 0$,

where the second property follows from Proposition 3.1. Moreover, a simple consequence of the third property is that $\partial_t u_\infty$ and $D^2 u_\infty$ are both uniformly bounded in $\mathbb{R}^n \times \mathbb{R}^-$. To see this, let

$$u_{\infty,R}(x,t) = \frac{u_\infty(Rx, R^2 t)}{R^2},$$

for some $R > 1$. Then, due to property 3. for u_∞, we have $u_{\infty,R}$ is uniformly bounded in Q_1^-, and thus we have local $C_x^{1,1} \cap C_t^{0,1}$-estimates for $u_{\infty,R}$. Rescaling back, we see that $\partial_t u_\infty$ and $D^2 u_\infty$ are uniformly bounded in Q_R^-, independently of R.

Hence, Theorem 8.4 in [11] implies that, up to rotations,

$$u_\infty(x,t) = u_\infty(x_1) = \frac{\lambda_1}{2}(x_1^+)^2 - \frac{\lambda_2}{2}(x_1^-)^2. \tag{4.4}$$

In particular, this implies $\partial_t u_\infty = 0$.

We will now prove that this implies $\partial_t u = 0$. The desired result will then follow from Lemma 2.2 in [4]. Obviously,

$$\sup_{Q_{1/2}^-} |\partial_t u_R| = \sup_{Q_{R/2}^-} |\partial_t u| \leq C.$$

Hence, $\partial_t u_R$ is bounded in $Q_{1/2}^-$. Now, clearly $\partial_t u_R$ converges locally uniformly in the set $\{x_1 \neq 0\} \cap \mathbb{R}^n \times \mathbb{R}^-$. Hence, $\partial_t u_R$ converges a.e., and thus

$$\sup_{Q_{R/2}^-} |\partial_t u| = \sup_{Q_{1/2}^-} |\partial_t u_R| \to 0.$$

The theorem follows. □

5 Estimates near free boundary points where $|\nabla u|$ vanishes

In this sub section, we continue using the notation $\Gamma(u)$ for the free boundary of global solutions as well, keeping in mind that Q_1^- is replaced by $\mathbb{R}^n$. The following result says that the spatial gradient of global solution vanishes on the whole free boundary if there are one-phase free boundary points.

Lemma 5.1. *Let λ_1 and λ_2 be constants. Suppose u satisfies*

1. $\sup_{Q_\rho^-} |u| \leq C\rho^2$ *for* $\rho > 1$,
2. $Hu = \lambda_1 \chi_{\{u>0\}} - \lambda_2 \chi_{\{u<0\}}$ *in* $\mathbb{R}^n \times \mathbb{R}^-$,
3. $(0,0) \in \Gamma^+(u) \setminus \Gamma^-(u)$,

for some constant C. Then $|\nabla u| = 0$ on $\Gamma(u)$.

Proof. We argue by contradiction. If the statement of the lemma does not hold then there must be a point $(y_0, t_0) \in \Gamma(u)$, where the gradient does not vanish. Then (y_0, t_0) must be a two-phase point.

We begin by observing that, since u is a solution to the two-phase problem with constant coefficients, the parabolic monotonicity formula applies to $(\partial_e u)^\pm$ as in [11] with e any unit vector in a spatial direction. Moreover, Theorem 4.1 in [5] applies and we have uniform $C^{1,1}_x \cap C^{0,1}_t$-estimates for u. Again, we study the limits of the rescaled functions

$$v_j(x,t) = \frac{u(r_j x, r_j^2 t)}{r_j^2},$$

for sequences $r_j \to \infty$. Clearly, $v_j(0,0) = 0$ and $\nabla v_j(0,0) = 0$. Moreover, by the maximum principle, the sets $\{\pm u > 0\}$ cannot be bounded. Therefore, from Proposition 3.1, there exists a constant $c > 0$ depending on M(the bound for λ_is) and the dimension such that for all r, we have

$$\sup_{Q_r^-(y_0,t_0)} \pm u \geq cr^2,$$

and in particular for $r > \sqrt{y_0^2 + |t_0|}$ we have

$$\sup_{Q_{2r}^-} \pm u \geq cr^2.$$

Hence

$$\sup_{Q_\rho^-} \pm v_j \geq \frac{c\rho^2}{4},$$

for all $\rho > 2\sqrt{y_0^2 + |t_0|}/r_j$. Besides, the functions v_j satisfy

1. $\sup_{Q_\rho^-} |v_j| \leq C\rho^2$ for $\rho > 1$ from assumption 1 for u ,
2. $Hv_j = \lambda_1 \chi_{\{v_j>0\}} - \lambda_2 \chi_{\{v_j<0\}}$ in $\mathbb{R}^n \times \mathbb{R}^-$.

Taking the limit as $r_j \to \infty$ and passing to a subsequence if necessary, we conclude that $v_j \to v_0$ locally in $C^{1,\alpha}_x \cap C^\alpha_t(\mathbb{R}^n \times \mathbb{R}^-)$ where v_0 satisfies

1. $\sup_{Q_\rho^-} \pm v_0 \geq \frac{c\rho^2}{4}$ for all ρ,
2. $\sup_{Q_\rho^-} |v_0| \leq C\rho^2$ for $\rho > 1$,
3. $v_0(0,0) = |\nabla v_0(0,0)| = 0$,
4. $Hv_0 = \lambda_1 \chi_{\{v_0>0\}} - \lambda_2 \chi_{\{v_0<0\}}$ in $\mathbb{R}^n \times \mathbb{R}^-$.

By (1), v_0 must have a two-phase point at the origin. Applying the classification of global solutions (Theorem 8.4 in [11]) we conclude that v_0 must be monotone and is one-dimensional, depending only on one spatial variable. This in turn implies that the function $\phi(r, (\partial_e v_0)^\pm)$ in the monotonicity formula of Theorem 3.2 vanishes for all r and all directions e, simply because one of the directional derivatives $(\partial_e v_0)^\pm$ is identically zero.

By the scaling properties of ϕ this implies in particular

$$0 = \phi(1, (\partial_e v_0)^\pm) = \lim_{r_j \to \infty} \phi(r_j, (\partial_e u)^\pm) \geq \phi(r, (\partial_e u)^\pm),$$

for all r and all e. Hence u must be monotone in all directions e. As a consequence, u must be one-dimensional in the spatial variables. Moreover, arguing exactly as in the proof of Lemma 4.1, we can prove that u is time independent. If u is one-dimensional and monotone, it is easy to see that

$$u = \frac{\lambda_1}{2}(x_1^+)^2,$$

and hence ∇u vanishes on $\Gamma(u)$. □

Before proving a certain type of quadratic growth close to points where the spatial gradient is small, we need the following estimate for solutions to the parabolic one-phase obstacle problem. The estimate is the parabolic counterpart to Lemma 2.2 in [1].

Lemma 5.2. *Let u solve*

$$\begin{cases} Hu = f\chi_{\{u>0\}} \text{ in } Q_1^-, \\ u \geq 0, \\ (0,0) \in \partial\{u > 0\}, \end{cases}$$

where $0 \leq f \leq L$. Then, there is a constant $C(n, L)$, depending only on dimension and L, such that

$$\sup_{Q_{\frac{1}{2}}^-} u \leq C.$$

Proof. The proof is simply a translation of the proof Lemma 2.2 in [1] into parabolic terms. Split u as $u = v + w$ where

$$\begin{cases} Hv = Hu \text{ in } Q_1^-, \\ v = 0 \text{ on } \partial_p Q_1^-, \end{cases}$$

where $\partial_p Q_1^- = \{0\} \times B_1 \cup (-1, 0) \times \partial B_1$ is the parabolic boundary of Q_1^-, and

$$\begin{cases} Hw = 0 & \text{in } Q_1^-, \\ w = u & \text{on } \partial_p Q_1^-. \end{cases}$$

Since $0 \leq Hv \leq L$ we have

$$-C(L, n) \leq v \leq 0.$$

In addition, w being caloric implies that $w > 0$ in Q_1^-. Then

$$w(0) = u(0) - v(0) = -v(0) \leq C(L, n).$$

By Harnack's inequality we have

$$\sup_{Q_{\frac{1}{2}}^-} u \leq \inf_{Q_{\frac{1}{2}}^-} u \leq u(0) \leq C(L, n).$$

□

We can now proceed with the growth result.

Proposition 5.3. (Quadratic growth) *Given M, M_0 and A, there are ε, r_0 and C such that for all $u \in P_1(M, M_0, \varepsilon)$ and any $(y, s) \in Q^-_{1/4} \cap \Gamma(u) \cap \{|\nabla u| < Ar\}$ we have*

$$S_r((y,s),u) = \sup_{(x,t)\in Q_r^-} |u(y+x, s+t) - \nabla u(y,s)\cdot x| \le Cr^2, \tag{5.1}$$

for all $r < r_0$. Here ε, r_0 and C depend on M, M_0, A and the dimension.

Proof. We argue by contradiction. Assume that $u_j \in P_1(M, M_0, M_j)$ with $M_j < \varepsilon_j \to 0$ so that the statement of the proposition fails for u_j, i.e.,given any constant $C > 0$, there exists $r_j > 0$, $(y_j, s_j) \in Q^-_{1/4} \cap \Gamma(u) \cap \{|\nabla u| < Ar_j\}$ such that

$$S_{r_j}((y_j,s_j),u_j) = \sup_{(x,t)\in Q_{r_j}^-} |u_j(y_j+x, s_j+t) - \nabla u_j(y_j,s_j)\cdot x| > Cr_j^2. \tag{5.2}$$

Due to the local uniform estimates for the class $P_1(M, M_0, M_1)$, for a subsequence, again labeled u_j, we know that u_j converges to a function $u_0 \in P_\rho(M, M_0, 0)$, i.e. a solution of the parabolic two-phase obstacle problem with constant coefficients in Q^-_ρ, for any $\rho < 1$. For this problem, there are local $C^{1,1}_x \cap C^{0,1}_t$-estimates (see Theorem 4.1 in [5]). Therefore, $S_r((y,t),u_0) \le C_0 r^2$ where $C_0 = C_0(M, M_0)$ for any $(y,t) \in Q^-_{\frac{1}{4}}$ and any $r \le 1/4$. This implies

$$S_r((y,t),u_j) \le C_0 r^2 + \tau(j) \tag{5.3}$$

where $\tau(j) \to 0$ as $j \to \infty$. In particular, for j large enough we have

$$S_r((y,t),u_j) \le 2C_0 r^2, \quad \text{for} \quad r = 1/4.$$

For a constant $C > C_0$ to be chosen later, let $(y_j, t_j) \in Q^-_{\frac{1}{4}} \cap \Gamma(u_j) \cap \{|\nabla u| < Ar_j\}$ denote sequence of points satisfying (5.2). Now define

$$\sigma_j = \sup\{r : S_r((y_j,t_j),u_j) > 2Cr^2\}.$$

Note that the set $\sup\{r : S_r((y_j,t_j),u_j) > 2Cr^2\}$ is non empty because of (5.2) and hence σ_j exists. Moreover, $\sigma_j \le 1/4$ and since $\sigma_j \ge r_j$, we have

$$|\nabla u_j(y_j,t_j)|/\sigma_j \le |\nabla u_j(y_j,t_j)|/r_j < A. \tag{5.4}$$

Hence, if we define

$$v_j(x,t) = \frac{u_j(\sigma_j x + y_j, \sigma_j^2 t + t_j) - \nabla u_j(y_j,t_j)\cdot \sigma_j x}{\sigma_j^2}$$

then v_j satisfies

1. $\sup_{Q_1^-} |v_j| = 2C$,
2. $\sup_{Q_\rho^-} |v_j| \le 2C\rho^2$ for $\rho > 1$,

3. $v_j(0,0) = |\nabla v_j(0,0)| = 0$,
4. $Hv_j = \lambda_1(\sigma_j x + y_j, \sigma_j^2 t + t_j)\chi_{\{v_j > -\nabla u(y_j,t_j)\cdot x/\sigma_j\}}$
   $-\lambda_2(\sigma_j x + y_j, \sigma_j^2 t + t_j)\chi_{\{v_j < -\nabla u(y_j,t_j)\cdot x/\sigma_j\}}$ in $Q^-_{\frac{1}{\sigma_j}}$.

Moreover, from (5.4)

$$\nabla u_j(y_j,t_j)/\sigma_j \to \nu, \quad \text{where} \quad 0 \le |\nu| \le A. \tag{5.5}$$

Hence, passing to a subsequence, again labeled v_j, we conclude that v_j converges locally in $C_x^{1,\alpha} \cap C_t^{0,\alpha}$ to a limit v_0 satisfying

$$\sup_{Q_1^-} |v_0| = 2C, \tag{5.6}$$

$$\sup_{Q_\rho^-} |v_0| \le 2C\rho^2 \quad \text{for some} \quad \rho > 1, \tag{5.7}$$

$$v_0(0,0) = |\nabla v_0(0,0)| = 0, \tag{5.8}$$

$$Hv_0 = \lambda_1(y_0,t_0)\chi_{\{v_0 > -\nu\cdot x\}} - \lambda_2(y_0,t_0)\chi_{\{v_0 < -\nu\cdot x\}} \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^- \tag{5.9}$$

where we assume $(y_j,t_j) \to (y_0,t_0)$. Clearly, (5.8) above implies that the origin is a free boundary point of v_0, however we cannot decide whether it is a one-phase or a two-phase point. We therefore need to treat following cases.

Case 1: $\nu \neq 0$: Then Lemma 4.1 applies and we obtain

$$\sup_{Q_1^-} |v_0| = \frac{\max(\lambda_i(y_0,t_0))}{2},$$

which contradicts (5.6) whenever C is large enough.

Case 2: $\nu = 0$ and the origin is two phase free boundary point, i.e., $(0,0) \in \Gamma^+(v_0) \cap \Gamma^-(v_0)$: In this case, Theorem 8.4 in [11] applies and a contradiction follows as in the Case 1.

Case 3: $\nu = 0$ and origin is a one phase free boundary point: Without loss of generality, suppose $(0,0) \in \Gamma^+(v_0)\backslash\Gamma^-(v_0)$. From Lemma 5.1 we know that ∇v_0 vanishes on $\Gamma(v_0)$. Therefore, $v_0^\pm$ are global solutions to the parabolic one-phase obstacle problem:

$$Hv_0^+ = \lambda_1(y_0,t_0)\chi_{\{v_0^+ > 0\}} \text{ in } \mathbb{R}^n \times \mathbb{R}^-,$$

and

$$Hv_0^- = \lambda_2(y_0,t_0)\chi_{\{v_0^- > 0\}} \text{ in } \mathbb{R}^n \times \mathbb{R}^-.$$

Now, by Lemma 5.2 we have

$$\sup_{Q_1^-} v_0^\pm \le C'(\lambda_1(y_0,t_0), \lambda_2(y_0,t_0), n),$$

and hence we have again reached a contradiction, for C large enough. □

6 Estimates near free boundary points where $|\nabla u|$ does not vanish

In this subsection we treat the case with nonzero gradient across the free boundary. Using the method of Von Mises transform as in [11], we will show that $\partial_t u$ is locally Hölder continuous. The proof relies on Krylov-Safonov type estimates for parabolic operators and hence we require to impose the condition 5. (Definition 1.1) on the functions λ_is. Then, shifting the derivative with respect to t on the right hand side of the equation, we can treat the problem as an elliptic one and obtain the desired estimate.

Lemma 6.1. *Let $f_1, f_2 \in C^\alpha(Q_1^-)$ such that*

$$||f_i||_{C^\alpha(Q_1^-)}, ||\partial_t f_i||_{L^p(Q_1^-)} \le C_1$$

with

$$p > \frac{2+n}{2},$$

and suppose v satisfies

1. $Hv = f_1\chi_{\{v>0\}} - f_2\chi_{\{v<0\}}$ *in* Q_1^-,
2. $\sup_{Q_1^-} |v| \le C_2$,
3. $|\nabla v(0,0)| = 1$,
4. $(0,0) \in \Gamma(v)$.

Then there are constants C, δ and $\beta \in (0,1)$ depending on C_1, C_2 and the dimension such that

$$||\partial_t v||_{C^\beta(Q_\delta^-)} \le C.$$

Proof. For simplicity we can suppose that $\nabla v(0,0) = e_n$. Moreover, for δ small, $|\nabla v| > \frac{1}{2}$ in Q_δ^-. We introduce the new coordinates $y_k = x_k$ for $k < n$ and $y_n = v(x)$. Let $\psi(t,y) = x_n$.

Transforming the equation for v into terms of ψ we obtain

$$\partial_t\psi - \sum \psi_{kk} + \frac{1}{\psi_n}\sum 2\psi_k\psi_{kn} - \frac{1}{\psi_n^2}(1 + \sum \psi_k^2)\psi_{nn} = g\psi_n,$$

where

$$g(x,t) = \begin{cases} f_1 & \text{when } y > 0, \\ f_2 & \text{when } y < 0. \end{cases}$$

The differential operator appearing can also be written in the form

$$L\psi = \partial_t\psi - \sum_{k<n} \partial_k\psi_k + \partial_n\left(\frac{1+\sum_{k<n}\psi_k^2}{\psi_n}\right) = \psi_t - \operatorname{div} A(\nabla\psi),$$

where

$$A(p_1,\ldots,p_n) = \left(p_1,\ldots,p_{n-1}, -\frac{1+\sum_{k<n} p_i^2}{p_n}\right).$$

Differentiating the equation with respect to t yields then

$$\partial_t \psi_t - \partial_k \frac{\partial A_k}{\partial p_j}(\psi_t)_j = g\partial_t \psi_n + \psi_n \partial_t g.$$

With $w = \psi_t$ this becomes

$$\partial_t w - \partial_k (B_{jk} w_j) = \partial_t(\psi_n g),$$

where B is a matrix with C^α coefficients and the right hand side is in L^p. Therefore, Theorem 1.1 on page 419 and Theorem 7.1 on page 181 in [7] imply that $w \in C^\beta$ for some $\beta \in (0,1)$. Thus, switching back to v ends the proof. $\square$

Remark 6.2. *The observant reader might wonder whether the computations above can be justified. This is indeed the case, by approximating the right hand side with a smooth function, all the involved functions becomes infinitely differentiable, and we obtain estimates that does not depend on the smoothness of the approximation.*

Proposition 6.3. *Given constants M, $M_0 > 0$, let $r_0 = r_0(M, M_0)$ and $\varepsilon = \varepsilon(M, M_0)$ be chosen as in Proposition 5.3. Then, there exists constants δ_0 and C depending on M, M_0 and the dimension such that for all $u \in P_1(M, M_0, \varepsilon)$ there holds*

$$||u||_{C^{1,1}_x \cap C^{0,1}_t (Q_{\delta_0 |\nabla u(y,t)|}(y,t))} \leq C \quad \text{for all} \quad (y,t) \in \Gamma(u) \cap \{0 < |\nabla u(y,t)| \leq r_0\}. \tag{6.1}$$

Proof. Take $(y,s) \in \Gamma(u) \cap \{0 < |\nabla u(y,t)| \leq r_0\}$ and put $r_y = |\nabla u(y,s)|$. Define moreover

$$v(x,t) = \frac{u(y + r_y x, s + r_y^2 t)}{r_y^2}.$$

Clearly,

$$(0,0) \in \Gamma(v) \quad \text{and} \quad |\nabla v(0,0)| = \frac{\nabla u(y,s)}{r_y} = 1. \tag{6.2}$$

Moreover, applying Proposition 5.3 we have

$$S_r((y,s),u) = \sup_{(x,t) \in Q_r^-} |u(y + x, s + t) - \nabla u(y,s) \cdot x| \leq Cr^2, \quad \text{for all} \quad r < r_0.$$

Hence, for any $(x,t) \in Q_1^-$ we have

$$v(x,t) = \frac{u(y + r_y x, s + r_y^2 t)}{r_y^2} \leq \frac{Cr_y^2 + r_y^2}{r_y^2} \leq C + 1, \tag{6.3}$$

where $C = C(M, M_0)$. Hence, v satisfies the hypotheses of Lemma 6.1 and we conclude that

$$||\partial_t v||_{C^\beta(Q^-_{\delta_1})} \leq C'(M, M_0).$$

Scaling back, we see that

$$||\partial_t u||_{C^\beta(Q^-_{\delta_1 |\nabla u(y,t)|}(y,s))} \leq C'(M, M_0). \tag{6.4}$$

Now we can treat the problem as an elliptic one by moving the term $\partial_t u$ to the right hand side of the equation satisfied by u. Thus, Proposition 2.6 in [4] implies that

$$||u(\cdot,t)||_{C^{1,1}_x(B_{\delta_2|\nabla u(y,s)|}(y,t))} \leq C(M, M_0), \quad \text{uniformly in} \quad t. \tag{6.5}$$

This together with the fact that $\partial_t u$ is Hölder continuous implies the desired estimate choosing δ_0 small enough. ☐

7 $C^{1,1}_x \cap C^{0,1}_t$-estimates

Since we now have obtained the correct estimates in neighbourhood of free boundary points both when the gradient is zero and otherwise, the main result follows by combining these two results in the right way. First we prove that we have the optimal regularity when the oscillation of the λ_is are small enough, precisely in case when $u \in P_1(M, M_0, \varepsilon)$.

Proposition 7.1. *Given M and M_0, there exists positive constants $\varepsilon > 0$, C and r_1 such that whenever $u \in P_1(M, M_0, \varepsilon)$ we have*

$$\|u\|_{C^{1,1}_x \cap C^{0,1}_t(Q^-_{r_1/2})} \leq C,$$

where C, ε and r_1 depend on M, M_0 and the dimension.

Proof. Given M, M_0 we choose $\varepsilon > 0$ such that the Proposition 5.3 applies. Now choose point $(y, s) \in Q^-_1$ such that

$$d := \operatorname{dist}((y, s), \Gamma(u)) < r_0(M, M_0) \tag{7.1}$$

where r_0 is the one obtained in Proposition 5.3.

Now if $|\nabla u| < d$ then put

$$v(x,t) = \frac{u(dx + y, d^2 t + s) - dx \cdot \nabla u(y, s)}{d^2}.$$

Then $Hv = \lambda_i(dx, d^2 t)$ in $Q^-_{1/2}$ for one of $i = 1, 2$, as the point (y, s) is away from the free boundary. Moreover, Proposition 3.1 implies that $|v| \leq C$ in Q^-_1. Hence, interior estimates imply

$$\sup_{Q^-_{d/2}(y,s)} |\partial_t u| + |D^2 u| = \sup_{Q^-_{1/2}} |\partial_t v| + |D^2 v| \leq C.$$

In the other case, if $|\nabla u| > d$, then Proposition 6.3 implies

$$||u||_{C^{1,1}_x \cap C^{0,1}_t(Q_{t_0|\nabla u(y,s)|}(y,s))} \leq C.$$

Hence, with $r_1 = \min(2t_0 d, d)$ the theorem follows. ☐

Rescaling this result we obtain the correct regularity without the restriction on the oscillation and we can then prove Theorem 1.2.

Proof of Theorem 1.2. For $u \in P_1(M_0, M, M_1)$, choose $0 < \rho < 1$ such that

$$\rho^\alpha M_1 < \varepsilon(C(M, M_0), M, M_1) = \varepsilon$$

where ε is as in Proposition 7.1 and C from Proposition 8.5. Then by the latter we have

$$v(x) = \frac{u(\rho x, \rho^2 t)}{\rho^2} \in P_1(C(M, M_0), M_0, \varepsilon)$$

and so by Proposition 7.1

$$||u||_{C_x^{1,1} \cap C_t^{0,1}(Q^-_{\rho r_1/2})} = ||v||_{C_x^{1,1} \cap C_t^{0,1}(Q^-_{r_1/2})} \leq C.$$

□

8 Appendix

In this section we present a simpler result which is important: the quadratic growth around branching points, without any restriction on the oscillation of the λ_is. This result is needed in order to complete the proof of Theorem 1.2. The result is the parabolic counterpart to Proposition 4.1 in [13]. The proof is very similar to the one given for the elliptic case with some technical details inspired from the proof of Lemma 8 in [3].

The first thing we need is a version of Weiss' monotonicity formula adapted to our case. Since the coefficients are Hölder continuous, the Weiss energy functional enjoys the same properties as shown in Proposition 1 in [3]. More precisely, let $\psi \in C_0^\infty(B_{\frac{3}{4}})$, $\psi = 1$ in $B_{\frac{1}{2}}$, $v = u\psi$ and

$$W(r; v) = \frac{1}{r^4} \int_{T_r} \left(|\nabla v|^2 + 2\lambda_1 v^+ + 2\lambda_- v^- + \frac{v^2}{t} \right) \mathrm{d}\gamma,$$

where

$$\mathrm{d}\gamma = G(x, -t)\, \mathrm{d}x\, \mathrm{d}t,$$

with $G(x, t)$ being the standard heat kernel, and

$$T_r = (-4r^2, -r^2] \times \mathbb{R}^n.$$

Then the following result holds:

Proposition 8.1. *Let $u \in P_1(M_0, M, M_1)$ and put $v = u\psi$. Then there is continuous function $F = F_{M_1,M_2}$ such that*

$$W(r; v, f) + F(r)$$

is a nondecreasing function for $0 < r < 1/2$.

The proof is almost identical to the one of Proposition 1 in [3]. The only difference is that we need to change the $E(r; v)$ to instead be

$$E(r; v) = \frac{1}{r^4} \int_{T_r} \left(|\nabla v|^2 + 2\lambda_1(0) v^+ + 2\lambda_2(0) v^- + \frac{v^2}{t} \right) \mathrm{d}\gamma.$$

Using this semi-monotonicity one can prove the quadratic growth result at branching points. Before doing that we need some technical tools given below.

We state here a result of Simon, namely Theorem 6 in [12]).

Lemma 8.2. *(Simon's lemma) Let* $X_0 \subset X \subset X_1$ *be Banach spaces such that* X_0 *is compactly embedded in* X *and* X *is continuously embedded in* X_1*. Moreover assume that* u_k *is a sequence of functions such that for some* $q > 1$

$$\|u_k\|_{L^q(I;X)} + \|u_k\|_{L^1(I;X_0)} + \|\partial_t u_k\|_{L^1(I;X_1)} \leq C,$$

where $I \subset \mathbb{R}$ *is a compact interval, then there is a subsequence* u_{k_j} *that converges in* $L^p(I;X_0)$ *for all* $1 \leq p < q$*.*

In what follows we will apply this lemma with $X_0 = H^1_{\gamma^{-1}}(\mathbb{R}^n)$, $X = L^2_{\gamma^{-1}}(\mathbb{R}^n)$ and $X_1 = (Y)^*$, the dual of the space Y equipped with the norm

$$\|\phi\|_Y = \left\|\frac{\phi}{G(x,1)}\right\|_{L^\infty(\mathbb{R}^n)} + \|\nabla\phi\|_{L^2_{(\gamma^{-1})^{-1}}(\mathbb{R}^n)},$$

where the subscript γ^{-1} or $(\gamma^{-1})^{-1}$ stands for that we use the weighted space with $\gamma^{-1} = G(x,1)$ or $(\gamma^{-1})^{-1}$ as weight. Then, the fact that the embedding $X \to X_0$ is compact is a special case of Theorem 3.1 in [6]. The fact that the embedding $X \to X_1$ is continuous is clear. We will also use that we have an embedding of the form

$$\nabla L^2_{\gamma^{-1}}(T_1) + L^1_{\gamma^{-1}}(T_1) \to L^1((-4,-1), Y^*).$$

The proof of the following results can all be found in [3]. The first one, which is Lemma 5 in [3], is a consequence of energy estimates. Below we use the notation $S_r = (-4r^2, 0] \times \mathbb{R}^n$.

Lemma 8.3. *Let* u *be a limit of functions that all have compact support. Suppose* u *solves*

$$Hu = f \text{ in } S_1, \tag{8.1}$$

and that

$$\iint_{T_1} \frac{u^2}{-t} + |\nabla u|^2 \,\mathrm{d}\gamma \leq C,$$

$$\|f\|_{L^\infty(S_1)} \leq C.$$

Then

$$\int_{\mathbb{R}^n} u^2(x,t_0)\,\mathrm{d}\gamma^{t_0} \leq C'$$

for any $t_0 \in (-4,0)$*.*

The second one is a consequence of the Almgren frequency lemma, and it is the parabolic counterpart to Lemma 4.1 in [13]. This is Lemma 7 in [3].

Lemma 8.4. *Let* $w \in L^2_\gamma(T_1)$ *be a caloric function in* S_1 *which is the limit of functions all having compact support and bounded heat operator. Assume further that* $w(0,0) = 0$ *and* $|\nabla w(0,0)|$. *Then*

$$\int_{T_r} |\nabla w|^2 \,\mathrm{d}\gamma \geq \int_{T_r} \frac{w^2}{-t} \,\mathrm{d}\gamma, \tag{8.2}$$

with equality if and only if w *is parabolic homogeneous of degree* 2.

Now we can prove the quadratic growth.

Proposition 8.5. *Let* $u \in P_4(M_0, M, M_1)$. *Then there is a constant* $C(M_0, M, M_1)$ *such that*

$$\sup_{Q_r^-} |u| \leq Cr^2$$

whenever r *is small enough.*

Proof. All along this proof we will use C for an abritary constant depending only on M_0, M and M_1.

It is clear that by parabolic estimates it is sufficient to prove that

$$\int_{T_r} |v| \,\mathrm{d}\gamma \leq Cr^4. \tag{8.3}$$

In what follows we denote by

$$\Pi = \{u \text{ caloric and parabolically homogeneous of degree } 2\}.$$

We will prove that

$$\mathrm{dist}_r(v, \Pi) = \inf_{p \in \Pi} \int_{T_r} \frac{(v-p)^2}{-t} \,\mathrm{d}\gamma \leq Cr^4. \tag{8.4}$$

By the use of the monotonicity functional this implies (8.3). Indeed, using that p fulfills

$$\int_{T_1} |\nabla p|^2 \,\mathrm{d}\gamma = \int_{T_1} \frac{p^2}{-t} \,\mathrm{d}\gamma$$

and $x \cdot \nabla p + 2t\partial_t p - 2p = 0$ we obtain for p_r being the minimizer for v_r

$$\int_{T_1} \lambda_1 v_r^+ + \lambda_2 v_r^- \,\mathrm{d}\gamma \leq W(r; v, \lambda_i) - \int_{T_r} |\nabla v_r|^2 + \frac{v_r^2}{t} \,\mathrm{d}\gamma$$

$$= W(r; v, \lambda_i) - \int_{T_1} |\nabla(v_r - p_r)|^2 + \frac{(v_r - p_r)^2}{t} \,\mathrm{d}\gamma \leq W(r_0, v, \lambda_i) + F(r_0) + C.$$

The proof is now by contradiction. If (8.4) is not true, then there exists $r_j \to 0$ such that

$$M_j = \mathrm{dist}_1(v_{r_j}, \Pi) = \int_{T_1} \frac{(v_{r_j} - p_j)^2}{t} \,\mathrm{d}\gamma \to \infty.$$

Let $w_j = v_{r_j}/M_j$. Then

$$\int_{T_1} \frac{w_j^2}{-t}\,\mathrm{d}\gamma = 1$$

and

$$\begin{aligned}
&\int_{T_1} |\nabla w_j|^2 + \frac{w_j^2}{t}\,\mathrm{d}\gamma = \frac{1}{M_j^2} W(r_j; v) - \frac{1}{M_j^2}\int_{T_1} |\nabla p_j|^2 + \frac{p_j^2}{t}\,\mathrm{d}\gamma - \\
&\int_{T_1} \frac{\lambda_1(r_j x, r_j^2 t) v_{r_j}^+ + \lambda_2(r_j x, r_j^2 t) v_{r_j}^-}{M_j}\,\mathrm{d}\gamma + \frac{1}{M_j^2}\int_{T_1} 2\nabla v_{r_j}\nabla p_j + \frac{2 v_{r_j} p_j}{t}\,\mathrm{d}\gamma \\
\leq\ & \frac{W(r_0; v, \lambda_i(r_j x, r_j^2 t)) + F(r_j)}{M_j} \to 0,
\end{aligned}$$

where we have used that p_j fulfills

$$\int_{T_1} |\nabla p|^2\,\mathrm{d}\gamma = \int_{T_1} \frac{p^2}{-t}\,\mathrm{d}\gamma$$

and $x \cdot \nabla p + 2t\partial_t p - 2p = 0$. Both these inequalities are simple consequences of p being a caloric polynomial of parabolic degree 2.

The assumptions on u implies (by local estimates) that $u \in C_x^{1,\alpha} \cap C_t^{\alpha}(Q_2^-)$ for $0 < \alpha < 1$. This in turn implies that $H(u\psi) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^-)$, which implies that

$$\|Hw_j\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^-)} = \frac{1}{M_j}\|Hv\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^-)} \to 0,$$

by scaling. Moreover, by Lemma 8.3 we have that

$$\int_{\mathbb{R}^n} w_j^2(x,t)\,\mathrm{d}\gamma^t \leq C$$

for all $t \in (-4, 0)$. Here $\mathrm{d}\gamma^t = G(x, -t)\,\mathrm{d}x$. We also have

$$\partial_t w_j = \Delta w_j - Hw_j = a + b,$$

where

$$a = \Delta w_j = \nabla \cdot \nabla w_j \subset \nabla L_\gamma^2(T_1)$$

and

$$b = -Hw_j \in L^\infty(T_1).$$

Now define $\hat{w}_j(x,t) = w(x\sqrt{-t}, t)$. Then

$$\begin{aligned}
\nabla \hat{w}_j(x,t) &= \sqrt{-t}\nabla w(x\sqrt{-t}, t) \\
\partial_t \hat{w}_j(x,t) &= \partial_t w(x\sqrt{-t}, t) + \frac{x}{2\sqrt{-t}}\nabla w(x\sqrt{-t}, t) \\
&\subset \nabla L^2_{\gamma^{-1}}(T_1) + L^1_{\gamma^{-1}}(T_1) \subset L^1((-4,-1), Y^*).
\end{aligned}$$

This implies that have the uniform estimates

$$\left\|\frac{\hat{w}}{(-t)^{\frac{1}{2}}}\right\|_{L^2_{\gamma^{-1}}} + \|\nabla \hat{w}\|_{L^2_{\gamma^{-1}}(T_1)} + \|\partial_t \hat{w}\|_{L^1((-4,-1),Y^*)} \leq C.$$

Simons lemma, Lemma 8.2, together with the locally uniform $C_x^{1,\alpha} \cap C_t^{\alpha}(\mathbb{R}^n \times (-4, 0])$-estimates, now implies that, up to a subsequence, $\hat{w_j} \to \hat{w_0}$ in $L^2_\gamma(T_1)$, and therefore also the same for w_j, and that w_j converges locally in $C_x^{1,\alpha} \cap C_t^{\alpha}(\mathbb{R}^n \times (-4, 0])$. Thus, passing to the limit we obtain that w_0 is caloric in T_1, $|\nabla w_0(0, 0)| = 0$, and w_0 satisfies

$$\int_{T_1} \frac{w_0^2}{-t} \, \mathrm{d}\gamma = 1, \tag{8.5}$$

$$\int_{T_1} \frac{w_0 p}{-t} \, \mathrm{d}\gamma = 0, \tag{8.6}$$

for all $p \in \Pi$, and

$$\int_{T_1} |\nabla w_0|^2 + \frac{w_0^2}{t} \, \mathrm{d}\gamma \leq 0$$

which by Lemma 8.4 implies that w_0 is parabolic homogeneous of degree 2, so $w_0 = 0$ by (8.6) which contradicts (8.5). $\square$

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