THE N-MEMBRANES PROBLEM

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ABSTRACT. In this paper we study the N-membranes problem for the laplacian. We prove the optimal growth of the consecutive differences $u_i - u_{i+1}$ and that the free boundaries $\partial\{u_i > u_{i+1}\}$ have zero Lebesgue measure, under some assumptions on the functions f_i that appear in the right hand side.

1. INTRODUCTION

1.1. **Problem.** Given N functions $g_i \in H^1(B_1) \cap L^\infty(B_1)$ we study the N-membranes problem in B_1, i.e. the problem of minimizing the functional

$$\sum_i \int_{B_1} |\nabla u_i|^2 + f_i u_i \, \mathrm{d}x,$$

over the admissible set

$$\left\{ u_i - g_i \in H_0^1(B_1), u_1 \geq u_2 \geq \ldots \geq u_N \right\}.$$

1.2. **Known results.** The existence and uniqueness of the solution of the N-membranes problem has been studied before. The 2-membranes problem was first studied by Vergara Caffarelli in [VC71] for the uniformly elliptic linear case. In Vergara Caffarelli [VC73] and [VC74] the case of the mean curvature equation was studied. Later, in [CVC85], Chipot and Vergara Caffarelli studied the case of N-membranes. There they prove the so far best regularity result known for the N-membranes problem when $N > 3$, the Lewy-Stampacchia type inequalities

$$\min_{j \leq i} f_j \leq \Delta u_i \leq \max_{j \geq i} f_j$$

together with the $C^{1,\alpha} \cap W^{2,p}$-regularity for all $\alpha < 1$ and all $p < \infty$. Moreover, in [CCVC05] Carillo, Chipot and Vergara Caffarelli studied the N-membranes problem when a nonlocality appears both in the coefficients of the operator and in the constraints. In the case of two membranes and for a very general class of nonlinear operators, Silvestre proved in [Sil05] the $C^{1,1}$-regularity for the pair of functions solving the problem and also the full regularity of the free boundary under a certain thickness assumption on the coincidence set. Finally in [ARS05b], Azevedo, Rodrigues and Santos studied the regularity of the solution of the variational inequality for the problem of N-membranes in equilibrium with a degenerate operator of p-Laplacian type, $1 < p < \infty$, for which they obtained the corresponding Lewy-Stampacchia inequalities. They studied the stability of the coincidence sets, by considering the problem as a system coupled through the characteristic functions

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of the sets where at least two membranes are in contact. They also obtain that the functions u_i satisfy the equations (in the case of the Laplace operator):

$$\Delta u_i = f_i + \sum_{1\leq j<k\leq N, j\leq i\leq k} b_i^{j,k}\chi_{j,k}, \tag{1.1}$$

where $b_i^{j,k}$ is a certain linear combination of the f_i:s (see Definition 4 in the Appendix) and

$$\chi_{j,k} = \{u_j = u_{j+1} = \cdots = u_k\}.$$

1.3. **Main result.** In order to formulate the the main result, we need to define our local class of solutions.

Definition 1. *We say that the N functions u_i belong to the class $P_r(M)$ if*

(1) *u_i solves the N-membranes problem in B_r,*
(2) $\sup_{B_r} |u_i| \leq M$,
(3) $\sup_{B_r} |f_i| \leq M$.

In what follows we will always denote by w_i the consecutive differences $u_i - u_{i+1}$. The main result of this paper is Theorem 1, where we prove that the differences w_i has quadratic growth from the free boundaries, i.e.

$$\sup_{B_r(x_0)} w_i \leq C_i r^2 ,$$

for $x_0 \in \partial\{w_i > 0\}$. This can be done with just the assumption that $f_i \in L^\infty$. Then under some more assumptions on the f_i:s we then prove non-degeneracy in Proposition 2, i.e.

$$\sup_{B_r(x_0)} w_i \geq \lambda_i r^2 ,$$

for some $0 < \lambda_i < C_i$.

Using this, we are able to prove Corollary 1, i.e. that the free boundaries $\partial\{w_i > 0\}$ is locally porous, and in particular that it has zero Lebesgue measure.

2. Quadratic growth of differences

In this section, we prove that the differences w_i have quadratic growth from the free boundary. This is done with a blow-up method.

Theorem 1. *Let $u_i \in P_1(M)$ and $x_0 \in \partial\{w_i > 0\}$. Then there is a C such that*

$$\sup_{B_r(x_0)} w_i \leq Cr^2,$$

for r small enough. Here $C = C(M, n)$.

Proof. We actually prove instead that either we have

$$\sup_{B_r} w_i \leq Cr^2 ,$$

or there is a k such that

$$\sup_{B_r} w_i \leq 4^{-k} \sup_{B_{2^k r}} w_i .$$

We argue by contradiction. If this is not true then there is a sequence of radii r_j and functions w_i^j such that

$$S_j = \sup_{B_{r_j}} w_i^j \geq Cjr_j^2 ,$$

and

$$\sup_{B_{r_j}} w_j \geq 4^{-k} \sup_{B_{2^k r_j}} w_j ,$$

for all k. Let

$$h_j(x) = \frac{w_i^j(r_j x + x_0)}{S_j}\,.$$

Then h_j fulfills:

(1) $h_j \geq 0$,
(2) $h_j(0) = 0$,
(3) $\sup_{B_{2^k r_j}} h_j \leq 4^k$,
(4) $\sup_{B_1} h_j = 1$, and
(5) $|\Delta h_j| \leq \frac{1}{Cj}|\Delta w_i^j(r_j x + x_0)| \leq \frac{C'(M)}{Cj}$.

Here, the last inequality in (5) follows from equation (1.1). By (1)-(5) the sequence h_j is uniformly bounded in $C^{1,\alpha}(B_{1/r_j})$ and therefore there is a subsequence again named h_j such that $h_j \to h$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ with h such that

(1) $h \geq 0$,
(2) $h(0) = 0$,
(3) $\sup_{B_{2^k}} h \leq 4^k$,
(4) $\sup_{B_1} h = 1$, and
(5) $\Delta h = 0$.

This contradicts the strong maximum principle for harmonic functions, and thus we have reach a contradiction. □

In a similar manner, we can prove the following growth result on the gradient:

Proposition 1. *Under the same assumptions as in Theorem 1 we have*

$$\sup_{B_r(x_0)} |\nabla w_i| \leq Cr\,.$$

for some C. Here again $C = C(n, M)$.

Proof. We argue by contradiction and prove that either

$$\sup_{B_r} |\nabla w_i| \leq Cr\,,$$

or there is a k such that

$$\sup_{B_r} |\nabla w_i| \leq 2^{-k} \sup_{B_{2^k}} |\nabla w_i|\,.$$

If this is not true then we can find a sequence of radii r_j and functions w_i^j such that

$$S_j = \sup_{B_{r_j}} |\nabla w_i^j| \geq Cjr_j\,,$$

and

$$\sup_{B_{r_j}} |\nabla w_i^j| \geq 2^{-k} \sup_{B_{2^k r_j}} |\nabla w_i^j|\,,$$

for all k. Now let

$$h_j(x) = \frac{w_i^j(r_j x + x_0)}{r_j S_j}\,.$$

Then h_j fullfills:

(1) $h_j \geq 0$,
(2) $|\nabla h_j|(0) = 0$,
(3) $\sup_{B_2} h_j \leq Cj^{-1}$,
(4) $\sup_{B_{2^k}} |\nabla h_j| \leq 2^k$,
(5) $\sup_{B_1} |\nabla h_j| = 1$, and
(6) $|\Delta h_j| \leq \frac{1}{Cj}|\Delta w_i^j(r_j x + x_0)|$.

Here the estimate (3) follows from applying Theorem 1 to the sequence w_i^j, since we know that $u_i^j \in P_1(M)$. Now from the C^1-estimates for Laplace equation we get

$$\sup_{B_1} |\nabla h_j| \le C(\sup_{B_2} h_j + \sup_{B_2} \Delta h_j) \le C' j^{-1} ,$$

which contradicts (5) for j large. □

Remark 1. *In the standard obstacle problem, and also in the 2-membrane problem the assumption that the f_i are Dini continuous together with the quadratic growth is enough to deduce $C^{1,1}$-estimates. However, in the case with three or more membranes this is not immediate, the main reason being that this does not ensure that Δw_i is Dini continuous (or not even continuous) in the set $\{w_i > 0\}$.*

3. Local regularity of the free boundary $\partial\{u_i > u_{i+1}\} \setminus \cup_{k\neq i}\{u_k = u_{k+1}\}$

A simple observation is that if we take $x_0 \in \partial\{u_i > u_{i+1}\} \setminus \cup_{k\neq i}\{u_k = u_{k+1}\}$ then there is a ball $B_r(x_0)$ such that with $w_i = u_i - u_{i+1}$ satisfies

$$\begin{aligned} &w_i \ge 0 && \text{in } B_r(x_0) \\ &\Delta w_i = f_i - f_{i+1} = g_i && \text{in } B_r(x_0) \cap \{w > 0\} \end{aligned}$$

So under the assumption that $g_i = f_i - f_{i+1} > 0$ at x_0 we have the standard obstacle problem and thus locally in B_r the free boundary $\partial\{w > 0\}$ is real analytic under a suitable thickness condition on the coincidence set $\{w_i = 0\}$, by the classical theory. See for instance [Caf98].

4. The free boundary has zero Lebesgue measure

In the appendix we show that under the assumption that $(f_i - f_{i+1}) > C > 0$ then $\Delta(u_i - u_{i+1}) > C' > 0$ where $C' = C'(n, C)$. Therefore we can prove non-degeneracy, like in the usual obstacle-type problems.

Proposition 2. *Let $u_i \in P_1(M)$ and $x_0 \in \partial\{w_i > 0\}$. Moreover, assume that $\inf f_i - f_{i+1} > C > 0$. Then there is a constant $\lambda = \lambda(C, n)$ such that*

$$\sup_{B_r(x_0)} w_i \ge \lambda r^2 .$$

Proof. From Lemma 2 in the appendix we know that $\Delta w_i \ge C'$ in $\{w_i > 0\}$. Take $y \in \{w_i > 0\}$ and r_0 such that $B_{r_0}(y) \subset \{w_i > 0\}$ let

$$v(x) = w_i(x) - \frac{C'}{2n}|x - y|^2 .$$

Then $\Delta v \ge 0$. Moreover $v(y) > 0$. Hence There is a $x_y \in \partial(B_{r_0}(y) \cap \{w_i > 0\})$ such that $v(x_y) > 0$. Now on $\partial\{w_i > 0\}$ $v \le 0$ so $x_y \in \partial B_{r_0}(y)$. Then let $y \to y_0 \in \partial\{w_i > 0\}$ then $x_y \to x \in \partial\{w_i > 0\}$. This implies the desired result. □

This together with the quadratic growth of w_i implies in that the free boundary $\partial\{w_i > 0\}$ is porous and in particular that it has zero Lebesgue measure.

Definition 2. Let $A \subset \mathbb{R}^n$, and define

$$\begin{aligned} \gamma(x, R, A) &= \sup\{r : B(z, r) \subset B(x, r) \setminus A \text{ for some } z \in \mathbb{R}^n\} . \\ p(x, A) &= \limsup_{R\to 0} \gamma(x, R, A)/R , \end{aligned}$$

Then A is said to be porous if $p(x, A) > 0$ for all $x \in A$.

Theorem 2. *Let $u_i \in P_1(M)$. Then the free boundary, $\partial\{w_i > 0\}$ is locally porous, i.e. there is a neighborhood U such that $U \cap \partial\{w_i > 0\}$ is porous.*

Proof. Take x_0 in $\partial\{w_i > 0\}$. Then by Proposition 2 there is a $z \in \partial B_r(x_0)$ such that $w_i(z) \geq \lambda r^2$ for r small enough. Now take $y \in B_{\delta r}(z)$. Then we have

$$w_i(y) \geq \lambda r^2 - \sup_{B_{\delta r}(z)} |\nabla w_i| \delta r \geq r^2(\lambda - C\delta(1+\delta)),$$

by Proposition 1. Now if we take δ small enough we will have that $B_{\delta r}(z) \subset B_{2r}(x_0)$. Hence we have that

$$\gamma(x, 2R) \geq \delta r,$$

when r is small enough which implies that $\partial\{w_i > 0\}$. □

Corollary 1. *Under the same assumptions as in Theorem 2 the free boundary has zero Lebesgue measure.*

Proof. Any porous set has Lebesgue density strictly less than 1 at any point, and thus it must have zero measure by the Lebesgue density theorem. □

Remark 2. *With classical methods it will be hard to prove the regularity of the whole free boundary* $\partial\{w_i > 0\}$ *even if we assume that* $f_i \in C^\infty$*, again by the same reasons mentioned in Remark 1.*

5. Appendix

Here we prove the combinatorial result needed in the proof of non-degeneracy. First we need to introduce some notations on linear combinations of f_i, and also some results.

Definition 3.

$$\langle f \rangle_{j,k} = \frac{1}{k-j+1} \sum_{i=j}^{k} f_i$$

Definition 4.

$$b_i^{j,k} = \begin{cases} \langle f \rangle_{j,k} - \langle f \rangle_{j,k-1} & \text{if } i = j, \\ \langle f \rangle_{j,k} - \langle f \rangle_{j+1,k} & \text{if } i = k, \\ \frac{2}{(k-j)(k-j+1)} \left(\langle f \rangle_{j+1,k+1} - \frac{1}{2}(f_j + f_k) \right) & \text{if } j < i < k \end{cases}$$

For these coefficients $b_i^{j,k}$ one can prove the following (see [ARS05b])

Lemma 1. *If* $j \leq l < r$ *then*

$$\sum_{k=l+1}^{r} b_j^{j,k} = \frac{r-l}{r-j+1} (\langle f \rangle_{l+1,r} - \langle f \rangle_{j,l}).$$

Lemma 2. *Assume as before* $\inf(f_i - f_{i+1}) > C > 0$*. Then with* $w_i = u_i - u_{i+1}$ *we have*

$$\Delta w_i > C' > 0$$

in the set $\{w_i > 0\}$*.*

Proof. In the paper [ARS05b] the equation for the N-membrane problem is found. It reads

$$\Delta u_i = f_i + \sum_{1 \leq j < k \leq N, j \leq i \leq k} b_i^{j,k} \chi_{j,k}.$$

Taking the difference between the equation for u_i with the one for u_{i+1} we get

$$\Delta w_i = f_i - f_{i+1} + \sum_{1 \leq j < k \leq N, j \leq i \leq k} b_i^{j,k} \chi_{j,k} - \sum_{1 \leq j < k \leq N, j \leq i+1 \leq k} b_i^{j,k} \chi_{j,k}, \tag{5.1}$$

which we can rewrite as

$$\Delta w_i = f_i - f_{i+1} + \sum_{j<k, j \leq i, k \geq i+1} (b_i^{j,k} - b_{i+1}^{j,k}) \chi_{j,k} + \sum_{j<k, k=i} b_i^{j,k} \chi_{j,k} - \sum_{j<k, j=i+1} b_{i+1}^{j,k} \chi_{j,k}.$$

We wish to study the right hand side in the set $\{w_i > 0\}$. In that set we observe that $\chi_{j,k} = 0$ if

(1) $j = i$ or
(2) $k = i + 1$ or
(3) $j < i$ and $k > i + 1$.

Thus, the first sum vanishes in the set $\{w_i > 0\}$ because of the properties just mentioned. Therefore we have

$$\Delta w_i = f_i - f_{i+1} + \sum_{j<k,k=i} b_i^{j,k}\chi_{j,k} - \sum_{j<k,j=i+1} b_{i+1}^{j,k}\chi_{j,k} = A + B + C\,,$$

in $\{w_i > 0\}$. That A is positive is trivial, so we now on focus on B and C. For B we use that

$$b_i^{j,i} = \langle f\rangle_{j,i} - \langle f\rangle_{j+1,i} > c > 0\,,$$

and hence $B > C' > 0$.

For C, assume that we are at a point where $\chi_{i+1,k} = 1$ for $k < M$ and 0 for $k \geq M$. Then we use Lemma 1 and get at this point

$$C = -\sum_{k=i+2}^{M} b_{i+1}^{i+1,k} = \frac{M-i-1}{M-i}(\langle f\rangle_{i+1,i+1} - \langle f\rangle_{i+2,M})\,.$$

Now we notice that

$$\langle f\rangle_{i+1,i+1} - \langle f\rangle_{i+2,M} = f_{i+1} - \frac{f_{i+2} + \cdots + f_M}{M-i-1} \geq 0.$$

Thus $C \geq 0$, and the result follows. □

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