A TWO-PHASE OBSTACLE-TYPE PROBLEM FOR THE p-LAPLACIAN

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Abstract. We study the so-called two-phase obstacle-type problem for the p-Laplacian when p is close to 2. We introduce a new method to obtain the optimal growth of the function from branch points, i.e. two-phase points in the free boundary where the gradient vanishes. As a by-product we can locally estimate the $(n-1)$-Hausdorff-measure of the free boundary for the special case when $p > 2$.

1. Introduction and main result

1.1. **Problem.** Given a smooth and bounded domain $D \subset \mathbb{R}^n$, $1 < p < \infty$ and $g \in W^{1,p}(D) \cap L^\infty(D)$ we study the minimizer of the functional

$$J_D(u) = \int_D \frac{|\nabla u|^p}{p} + \lambda_1 u^+ + \lambda_2 u^- \,\mathrm{d}x \tag{1.1}$$

over the set $\{u : u - g \in W_0^{1,p}(D)\}$, and its corresponding Euler-Lagrange equation

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda_1 \chi_{\{u>0\}} - \lambda_2 \chi_{\{u<0\}} \quad \text{in } D. \tag{1.2}$$

It is not obvious that they are equivalent, however we prove in Section 2 that this is the case when $p > 2$.

In the sets $\Omega^+(u) = \{u > 0\} \cap D$ and $\Omega^-(u) = \{u < 0\} \cap D$ the right hand side of (1.2) is constant. However on the boundary of these sets, $\Gamma^\pm(u) = \partial\Omega^\pm(u) \cap D$ we have a jump discontinuity, and therefore we call $\Gamma^\pm(u)$ free boundaries. Moreover, due to the non-linear nature of the p-Laplacian we cannot use the standard methods adapted to the linear cases.

1.2. **Known results.** For $p = 2$ the operator reduces to the Laplacian and hence we arrive at the two-phase obstacle problem. Independently of each other Shahgholian ([Sha03]) and Uraltseva ([Ura01]) proved that solutions to this problem are in $C^{1,1}_{\mathrm{loc}}(D)$. In both of these papers, the powerful monotonicity formula from [CJK02] plays an important role. Later, in [SUW04] Shahgholian, Uraltseva and Weiss proved that the free boundary is C^1-regular near so-called branching points, i.e. two-phase free boundary points where the gradient vanishes.

If we impose a sign restriction on u, then the problem is like the obstacle problem for the p-Laplacian which has been studied earlier: In [KKPS00] the optimal growth near the free boundary is proved, and in [LS03] it is proved that the free boundary, $\partial\{u > 0\} \cap D$ has locally finite $(n-1)$-Hausdorff measure.

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1.3. **Main result.** The main result of the paper is that minimizers of (1.1) has, around branching points, i.e. around points in the set $\Gamma(u) = \Gamma^+(u) \cap \Gamma^-(u) \cap \{|\nabla u| = 0\}$, the optimal growth of order r^{p^*}, where $p^* = \frac{p}{p-1}$, when p is sufficiently close to 2. That this is optimal follows from the one-dimensional minimizer

$$u(x) = \frac{\lambda_1}{p^*}(x_+)^{p^*} - \frac{\lambda_2}{p^*}(x_-)^{p^*}.$$

Since our main results concern only local properties, we will state them for minimizers in balls instead of general smooth and bounded domains. In order to state our results we first need to define our class of minimizers. In what follows we will denote by $B_r(y)$ the ball of radius r centered at y and for simplicity use the notation $B_r = B_r(0)$.

Definition 1. *We say that $u \in P_r(M, p, y)$ if*

(1) *u is a minimizer of $J_{B_r(y)}$ with parameter p.*
(2) *$||u||_{L^\infty(B_r(y))} \leq M$.*
(3) *$y \in \Gamma(u)$.*

Here we will also use the notation $P_r(M, p) = P_r(M, p, 0)$.

Remark 1. *The reader may observe that if $u \in P_r(M, p)$ and $y \in \Gamma(u)$ then $u \in P_\rho(M, p, y)$ for $\rho < \operatorname{dist}(y, \partial B_1)$.*

Theorem 1. *(Optimal growth) Let $u \in P_1(M, p)$. Then there are constants C, δ and r_0, depending on λ_i, M and the dimension such that $|p-2| < \delta$ implies*

$$\sup_{B_r(y)} |u| \leq C r^{p^*} \tag{1.3}$$

for $r < r_0$ and any $y \in B_{1/2} \cap \Gamma(u)$.

As a corollary to Theorem 1 we obtain also that $\Gamma^+(u) \cup \Gamma^-(u)$ has locally finite $(n-1)$-Hausdorff measure when $p > 2$. We exploit the method used in [SW06] and adapt it to our case.

Theorem 2. *Under the same hypotheses as in Theorem 1 and for $p > 2$ we have*

$$\mathcal{H}^{n-1}((\Gamma^+(u) \cup \Gamma^-(u)) \cap B_{1/2}) < C, \tag{1.4}$$

where C depends on M, λ_i and the dimension.

2. Existence and uniqueness

In this section we discuss the existence of minimizers of (1.1) and solutions of (1.2).

Since the functional (1.1) is weakly lower semicontinuous and convex there exists a unique u that minimizes J.

Proposition 1. *Let u be a minimizer of (1.1) and let $p \geq 2$. Then u satisfies equation (1.2) almost everywhere.*

Proof. Let u be a minimizer of (1.1) and take $\phi \in C_0^\infty(D)$. Then for $t > 0$

$$\begin{aligned}
0 &\leq \frac{J_D(u + t\phi) - J_D(u)}{t} \\
&= \int_D \frac{1}{p} \frac{|\nabla u + t\phi|^p - |\nabla u|^p}{t} \,\mathrm{d}x + \int_D \lambda_1 \frac{(u + t\phi)^+ - u^+}{t} \,\mathrm{d}x \\
&+ \int_D \lambda_2 \frac{(u + t\phi)^- - u^-}{t} \,\mathrm{d}x \\
&= A + \lambda_1 B + \lambda_2 C.
\end{aligned}$$

For B we have

$$B=\frac{1}{t}\int_D (u+t\phi)^+-u^+\,\mathrm{d}x=\frac{1}{t}\int_{\{u+t\phi>0\}\cap D}u+t\phi\,\mathrm{d}x-\frac{1}{t}\int_{\{u>0\}\cap D}u\,\mathrm{d}x.$$

But we can rewrite $\{u>0\}\cap D$ as

$$\{u>0\}\cap D=(\{u+t\phi>0\}\cup\{0<u\leq -t\phi\}\setminus\{-t\phi<u\leq 0\})\cap D.$$

Using this relation we obtain

$$\begin{aligned}
B&=\frac{1}{t}\int_{\{u+t\phi>0\}\cap D}u+t\phi\,\mathrm{d}x-\frac{1}{t}\int_{\{u+t\phi>0\}\cap D}u\,\mathrm{d}x\\
&-\frac{1}{t}\int_{\{0<u\leq -t\phi\}\cap D}u\,\mathrm{d}x-\frac{1}{t}\int_{\{-t\phi<u\leq 0\}\cap D}u\,\mathrm{d}x\\
&=\int_{\{u+t\phi>0\}\cap D}\phi\,\mathrm{d}x-\frac{1}{t}\int_{\{0<u\leq -t\phi\}\cap D}u\,\mathrm{d}x\\
&+\frac{1}{t}\int_{\{-t\phi<u\leq 0\}\cap D}u\,\mathrm{d}x\\
&\leq\int_{\{u+t\phi>0\}\cap D}\phi\,\mathrm{d}x.
\end{aligned}$$

The analogous result for C will be

$$\frac{1}{t}\int_D (u+t\phi)^- - u^-\,\mathrm{d}x\leq\int_{\{u+t\phi<0\}\cap D}-\phi\,\mathrm{d}x.$$

Putting this into the initial inequality we have

$$\begin{aligned}
0&\leq\frac{J_D(u+t\phi)-J_D(u)}{t}\\
&\leq\int_D\frac{1}{p}\frac{|\nabla u+t\phi|^p-|\nabla u|^p}{t}\,\mathrm{d}x+\lambda_1\int_{\{u+t\phi>0\}\cap D}\phi\,\mathrm{d}x\\
&-\lambda_2\int_{\{u+t\phi<0\}\cap D}\phi\,\mathrm{d}x,
\end{aligned}$$

so when $t\to 0$

$$\int_D|\nabla u|^{p-2}\nabla u\nabla\phi\,\mathrm{d}x+\lambda_1\int_{\{u\geq 0\}\cap D}\phi\,\mathrm{d}x-\lambda_2\int_{\{u\leq 0\}\cap D}\phi\,\mathrm{d}x\geq 0.$$

Now this is true for all $\phi\in C_0^\infty(D)$. This implies that $\Delta_p u$ is a bounded distribution and thus it is a measure. Using the Radon-Nikodym theorem we know that there is an $f\in L^1(D)$ that represents $\Delta_p u$ as a distribution and thus $\Delta_p u$ is in $L^1(D)$. From Proposition 1 in [Tol84] we have that $|\nabla u|^{p-2}\nabla u\in W^{1,1}_{\text{loc}}(D)$ which implies that $\Delta_p u=0$ a.e. in the set where the gradient vanishes. Moreover, the set $\Gamma^+(u)\cup\Gamma^-(u)\cap\{|\nabla u|\neq 0\}$ is locally a differentiable manifold (since u is C^1) and therefore it has measure zero. From there it follows that u solves the equation a.e. □

Proposition 2. *Any solution to equation (1.2) is unique.*

Proof. Assume that we have two solutions u and v and also that at least one of them is not identically zero. Then we make use of the following well-known inequality

$$C(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)\geq\begin{cases}|\xi-\eta|^2(|\xi|+|\eta|)^{p-2} & \text{if } 1<p\leq 2,\\ |\xi-\eta|^p & \text{if } p\geq 2,\end{cases}$$

where $C = C(n,p) > 0$. With $\xi = \nabla u$ and $\eta = \nabla v$ we obtain

$$C\int_D (\Delta_p u - \Delta_p v)(v-u)\,\mathrm{d}x \geq \int_D |\nabla u - \nabla v|^2(|\nabla u| + |\nabla v|)^{p-2}\,\mathrm{d}x,$$

if $1 < p \leq 2$, and

$$C\int_D (\Delta_p u - \Delta_p v)(v-u)\,\mathrm{d}x \geq \int_D |\nabla u - \nabla v|^p\,\mathrm{d}x,$$

if $p \geq 2$.

Now we study the sign of $(\Delta_p u - \Delta_p v)(v-u)$. Since we can interchange the roles of u and v it suffices to investigate the case when $u \geq v$. We split it into seven different cases

(1) $v > 0$ then $\Delta_p u - \Delta_p v = \lambda_1 - \lambda_1 = 0$ so the expression is zero.
(2) $v < 0$ and $u > 0$ then $(\Delta_p u - \Delta_p v) = \lambda_1 + \lambda_2$ so the expression is non-positive.
(3) $v < 0$ and $u = 0$ then $(\Delta_p u - \Delta_p v) = \lambda_2$ so the expression is non-positive.
(4) $v < 0$ and $u < 0$ then $(\Delta_p u - \Delta_p v) = \lambda_2 - \lambda_2$ so the expression is zero.
(5) $v = 0$ and $u > 0$ then $(\Delta_p u - \Delta_p v) = \lambda_1$ so the expression is non-positive.
(6) $v = 0$ and $u = 0$ then $(\Delta_p u - \Delta_p v) = 0$ so the expression is non-positive.
(7) $v = 0$ and $u < 0$ then $(\Delta_p u - \Delta_p v) = -\lambda_2$ so the expression is non-positive.

Therefore

$$(\Delta_p u - \Delta_p v)(v-u) \leq 0,$$

which by the two inequalities above implies $\nabla u = \nabla v$, and since u and v have the same boundary values this implies that $u = v$. □

3. Stability results

In this section we present some stability results we will need in order to prove our main result. First of all we prove that the minimizer of (1.1) does not grow too slow around branching points.

Proposition 3. *(Nondegeneracy) Let $u \in P_R(M,p)$. Then for every $y \in \Gamma(u)$ we have*

$$\sup_{B_r(x)\cap\Omega^+(u)} u \geq \frac{1}{p^*}\left(\frac{\lambda_1}{n}\right)^{p'} r^{p^*}, \tag{3.1}$$

and

$$\inf_{B_r(x)\cap\Omega^-(u)} u \leq -\frac{1}{p^*}\left(\frac{\lambda_2}{n}\right)^{p'} r^{p^*}, \tag{3.2}$$

for $r < \operatorname{dist}(y, \partial B_R)$. Here $p^* = \frac{p}{p-1}$ as before and from now on $p' = \frac{1}{p-1}$.

Proof. Let $y \in \Omega^+(u)$, and take r such that $B_r(y) \subset D$. Define

$$w(x) = \frac{1}{p^*}\left(\frac{\lambda_1}{n}\right)^{p'} |x-y|^{p^*}.$$

Then we have

$$\Delta_p u \geq \Delta_p w,$$

in $\Omega^+(u) \cap B_r(y)$. Since $u(y) > w(y) = 0$ there is, by the comparison principle, $x_y \in \partial(B_r \cap \Omega^+(u))$ such that $u(x_y) > w(x_y)$. Moreover, $u \leq w$ on $\Gamma^+(u)$ so $x_y \in \partial B_r \cap \Omega^+(u)$. Therefore

$$\sup_{\partial B_r(y)\cap\Omega^+(u)} u > \frac{1}{p^*}\left(\frac{\lambda_1}{n}\right)^{p'} r^{p^*}.$$

Now if $x \in \Gamma^+(u)$ we can take $x^j \in \Omega^+(u)$ such that $x^j \to x$. Therefore we obtain (3.1) by continuity. Also, (3.2) can be obtained in a similar manner. □

Now we prove that the class is stable when $p \to 2$.

Proposition 4. *Let $u_j \in P_R(M, p_j)$ where $p_j \to 2$. Then there is a subsequence, again labelled u_j and a function $u_2 \in P_r(M, 2)$ for all $r < R$, such that $u_j \to u_2$ in $C^{1,\alpha}(B_r)$ for all $r < R$.*

In order to prove Proposition 4 we will need the following Lemma:

Lemma 1. *Let $u_j \in P_r(M, p_j)$ such that $u_j \to u$ in $C^{1,\alpha}(B_r)$ for some $\alpha > 0$. Then*

(1) $\limsup_j \{u_j = |\nabla u_j| = 0\} \subset \{u = |\nabla u| = 0\}$,
(2) $\Omega^\pm(u) \subset \liminf_j \Omega^\pm(u_j)$,
(3) $\limsup_j \Gamma^\pm(u_j) \subset \Gamma^\pm(u)$.

Proof. The first two inclusions immediately follow from the C^1-convergence $u_j \to u$. To prove the third inclusion we take a convergent subsequence $x_j \in \Gamma(u_j)$ such that $x_j \to x$. Then $u_j(x_j) = 0$ and hence $u(x) = 0$. According to Proposition 3 for any $r > 0$ there exists $y_j \in \partial B_r(x)$ such that $u_j(y_j) \geq Cr^{p^*} > 0$ for all j. Thus there exists $y \in \partial B_r(x)$ such that $u(y) \geq Cr^{p^*} > 0$. Since r is arbitrary we conclude that $x \in \Gamma^+(u)$ and $\limsup_j \Gamma^+(u_j) \subset \Gamma^+(u)$. We use analogous arguments to show that $\limsup_j \Gamma^-(u_j) \subset \Gamma^-(u)$. □

Proof of Proposition 4. Using the same argument as in Proposition 1 we have that $\Delta_{p_j} u \leq C$ in distribution sense. By Theorem 1 in [Tol84] we have that $u_j \in C^{1,\alpha}(B_r)$ uniformly. This together with Lemma 1 implies that for a subsequence u_j converges in $C^{1,\alpha}(B_r)$ to some $u_2 \in P_r(M, 2)$. □

4. Optimal growth

Proof of Theorem 1. By covering arguments and Remark 1 it is sufficient to prove the Theorem for $y = 0$.

Take a sequence $p_j \to 2$, with $u_j \in P_1(M, p_j)$ such that $u_j \to u_2$ in $C^{1,\alpha}(B_r)$, where $u_2 \in P_r(M, 2)$ for all $r < 1$. This is possible because of Proposition 4. Then for any $\tau > 0$ there is a j_τ such that $j > j_\tau$ implies

$$\sup_{B_r} |u_j - u_2| \leq \tau,$$

for $r < 1$. Now from [Sha03] and [Ura01] we know that

$$\sup_{B_r} |u_2| \leq C_1 r^2, \tag{4.1}$$

for $r \leq 1/4$ and for some constant C_1 depending only on the dimension. Hence

$$\sup_{B_r} |u_j| \leq C_1 r^2 + \tau,$$

which implies that for $r = 1/4$ and τ small enough we will have

$$\sup_{B_r} |u_j| \leq 2C_1 r^{p_j^*}.$$

Take $C > C_1$ to be chosen later and let

$$r_j = \sup\{r : \sup_{B_r} |u_j| > 2Cr^{p_j^*}\}.$$

If we can show that $r_j = 0$ for j big enough the theorem is proved. Obviously we have $\sup_{B_{r_j}} |u_j| = 2Cr_j^{p^*}$. We argue by contradiction. If the statement of the

Theorem fails for any $\delta > 0$ then $r_j \to r_0$ where all $r_j > 0$, for r_j associated to some sequence of p_j:s. Define

$$v_j(x) = \frac{u_j(r_j x)}{r_j^{p_j^*}}.$$

We must have $r_0 = 0$ since we otherwise will have a contradiction to (4.1). Now we have

(1) $\sup_{B_1} |v_j| = 2C$
(2) $\sup_{B_r} |v_j| \leq 2Cr^{p_j^*}$ for $1 < r < 1/r_j$,
(3) $v_j \in P_r(2C|r|^{p_j^*}, p_j)$ for all $r \geq 1$.

By Proposition 4 we can take a subsequence v_j converging in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ to a function v_2 such that

(1) $\sup_{B_1} |v_2| = 2C$
(2) $\sup_{B_r} |v_2| \leq 2Cr^2$ for $r > 1$,
(3) $v_2 \in P_r(2C|r|^2, 2)$ for all $r \geq 1$.

Now (3) together with Theorem 1 in [Ura01] implies that $D^2 v_2 \in L^\infty(\mathbb{R}^n)$. Therefore Theorem 4.3 in [SUW04] applies and we have

$$v_2 = \frac{\lambda_1}{2}(x_1^+)^2 - \frac{\lambda_2}{2}(x_1^-)^2,$$

in some system of coordinates. Therefore

$$\sup_{B_1} |v_2| = \frac{1}{2}\max(\lambda_1, \lambda_2).$$

This is clearly a contradiction to

$$\sup_{B_1} |v_2| = 2C,$$

for

$$C > \frac{\max(\lambda_1, \lambda_2)}{4}.$$

□

Corollary 1. *Under the same assumptions as in Theorem 1 and with $p > 2$ the result in Theorem 1 also holds for solutions to (1.2).*

Remark 2. *In the proof of the optimal growth, the only thing we exploit is the classification of global solutions for $p = 2$ (or the uniform bound on the B_1-norm of global solutions). If we would be able to prove something similar for any p then we could prove the optimal growth for any p.*

5. $\mathcal{H}^{n-1}$-estimates of the free boundary

As a by-product of the optimal growth result, we obtain in this section estimates of the $(n-1)$-Hausdorff measure of the free boundary. However, the method used makes it necessary to restrict ourselves to the case where $p > 2$.

5.1. **Growth results.** Here we present some technical results concerning the growth of certain quantities that we need in order to prove Theorem 2. Observe that $p' = \frac{1}{p-1}$ is the expected growth rate of $|\nabla u|$, and indeed it is the growth rate as we see below.

Lemma 2. *Let $u \in P_1(M,p)$, where $|p-2|<\delta$ with δ is as in Theorem 1. Then there are C and r_0 depending on M and the dimension such that*

$$\sup_{B_r} |\nabla u| \leq C r^{p'}, \tag{5.1}$$

for $r < r_0$.

Proof. For any function f, let

$$S_r(f) = \sup_{B_r} |f|.$$

Suppose that the assertion fails. Then there exist $u_j \in P_1(M,p)$ such that

$$S_{2^{-j-1}}(|\nabla u_j(x)|) > \max(j2^{-jp'}, 2^{-p'} S_{2^{-j}}(|\nabla u_j|)). \tag{5.2}$$

Let

$$v_j(x) = \frac{u_j(2^{-j}x)}{2^{-j} S_{2^{-j-1}}(|\nabla u_j|)},$$

where $x \in B_1$. Then by Theorem 1 and (5.2)

$$|v_j| \leq \frac{(2^{-j})^{\frac{p}{p-1}}}{2^{-j} S_{2^{-j-1}}(|\nabla u_j|)} \leq \frac{1}{j} \quad \text{in } B_1.$$

Moreover

$$S_{1/2}(|\nabla v_j|) = S_{1/2}\left(\frac{|\nabla u(2^{-j}x)|}{S_{2^{-j-1}}(|\nabla u_j|)}\right) = 1, \quad \text{and}$$

$$S_1(|\nabla v_j|) = S_1\left(\frac{|\nabla u(2^{-j}x)|}{S_{2^{-j-1}}(|\nabla u_j)|}\right) = \frac{S_{2^{-j}}(|\nabla u_j)|)}{S_{2^{-j-1}}(|\nabla u_j|)} \leq 2^{p'}.$$

Since we also have

$$||\Delta_p v_j||_\infty \leq j^{1-p},$$

the uniform C^1-estimates for p-Laplace type equations (see Theorem 1 in [Tol84]) implies that

$$1 = S_{1/2}(|\nabla v_j|) \leq C(S_1(v_j) + j^{1-p}) \leq C(j^{-1} + j^{1-p}),$$

which gives a contradiction for large j. □

In the following Lemma we will show that the quantity

$$\frac{1}{|B_r|}\int_{B_r} (|\nabla u|^{p-2}|D^2 u|)^2 \, \mathrm{d}x,$$

is uniformly bounded and therefore the integrand is locally in L^∞.

Lemma 3. *Assume that the hypotheses of Theorem 2 hold. Then there are C and r_0 depending on M, the dimension and λ_i such that*

$$\frac{1}{|B_r|}\int_{B_r} (|\nabla u|^{p-2}|D^2 u|)^2 \, \mathrm{d}x \leq C,$$

for $r < r_0$.

Proof. Let

$$S(r,u) = \int_{B_1} [|\nabla u(rx)|^{p-2}|D^2 u(rx)|]^2 \, \mathrm{d}x.$$

Then it suffices to show that there exists a constant C_0 such that for all non-negative integers k, the following holds

$$S(2^{-k-1}, u) \leq \max(C_0, S(2^{-k}, u)). \tag{5.3}$$

We argue by contradiction. Let $u_k \in P_1(M,p)$ and assume there exist positive integers j_k such that (5.3) fails, that is

$$S(2^{-j_k-1}, u_k) \geq \max(k, S(2^{-j_k}, u_k)).$$

Define

$$v_k(x) = \frac{u_k(2^{-j_k}x)}{S(2^{-j_k-1}, u_k)},$$

where $x \in B_1$. It easily follows that

$$||v_k||_{L^\infty(B_1)} \leq \tfrac{M}{k}, \quad S(\tfrac{1}{2}, v_k) = 1, \quad S(1, v_k) \leq 1, \quad ||\Delta_p v_k|| \leq k^{1-p}.$$

Since $p > 2$, the uniform C^1-estimates and L^2-bounds of the second derivatives from Proposition 1 and Theorem 1 in [Tol84] imply

$$\sup_{B_{1/2}} |\nabla v_k| \leq C(k^{-1} + k^{1-p}),$$

and

$$\int_{B_1} |D^2 v_k(x)|^2 \, \mathrm{d}x < C < \infty,$$

which for k large enough and $p > 2$ gives the following contradiction

$$1 = S(\frac{1}{2}, v_k) < C(k^{-1} + k^{1-p})^{p-2} < 1.$$

□

5.2. **Estimates of the measure of level sets.** Now we perform some integration by parts with some good choice of test functions in order to get the estimates needed for Theorem 2. Let

$$\beta(u) = \lambda_1 \chi_{\Omega^+(u)} - \lambda_2 \chi_{\Omega^-(u)},$$

and

$$\psi_\varepsilon(t) = \begin{cases} 1 & \text{for } t > \varepsilon^{p'}, \\ \frac{|t|^{p-1} \operatorname{sgn} t}{\varepsilon} & \text{for } |t| \leq \varepsilon^{p'}, \\ -1 & \text{for } t < -\varepsilon^{p'}, \end{cases}$$

where again $p' = \frac{1}{p-1}$.

Lemma 4. *Assume that the hypotheses of Theorem 2 hold and take r_0 as in Lemma 2 and Lemma 3. Then we have for any nonnegative $\eta \in C_0^\infty(B_r)$*

$$\int_{B_r} |\nabla \beta(u)| \eta \, \mathrm{d}x = \int_{B_r} |\nabla \Delta_p u| \eta \, \mathrm{d}x \leq C \int_{B_r} |\nabla \eta| \, \mathrm{d}x \tag{5.4}$$

for $r < r_0$. Here C depends on M, the dimension and λ_i and the inequality is to be understood in the BV-sense.

Proof. From now on we use the notation $\partial_i = \frac{\partial}{\partial x_i}$. Take u_δ to be the solution of

$$\Delta_p u_\delta = \beta_\delta(u)$$

with $u_\delta = u$ on ∂B_r and where β_δ is a non-decreasing smooth approximation of β. Then we have

$$\partial_i \Delta_p u_\delta = \beta_\delta'(u_\delta) \partial_i u_\delta.$$

Multiplying by $\psi_\varepsilon(\partial_i u_\delta)\eta$ and integrating we obtain

$$\int_{B_r} \beta_\delta'(u_\delta) \partial_i u_\delta \psi_\varepsilon(\partial_i u_\delta) \eta \, \mathrm{d}x = -\int_{B_r} \partial_i (|\nabla u_\delta|^{p-2} \nabla u_\delta)(\nabla \eta \psi_\varepsilon(\partial_i u_\delta) + \nabla \psi_\varepsilon(\partial_i u_\delta)\eta) \, \mathrm{d}x = A + B,$$

where

$$A = -\int_{B_r} \psi_\varepsilon(\partial_i u_\delta) \nabla \eta (|\nabla u_\delta|^{p-2} \nabla \partial_i u_\delta + (p-2)|\nabla u_\delta|^{p-4} (\nabla \partial_i u_\delta \cdot \nabla u_\delta) \nabla u_\delta) \, \mathrm{d}x,$$

and

$$B = -\int_{B_r} \eta \psi_\varepsilon'(\partial_i u_\delta) |\nabla u_\delta|^{p-4} (|\nabla u_\delta|^2 |\nabla \partial_i u_\delta|^2 + (p-2)(\nabla \partial_i u_\delta \cdot \nabla u_\delta)^2) \, \mathrm{d}x \leq 0.$$

Using this, that β_δ is non-decreasing and that $t\psi_\varepsilon(t)$ nonnegative we have

$$\big|\int_{B_r} \beta_\delta'(u_\delta)\partial_i u_\delta \psi_\varepsilon(\partial_i u_\delta)\eta \,\mathrm{d}x\big| \leq |A|$$
$$\leq (p-1)\int_{B_r} |\nabla\eta||\nabla u_\delta|^{p-2}|D^2 u_\delta|\,\mathrm{d}x \leq C\int_{B_r}|\nabla\eta| \;\mathrm{d}x,$$

where in the last step we used Lemma 3. Now we observe that ψ_ε converges to the sign function when $\varepsilon \to 0$ and that for a subsequence $u_\delta \to u$ weakly in $W^{2,2}_{\mathrm{loc}}(B_r)$ and strongly in $C^{1,\alpha}_{\mathrm{loc}}(B_r)$ (by Proposition 1 and Theorem 1 in [Tol84]). Due to the lower semicontinuity of the BV-norm we now obtain (5.4) by first letting ε tend to zero and then letting δ tend to zero. □

Lemma 5. *Assume that the hypotheses of Theorem 2 hold and take r_0 as in Lemma 4. Then we have for any $\eta \in C_0^\infty(B_r)$*

$$\int_{B_r} \eta\psi_\varepsilon'(\partial_i u)(|\nabla u|^{p-2}\|\nabla\partial_i u|^2 + (p-2)|\nabla u|^{p-4}(\nabla\partial_i u\cdot\nabla u)^2)\,\mathrm{d}x$$
$$\leq C\int_{B_r}|\nabla\eta|\,\mathrm{d}x$$

for $r < r_0$. Here C depends on M, the dimension and λ_i.

Proof. Taking $\psi_\varepsilon(\partial_i u)\eta$ as a test function we have

$$\int_{B_r}(\partial_i\Delta_p u)\psi_\varepsilon(\partial_i u)\eta\,\mathrm{d}x = -\int_{B_r}\partial_i(|\nabla u|^{p-2}\nabla u)\nabla(\psi_\varepsilon(\partial_i u)\eta)\,\mathrm{d}x$$
$$= -\int_{B_r}(\nabla u|^{p-2}\nabla\partial_i u + (p-2)|\nabla u|^{p-4}(\nabla\partial_i u\cdot\nabla u\nabla u))\nabla\eta\psi_\varepsilon(\partial_i u)\,\mathrm{d}x$$
$$-\int_{B_r}(|\nabla u|^{p-2}\nabla\partial_i u + (p-2)|\nabla u|^{p-4}(\nabla\partial_i u\cdot\nabla u\nabla u))\psi_\varepsilon'(\partial_i u)\nabla\partial_i u\eta\,\mathrm{d}x.$$

Therefore

$$\int_{B_r}\eta\psi_\varepsilon'(\partial_i u)(|\nabla u|^{p-2}\|\nabla\partial_i u|^2 + (p-2)|\nabla u|^{p-4}(\nabla\partial_i u\cdot\nabla u)^2)\,\mathrm{d}x$$
$$= -\int_{B_r}(\partial_i\Delta_p u)\psi_\varepsilon(\partial_i u)\eta\,\mathrm{d}x$$
$$-\int_{B_r}(\nabla u|^{p-2}\nabla\partial_i u + (p-2)|\nabla u|^{p-4}(\nabla\partial_i u\cdot\nabla u\nabla u)\nabla\eta\psi_\varepsilon(\partial_i u)\,\mathrm{d}x$$
$$= A + B.$$

We can estimate A using (5.4)

$$|A| \leq C\int_{B_r}|\nabla\eta|\,\mathrm{d}x.$$

For B we obtain

$$|B| \leq (p-1)\int_{B_r}|\nabla\eta||\nabla u|^{p-2}|D^2 u|\,\mathrm{d}x \leq C\int_{B_r}|\nabla\eta|\,\mathrm{d}x,$$

where we used Lemma 3. □

Lemma 6. *Assume that the hypotheses of Theorem 2 hold and take r_0 as in Lemma 5. Then we have for any $r < r_0$*

$$\frac{1}{\varepsilon}|B_r\cap\{0<|\nabla u|<\varepsilon^{p'}\}| \leq C.$$

Here C depends on M, the dimension and λ_i.

Proof. Take $r < \rho < r_0$. Then we have for non-negative $\eta \in C_0^\infty(B_r)$ such that $\eta = 1$ on $B_{r/2}$

$$\frac{1}{\varepsilon}|B_{r/2} \cap \{0 < |\nabla u| < \varepsilon^{p'}\}| \leq C\frac{1}{\varepsilon}\sum_i \int_{B_{r/2}\cap\{0<|\partial_i u|<\varepsilon^{p'}\}} (\Delta_p u)^2 \,\mathrm{d}x$$

$$\leq C\frac{1}{\varepsilon}\int_{B_r\cap\{0<|\partial_i u|<\varepsilon^{p'}\}} \eta(|\nabla u|^{p-2}|\Delta u| + (p-2)|\nabla u|^{p-4}|u_i u_j u_{ij}|)^2 \,\mathrm{d}x$$

$$\leq C\int_{B_r\cap\{0<|\partial_i u|<\varepsilon^{p'}\}} \eta\psi'_\varepsilon(\partial_i u)|\nabla u|^{p-2}|\nabla\partial_i u|^2 \,\mathrm{d}x \leq C\int_{B_r} |\nabla\eta| \,\mathrm{d}x.$$

Here we have used that $\psi'_\varepsilon(t) = \frac{|t|^{p-2}}{\varepsilon}\chi_{\{|t|<\varepsilon^{p'}\}}$ and Lemma 5.

□

5.3. **The estimate of $\mathcal{H}^{n-1}(\Gamma(u) \cap B_{1/2})$.**

Proof of Theorem 2. By covering arguments and Remark 1 it is sufficient to prove the theorem for $\Gamma(u) \cap B_{r_0}$ where r_0 are as in Lemma 6.

We observe that (5.4) implies that $\Gamma^\pm(u)$ is of locally finite perimeter. Moreover, we know that the set $\Gamma^\pm(u) \cap \{|\nabla u| > 0\}$ is locally a C^1-manifold so therefore we get

$$\mathcal{H}^{n-1}(\Gamma^\pm(u) \cap \{|\nabla u| > 0\} \cap B_{1/2}) \leq C.$$

Thus we need now to focus on $\Gamma(u)$, the part of the free boundary where the gradient vanishes.

Because of Proposition 3 and Lemma 2 there is a c depending only on the dimension and λ_i such that for all $\varepsilon > 0$ we have

$$B_{c\varepsilon} \subset B_\varepsilon \cap \{0 < |\nabla u| < \varepsilon^{p'}\}. \tag{5.5}$$

Then

$$\mathcal{H}^{n-1}(\Gamma(u) \cap B_{r_0}) \leq C \liminf_{\varepsilon\to 0} \sum_i \varepsilon^{n-1}.$$

Take a finite cover B^i of $B_{r_0} \cap \Gamma(u)$, being balls of radii ε centered at $\Gamma(u)$ with at most N overlaps. Using (5.5) and Lemma 6 we obtain for ε small enough

$$\frac{1}{\varepsilon}\sum_i \varepsilon^n \leq \frac{1}{\varepsilon}C\sum_i |B^i \cap \{0 < |\nabla u| < \varepsilon^{p'}\}| \leq \frac{1}{\varepsilon}CN|B_{r_0} \cap \{0 < |\nabla u| < \varepsilon^{p'}\}| \leq C.$$

Therefore,

$$\mathcal{H}^{n-1}(\Gamma(u) \cap B_{r_0}) \leq C,$$

hence, the proof of the theorem is complete. □

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