REGULARITY OF A PARABOLIC FREE BOUNDARY PROBLEM WITH HÖLDER CONTINUOUS COEFFICIENTS

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Abstract. We consider the parabolic obstacle type problem

$$\begin{cases} Hu = f\chi_\Omega \text{ in } Q_1^-, \\ u = |\nabla u| = 0 \text{ on } Q_1^- \setminus \Omega, \end{cases}$$

where Ω is an unknown open subset of Q_1^-. This problem has its origin in parabolic potential theory. When f is Hölder continuous we can, under a combination of energetic and geometric assumptions, prove the optimal $C_x^{1,1} \cap C_t^{0,1}$ regularity of the solution.

1. Introduction and main result

1.1. **Problem.** Let u solve

$$\begin{cases} Hu = f\chi_\Omega & \text{in} \quad Q_1^-, \\ u = |\nabla u| = 0 & \text{in} \quad Q_1^- \setminus \Omega. \end{cases} \tag{1.1}$$

Here $Q_R^- = B_R \times (-R^2, 0]$, H is the heat operator, $\Lambda = \{u = |\nabla u| = 0\}$ and $\Omega = Q_1^- \setminus \Lambda$. For the function f we assume Hölder continuity in both space and time.

This problem arises in parabolic potential theory as follows. Let $g(x,t)$ be a function with support in $\Omega \subset Q_1^-$, and

$$G(x,s) = \frac{1}{(4\pi s)^{n/2}} e^{\frac{-|x|^2}{4s}}$$

be the heat kernel.

The heat potential can be defined as

$$U[g](x,t) = g(x,t) * G(x,-t),$$

where the convolution is taken over $\mathbb{R}^n \times \mathbb{R}^+$.

Since G is the heat kernel we have that

$$HU[g](x,t) = c_n g(x,t) = f\chi_\Omega \text{ in } \mathbb{R}_n \times \mathbb{R}^+,$$

where c_n is a negative constant depending only on the dimension and f is a nonzero function. If there is a function v that fulfills

$$\begin{aligned} Hv = 0 \quad & \text{in} \quad Q_1^-, \\ v = U[g] \quad & \text{in} \quad Q_1^- \setminus \Omega, \end{aligned}$$

Date: June 17, 2010.

This research project is a part of the ESF program Global. Moreover, E. Lindgren has partially been supported by grant KAW 2005.0098 from the Knut and Alice Wallenberg foundation, the ANR project "MICA", grant ANR-08-BLAN-0082 and by the Chair "Mathematical modelling and numerical simulation, F-EADS - Ecole Polytechnique - INRIA" during this work.

then we refer to v as the caloric continuation of $U[g]$. When the boundary of a domain is analytic the existence of such a caloric continuation is locally guaranteed by Cauchy-Kowalevskaya theorem.

Let $u = U[g] - v$, then u will satisfy

$$\begin{array}{lll} Hu = f\chi_\Omega & \text{in} & Q_1^- \\ u = |\nabla u| = 0 & \text{in} & Q_1^- \setminus \Omega, \end{array} .$$

which explains why we study (1.1).

1.2. **Notation.** Throughout the paper we will use the following notation:
$B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0|^2 < r\}$ the open spatial ball
$Q_r^-(x_0, t_0) = B_r(x_0) \times (-r^2 + t_0, 0 + t_0]$ the lower half cylinder
$S_r = \mathbb{R}^n \times (-4r^2, 0]$ an infinite strip
$T_r = \mathbb{R}^n \times (-4r^2, -r^2]$ another infinite strip
$G(x, s) = \frac{1}{(4\pi s)^{n/2}} e^{\frac{-|x|^2}{4s}}$ the heat kernel
$\mathrm{d}\gamma(x, s) = G(x, -s)\,\mathrm{d}x\,\mathrm{d}t$
$\mathrm{d}\gamma^s(x) = G(x, -s)\,\mathrm{d}x$
$\|f\|_{L^p_\mu(D)} = \left(\int_D |f(y)|^p \,\mathrm{d}\mu\right)^{1/p}$ Lebesgue norm for a measure μ
$\|f\|_{H_\mu(D)} = \left(\int_D |f(y)|^2 + |\nabla f(y)|^2 \,\mathrm{d}\mu\right)^{1/2}$ Sobolev norm for a measure μ
$\|f\|_{N(D_1;M(D_2))} = \|g(t)\|_{M(D_2)}$ where $g(t) = \|f(\cdot, t)\|_{N(D_1)}$
$\|f\|_{W^{2,1}_p(Q_r^-)} = \|f\|_{L^p(Q_r^-)} + \|\nabla f\|_{L^p(Q_r^-)} + \|D^2 f\|_{L^p(Q_r^-)} + \|f_t\|_{L^p(Q_r^-)}$
$u_r(x, t) = \frac{u(rx, r^2 t)}{r^2}$ the parabolic rescaling

1.3. **Known results.** In [CPS04] the same parabolic problem was studied but with the restriction $f = 1$. For this case the local $C^{1,1}_x \cap C^{0,1}_t$ regularity of the solution and the analyticity of the free boundary under some geometric assumption was proved.

The corresponding elliptic problem has been studied earlier. In [CKS00] the $C^{1,1}$ regularity of the solution was proved for constant f. Under a geometric condition on Ω near the origin it was also proved that the free boundary is analytic in a neighborhood of the origin.

A more general elliptic problem was studied in [PS07a]. The authors made the following double Dini assumption on the modulus of continuity of f:

$$\int_{0^+} \frac{\omega(r) \log \frac{1}{r}}{r} \,\mathrm{d}r < \infty$$

Under a combination of energetic and geometric assumptions they were able to prove $C^{1,1}$ regularity of the solution and C^1 regularity of the free boundary. In our case, due to certain technical issues for the heat equation we have to make the condition on f slightly stronger.

The parabolic obstacle problem, i.e. the case $u \geq 0$, which has an important application in valuation of stock options of American type, has been studied before. In [PS07b], it was proved that the local regularity of the solution is $C^{1,1} \cap C^{0,1}$. The option application was also studied in [LS09]. Under the assumption that the pay-off of an American option on serveral assets is convex, it was proved that the free boundary is the graph of a C^∞ function.

1.4. **Main result.** The main result in this paper is that solutions of (1.1) are, under geometric and energetic assumptions, $C^{1,1}_x \cap C^{0,1}_t$ regular if f is Hölder continuous. The regularity of the free boundary is left as an open problem, the reason being that there is a bunch of results that needs to be proved on the way to the regularity of the free boundary, for instance the results in [Bla01] and [BS03]. This could be an interesting future project.

In order to state our main theorem more precisely we define the class of solution we consider in this paper.

Definition 1. *We say u is a local solution in $P_R(M_1, M_2)$ if*

1. $Hu = f\chi_\Omega$, *in* Q_R^-, *in the sense of distributions,*
2. $u = |\nabla u| = 0$ *in* $Q_R^- \setminus \Omega$,
3. $\sup_{Q_R^-} |u| \le M_1$,
4. *for some* $0 < \beta < 1$ *there holds*
$$\sup_{|x-x_0|\le r, |t-t_0|\le r^2} |f(x,t) - f(x_0,t_0)| \le M_2 r^\beta,$$
for all r *small enough.*
5. $f(0,0) = 1$,
6. $(0,0) \in \Gamma = \partial\Omega \cap Q_R^-$.

We also need the following definition:

Definition 2. *The parabolic thickness function is given by*

$$\delta(r,u) = \frac{\min \operatorname{diam}(\Lambda(u(\cdot,-r^2)) \cap B_r)}{r}.$$

Our main theorem is the following:

Theorem 1. *Assume* $u \in P_4(M_1, M_2)$, $\psi \in C_0^\infty(B_1)$, $\psi = 1$ *in* $B_{\frac{1}{2}}$ *and* $v = u\psi$. *Then for given* $\varepsilon > 0$ *there exists* $r_{\varepsilon,M_1,M_2} > 0$ *such that if for some* $0 < r_0 < r_{\varepsilon,M_1,M_2}$

$$\delta(r_0/2, v) \ge \varepsilon \text{ and } W(r_0; v, f) < 2A_n - \varepsilon, \tag{1.2}$$

then

$$\|v\|_{L^\infty(Q_r^-)} \le Cr^2 \text{ for every } 0 < r \le r_0,$$

and

$$\|v\|_{C_x^{1,1} \cap C_t^{0,1}(Q_{c_0}^-)} \le C, \tag{1.3}$$

for some small $c_0 = c(\varepsilon, M_1, M_2, r_0)$ *and* C *a fixed constant. Here* $A_n = \frac{15}{4}$ *is related to the energy of global solutions, see (Theorem I in* [CPS04]*).*

2. Weiss' monotonicity formula

Define the Weiss energy for $v(x,t)$: $\mathbb{R}^n \times \mathbb{R}^- \to \mathbb{R}$ to be

$$W(r; v, f) = \frac{1}{r^4}\int_{T_r}\left(|\nabla v|^2 + 2fv + \frac{v^2}{t}\right) \mathrm{d}\gamma,$$

and let

$$v_r(x,t) = \frac{v(rx, r^2t)}{r^2},$$

so that $W(r; v, f) = W(1; v_r, f_r)$, where $f_r(x,t) = f(rx, r^2t)$.

The following proposition is a parabolic version of Weiss' monotonicity formula.

Proposition 1. *Let* $u \in P_1(M_1, M_2)$, $\psi \in C_0^\infty(B_{3/4})$ *such that* $\psi = 1$ *on* $B_{1/2}$, *and set* $v = u\psi$. *Then there is a* $W^{1,1}$*-function* $F = F_{M_1,M_2}$ *such that*

$$W(r; v, f) + \int_0^r F(s)\,\mathrm{d}s$$

is a nondecreasing function for $0 < r < 1/2$, *and in particular*

$$W'(r; v, f) \geq -F(r) + \frac{1}{r^4}\int_{T_r} \frac{(Lv)^2}{-t}\,\mathrm{d}\gamma.$$

We make the following definition of a parabolic homogeneous function.

Definition 3. *We say that* h *is parabolic homogeneous of degree* α *if* $h(kx, k^2 t) = k^\alpha h(x, t)$.

Using this definition we state a remark for the Weiss energy for $M_2 = 0$.

Remark 1. *For* $M_2 = 0$ *we have* $f = 1$. *Let* $u \in P_\infty(M_1, 0)$ *then* $W(r; u, 1)$ *is nondecreasing in* r. *Furthermore* $W(r; u, 1)$ *is constant for* $r_0 < r < r_1$ *if and only if* u *is parabolic homogeneous of degree* 2 *in* $Q^-_{r_1} \setminus Q^-_{r_0}$. *This was proved in* [Wei99].

In order to prove the proposition we first establish two technical results. The first result is an estimate for a linear operator applied to a local solution.

Lemma 1. *Let* $u \in P_1(M_1, M_2)$ *and define*

$$Lu = x \cdot \nabla u + 2tu_t - 2u.$$

Then for any $\alpha \in (0, 1)$ *there is a constant* C_α *such that*

$$\frac{1}{|Q_r^-|}\int_{Q_r^-} \frac{Lu}{r^2}\,\mathrm{d}x\,\mathrm{d}t \leq \frac{C_\alpha}{r^\alpha},$$

whenever $r < \frac{1}{2}$.

Proof. As a special case of Theorem 4.8 in [Lie96] we find that $u \in C_x^{1,1-\alpha} \cap C_t^{0,1-\alpha}$ for any $\alpha \in (0, 1)$ with the estimates depending on α, M_1 and M_2.

Define

$$v_r(x, t) = \frac{u(rx, r^2 t)}{r^2}.$$

Then, by the regularity of u

$$\sup_{Q_1^-} |v_r(x, t)| \leq \frac{C}{r^\alpha},$$

when $r < \frac{1}{2}$. Moreover,

$$\sup_{Q_1^-} |Hv_r| \leq C.$$

Therefore, by L^p-estimates (Theorem 7.22 in [Lie96])

$$\|v_r\|_{W_p^{2,1}(Q_1^-)} \leq \frac{C}{r^\alpha}.$$

In particular for $p = 1$ we have

$$\|\nabla v_r\|_{L^1(Q_1^-)} + \|v_r\|_{L^1(Q_1^-)} + \|\partial_t v_r\|_{L^1(Q_1^-)} \leq \frac{C}{r^\alpha}.$$

Scaling back, this gives the desired estimate for Lu. □

The next technical result gives integral estimate over $B_1 \times (-4r^2, -r^2)$ for an integrable function.

Lemma 2. *Suppose* $k \in C^{\beta}(Q_1^-)$ *with* $k(0,0) = 0$, *and* $g \in L^1(Q_1^-)$ *with*

$$\left(\frac{1}{|Q_r^-|}\int_{Q_r^-}\left|\frac{g}{r^2}\right|\,\mathrm{d}x\,\mathrm{d}t\right) \le \frac{C_\alpha}{r^\alpha}.$$

Then

$$\frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{B_1}|kg|\,\mathrm{d}\gamma \le C_0 r^{\beta-\alpha},$$

where $C_0 = C(\alpha, \beta, n)$.

Proof. The proof is split into two parts, estimates inside and outside B_r. We start with the part that comes from integration only over B_r. For $(x, s) \in T_r$ we have

$$|G(x,s)| \le \frac{C}{r^n}.$$

Therefore, we obtain

$$\frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{B_r}|kg|\,\mathrm{d}\gamma \le C\frac{1}{r^{n+2}}\int_{-4r^2}^{0}\int_{B_r}C_\beta r^\beta\frac{|g|}{r^2}\,\mathrm{d}x\,\mathrm{d}t \le C_0 r^{\beta-\alpha}. \tag{2.1}$$

Now we turn our attention to what happens outside B_r. We split $B_1 \setminus B_r$ into dyadic rings of the form $B_{2^{k+1}r}\setminus B_{2^k r}$. We observe that for $(x,s) \in B_{2^{k+1}r}\setminus B_{2^k r} \times (-4r^2, -r^2) = C_r$ we have for some $c > 0$,

$$c\frac{e^{-\frac{1}{c}4^k}}{r^n} \le G(x,s) \le \frac{e^{-c4^k}}{cr^n}.$$

Therefore,

$$\begin{aligned}\frac{1}{r^4}\int_{C_r}|kg|\,\mathrm{d}\gamma &\le \frac{e^{-c4^k}}{cr^{n+4}}\int_{C_r}|kg|\,\mathrm{d}x\,\mathrm{d}t\\ \le \frac{e^{-c4^k}}{cr^{n+2}}(2^{k+1})^2&\int_{Q^-_{2^{k+1}r}}C_\beta r^\beta\frac{|g|}{(2^{k+1}r)^2}\,\mathrm{d}x\,\mathrm{d}t\\ &\le Ce^{-c4^k}r^{\beta-\alpha}2^{(k+1)(n+2)}.\end{aligned}$$

Adding up all the rings we obtain

$$\begin{aligned}\frac{1}{r^4}\int_{B_1\setminus B_r}|kg|\,\mathrm{d}\gamma^s &\le \sum_{k=1}^{2^{k+1}r=1}\frac{1}{r^4}\int_{C_r}|kg|\,\mathrm{d}\gamma\\ &\le r^{\beta-\alpha}\sum_1^\infty Ce^{-c4^k}2^{(k+1)(n+2)} \le Cr^{\beta-\alpha}.\end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2) gives the desired estimate. ☐

We use Lemma 1 and 2 to prove of Proposition 1.

Proof of Proposition 1. Let

$$E(r;v) = \frac{1}{r^4}\int_{T_r}\left(|\nabla v|^2 + 2v + \frac{v^2}{t}\right)\mathrm{d}\gamma.$$

We observe that

$$E(rs;v) = E(s;v_r),$$

with

$$v_r(x,t) = \frac{v(rx, r^2t)}{r^2}.$$

Moreover, it is clear that

$$W(r;v,f)-E(r;v)=\frac{1}{r^4}\int_{T_r}2(f-1)v\,\mathrm{d}\gamma.$$

Define

$$Lv=\frac{\mathrm{d}}{\mathrm{d}r}v_r\Big|_{r=1}=x\cdot\nabla v+2tv_t-2v.$$

Using that $\nabla G(x,-t)=\frac{x}{2t}G(x,-t)$ and $\nabla v\nabla Lv=\nabla\cdot(\nabla vLv)-\Delta vLv$, integration by parts gives

$$\begin{aligned}
E'(r,v)r&=\frac{\mathrm{d}}{\mathrm{d}p}E(pr;v)\Big|_{p=1}\\
&=\frac{1}{r^4}\int_{T_r}\left(2\nabla v\nabla Lv+2Lv+\frac{2vLv}{t}\right)G(x,-t)\,\mathrm{d}x\,\mathrm{d}t\\
&=\frac{1}{r^4}\int_{T_r}\left(2\nabla\cdot\nabla vLv-2\Delta vLv+2Lv+\frac{2vLv}{t}\right)G(x,-t)\,\mathrm{d}x\,\mathrm{d}t\\
&=\frac{1}{r^4}\int_{T_r}Lv\left(-\frac{x}{t}\cdot\nabla v-2Hv-2v_t+2Lv+\frac{2v}{t}\right)G(x,-t)\,\mathrm{d}x\,\mathrm{d}t\\
&=\frac{1}{r^4}\int_{T_r}Lv\left(2-2Hv+\frac{2v-2tv_t-x\cdot\nabla v}{t}\right)G(x,-t)\,\mathrm{d}x\,\mathrm{d}t\\
&=\frac{1}{r^4}\int_{T_r}Lv(2(1-Hv)-\frac{Lv}{t})G(x,-t)\,\mathrm{d}x\,\mathrm{d}t\\
&=\frac{2}{r^4}\int_{T_r}Lv(1-Hv)\,\mathrm{d}\gamma+\frac{1}{r^4}\int_{T_r}\frac{(Lv)^2}{-t}\,\mathrm{d}\gamma. \qquad (2.3)
\end{aligned}$$

These calculations can be justified by taking a smooth approximation of v. We have to prove that the first term is integrable. We split it into two pieces:

$$\begin{aligned}
&\frac{1}{r^4}\int_{T_r}Lv(f(0)-f)\,\mathrm{d}\gamma\\
&+\frac{1}{r^4}\int_{T_r}Lv(f-Hv)\,\mathrm{d}\gamma=A+B
\end{aligned}$$

We observe that Lv vanishes outside B_1. Therefore, by Lemma 1 combined with Lemma 2

$$\int_0^1 A\le C\int_0^1 r^{\beta-\alpha}\le C,$$

if $\alpha<\beta$. The term B can be estimated quite easily since $f-Hv=0$ in $B_{1/2}$. Hence, we have only contributions from $B_{\frac{3}{4}}\setminus B_{\frac{1}{2}}$, where $G(x,-s)\le Cr^{-n}e^{-\frac{1}{64r^2}}$ and clearly $(f-Hv)\in L^\infty(Q_1^-)$. Therefore

$$\begin{aligned}
|B|&=\frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{B_{\frac{3}{4}}\setminus B_{\frac{1}{2}}}|Lv(f-Hv)|\,\mathrm{d}\gamma\\
&\le C\frac{e^{-\frac{1}{64r^2}}}{r^{n+4}}\int_{Q_1^-}|Lv|\,\mathrm{d}x\,\mathrm{d}t\le\frac{C}{r^{n+4}}e^{-\frac{1}{64r^2}}, \qquad (2.4)
\end{aligned}$$

which is integrable. It remains to prove that $W(r;v,f)-E(r;v)\to 0$ as $r\to 0$. Since v has support only in B_1, $v\in C^{1,\alpha}$ and $f\in C^\beta$ we can apply Lemma 2 with $\alpha=0$, and obtain

$$|W(r;v,f)-E(r;v)|=\left|\frac{1}{r^4}\int_{T_r}2(f-1)v\,\mathrm{d}\gamma\right|\le Cr\to 0.$$

□

3. Technical tools

In this section we will prove and state some technical results needed for the proofs later. The first one, given below, is a result due to Simon (see Theorem 6 p. 86 in [Sim87]).

Lemma 3. *(Simon's lemma) Let $X_0 \subset X \subset X_1$ be Banach spaces such that X_0 is compactly embedded in X and X is continuously embedded in X_1. Moreover assume that u_k is a sequence of functions such that for some $q > 1$*

$$\|u_k\|_{L^q(I;X)} + \|u_k\|_{L^1(I;X_0)} + \|\partial_t u_k\|_{L^1(I;X_1)} \leq C,$$

where $I \subset \mathbb{R}$ is a compact interval, then there is a subsequence u_{k_j} that converges in $L^p(I;X_0)$ for all $1 \leq p < q$.

In what follows we will apply this lemma with $X_0 = H^1_{\gamma^{-1}}(\mathbb{R}^n)$, $X = L^2_{\gamma^{-1}}(\mathbb{R}^n)$ and $X_1 = (Y)^*$, the dual of the space Y equipped with the norm

$$\|\phi\|_Y = \left\|\frac{\phi}{G(x,1)}\right\|_{L^\infty(\mathbb{R}^n)} + \|\nabla\phi\|_{L^2_{(\gamma^{-1})^{-1}}},$$

where the subscript γ^{-1} or $(\gamma^{-1})^{-1}$ stands for that we use the weighted space with $\gamma^{-1} = G(x,1)$ or $(\gamma^{-1})^{-1}$ as weight. Then, the fact that the embedding $X \to X_0$ is compact is a special case of Theorem 3.1 in [Hoo81]. The fact that the embedding $X \to X_1$ is continuous is clear. We will also use that we have an embedding of the form

$$\nabla L^2_{\gamma^{-1}}(T_1) + L^1_{\gamma^{-1}}(T_1) \to (Y)^*.$$

The following result is a quite standard energy-type estimate.

Lemma 4. *Let u be a limit of functions that all have compact support. Suppose u solves*

$$Hu = f \text{ in } S_1, \tag{3.1}$$

and that

$$\iint_{T_1} \frac{u^2}{-t} + |\nabla u|^2 \, \mathrm{d}\gamma \leq C,$$
$$\|f\|_{L^\infty(S_1)} \leq C.$$

Then

$$\int_{\mathbb{R}^n} u^2(x,t_0)\, \mathrm{d}\gamma^{t_0} \leq C'$$

for any $t_0 \in (-4,0)$.

Proof. We first assume that u has compact support. Take $\eta(t) \in C_0^\infty(\mathbb{R})$ such that $\eta = 1$ for $t \in (-1,-2)$ and $\eta(-4) = 0$.

Multiplying (3.1) with $u\eta$ and integrating implies for $t_0 \in (-1,-2)$:

$$\int_{-4}^{t_0}\int_{\mathbb{R}^n} (u\eta\Delta u - u_t u\eta)\, \mathrm{d}\gamma = \int_{-4}^{t_0}\int_{\mathbb{R}^n} fu\eta \, \mathrm{d}\gamma.$$

Integration by parts implies

$$\int_{-4}^{t_0}\int_{\mathbb{R}^n} \left(-|\nabla u|^2\eta - \frac{(u^2\eta)_t}{2} + \frac{u^2\eta_t}{2}\right) \mathrm{d}\gamma \tag{3.2}$$

$$= \int_{-4}^{t_0}\int_{\mathbb{R}^n} fu\eta \, \mathrm{d}\gamma. \tag{3.3}$$

Rearranging the terms, and using the properties of η we finally obtain

$$\int_{\mathbb{R}^n} (u^2\eta)(x,t_0)\,\mathrm{d}\gamma^{t_0} = \int_{-4}^{t_0}\int_{\mathbb{R}^n} \big(-2|\nabla u|^2\eta + u^2\eta_t - 2fu\eta\big)\,\mathrm{d}\gamma \leq C.$$

Now let $t_0 \in (-2,-4)$. Multiplying the equation with u and integrating over $\mathbb{R}^n \times (t_0,-1)$ yields

$$\int_{\mathbb{R}^n} u^2(x,t_0)\,\mathrm{d}\gamma^{t_0} - \int_{\mathbb{R}^n} u^2(x,-1)\,\mathrm{d}\gamma^{-1} \leq \int_{t_0}^{-1}\int_{\mathbb{R}^n} fu\eta\,\mathrm{d}\gamma \leq C.$$

We can do the same computations without a function η, which yields for $t_0 \in (0,-4)$

$$\int_{\mathbb{R}^n} \big(u^2(x,t_0) - u^2(x,-4)\big)\,\mathrm{d}\gamma^{t_0} = 2\int_{-4}^{t_0}\int_{\mathbb{R}^n} \big(-|\nabla u|^2 - fu\big)\,\mathrm{d}\gamma \leq C\int_{S_1} |u|\,\mathrm{d}\gamma. \tag{3.4}$$

To estimate this, we use that

$$H(|u| - \|f\|_{L^\infty(S_1)}t) \geq 0,$$

which implies for $C = \|f\|_{L^\infty(S_1)}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} (|u(x,t)| - Ct)\,\mathrm{d}\gamma^t \leq 0,$$

so that

$$\int_{\mathbb{R}^n} (|u(x,t)| - Ct)\,\mathrm{d}\gamma^t \leq \int_{\mathbb{R}^n} (|u(x,s)| - Cs)\,\mathrm{d}\gamma^s,$$

whenever $t \geq s$. As a consequence

$$\int_{S_1} |u|\,\mathrm{d}\gamma \leq C + 4\int_{\mathbb{R}^n} u(x,-4)\,\mathrm{d}\gamma^{-4} \leq C + 4\int_{\mathbb{R}^n} u^2(x,-4)\,\mathrm{d}\gamma^{-4} \leq C.$$

Inserted into (3.4) this implies

$$\int_{\mathbb{R}^n} u^2(x,t_0)\,\mathrm{d}\gamma^{t_0} \leq C.$$

To finish the proof, take u to be a limit of functions u_j with compact support satisfying the hypotheses of the Lemma. Then Fatou's lemma implies that for any $t_0 \in (-4,0)$

$$\int_{\mathbb{R}^n} u^2(x,t_0)\,\mathrm{d}\gamma^{t_0} \leq \liminf_{j\to\infty}\int_{\mathbb{R}^n} u_j^2(x,t_0)\,\mathrm{d}\gamma^{t_0} \leq C.$$

□

In the following lemma we obtain a parabolic counterpart to Almgren's frequency lemma.

Lemma 5. *Let* $u \in H^1_\gamma(S_1)$ *be caloric in* S_1 *and*

$$N(r) = N(r,u) = \frac{\int_{T_r} |\nabla u|^2\,\mathrm{d}\gamma}{\int_{T_r} \frac{u^2}{-t}\,\mathrm{d}\gamma}. \tag{3.5}$$

Then N *is nondecreasing for* $r \in (0,1)$*. Furthermore,* N *is constant if and only if* u *is parabolic homogeneous.*

Proof. The proof is quite standard, and the two main ingredients are partial integration and the Cauchy-Schwarz inequality.

Define $N(r) = A(r)/B(r)$, put $v(x,t) = u(rx, r^2t)$ and

$$v'(x,t) = \frac{d}{dr}v(x,t) = \frac{x}{r}\nabla v(x,t) + \frac{2t}{r}v_t(x,t).$$

Moreover, let

$$A_R(r) = \int_{T_1} |\nabla v|^2 \eta \,\mathrm{d}\gamma, B_R(r) = \int_{T_1} \frac{v^2}{-t} \eta \,\mathrm{d}\gamma$$

and also

$$N_R(r) = \frac{A_R(r)}{B_R(r)},$$

where $\eta \in C_0^\infty(B_{2R})$ with $\eta = 1$ in B_R and $0 \leq \eta \leq 1$. Now we compute, using partial integration and the fact that $\partial_t G(x, -t) + \Delta G(x, -t) = 0$

$$A_R(r) = \frac{r}{2} \int_{T_1} \frac{vv'}{-t} \eta \,\mathrm{d}\gamma + \int_{T_1} -v \nabla v \nabla \eta \,\mathrm{d}\gamma,$$

$$A_R'(r) = r \int_{T_1} \frac{(v')^2}{-t} \eta \,\mathrm{d}\gamma - 2 \int_{T_1} v' \nabla v \nabla \eta \,\mathrm{d}\gamma,$$

and

$$B_R'(r) = 2 \int_{T_1} \frac{vv'}{-t} \eta \,\mathrm{d}\gamma.$$

Using the expression for $A_R'(r)$ we obtain for any $\varepsilon > 0$ and any $\alpha > \varepsilon$

$$\begin{aligned}
A(\beta) - A(\alpha) &= \lim_{R \to \infty} A_R(\beta) - A_R(\alpha) \\
&= \lim_{R \to \infty} \int_\alpha^\beta r \int_{T_1} \frac{(v')^2}{-t} \eta \,\mathrm{d}\gamma - 2 \int_{T_1} v' \nabla v \nabla \eta \,\mathrm{d}\gamma \,\mathrm{d}r \\
&\geq \lim_{R \to \infty} \int_\alpha^\beta \left((r - \varepsilon) \int_{T_1} \frac{(v')^2}{-t} \eta \,\mathrm{d}\gamma - \frac{1}{\varepsilon} \int_{T_1} |\nabla v|^2 (-t) \frac{|\nabla \eta|^2}{\eta} \,\mathrm{d}\gamma \right) \mathrm{d}r \\
&\geq \int_\alpha^\beta \left((r - \varepsilon) \int_{T_1} \frac{(v')^2}{-t} \,\mathrm{d}\gamma \right) \mathrm{d}r,
\end{aligned}$$

where we used Young's inequality and the fact that $\nabla v \in L_\gamma^2$, and Fatou's lemma. Hence, letting $\varepsilon \to 0$ we obtain

$$A'(r) \geq r \int_{T_1} \frac{(v')^2}{-t} \,\mathrm{d}\gamma, \tag{3.6}$$

which shows that

$$r \int_{T_1} \frac{(v')^2}{-t} \,\mathrm{d}\gamma$$

is integrable. Therefore,

$$\begin{aligned}
N_R(s) - N_R(t) &= \int_s^t \frac{1}{(B_R(r))^2} (A_R'(r) B_R(r) - B_R'(r) A_R(r)) \\
&= \int_s^t \frac{r}{(B_R(r))^2} \left(\int_{T_1} \frac{(v')^2}{-t} \eta \,\mathrm{d}\gamma \int_{T_1} \frac{v^2}{-t} \eta \,\mathrm{d}\gamma - \left(\int_{T_1} \frac{vv'}{-t} \eta \,\mathrm{d}\gamma \right)^2 \right) \\
&+ \int_s^t \frac{r}{(B_R(r))^2} \left(-2 \int_{T_1} v' \nabla v \nabla \eta \,\mathrm{d}\gamma B_R(r) + \int_{T_1} -v \nabla v \nabla \eta \,\mathrm{d}\gamma B_R'(r) \right) \\
&= \int_s^t \frac{r}{(B_R(r))^2} \left(\int_{T_1} \frac{(v')^2}{-t} \,\mathrm{d}\gamma \int_{T_1} \frac{v^2}{-t} \,\mathrm{d}\gamma - \left(\int_{T_1} \frac{vv'}{-t} \,\mathrm{d}\gamma \right)^2 \right) \\
&+ \int_s^t \frac{r}{(B_R(r))^2} \left(-2 \int_{T_1} v' \nabla v \nabla \eta \,\mathrm{d}\gamma B_R(r) + \int_{T_1} -v \nabla v \nabla \eta \,\mathrm{d}\gamma B_R'(r) \right),
\end{aligned}$$

In addition, the last two terms integrable, due to the estimates

$$\int_s^t \frac{r}{B_R(r)} \int_{T_1} |v'| |\nabla v| |\nabla \eta| \,\mathrm{d}\gamma \,\mathrm{d}r \leq \int_s^t \frac{r \|\nabla v|_{L_\gamma^2}}{B_R(r)} \int_{T_1} |v'|^2 \,\mathrm{d}\gamma \,\mathrm{d}r \leq C$$

and

$$\int_s^t \frac{r}{B_R^2(r)}\int_{T_1}|v||\nabla v|\,\mathrm{d}\gamma B_R'(r)\,\mathrm{d}r \leq \int_s^t \frac{r\|v\|_{L^2_\gamma}^2\|\nabla v\|_{L^2_\gamma}}{B_R^2(r)}\int_{T_1}(v')^2\,\mathrm{d}\gamma\,\mathrm{d}r \leq C.$$

Also due to (3.6), the first two terms converge. Therefore we obtain

$$\begin{aligned} N(s)-N(t) &= \lim_{R\to\infty} N_R(s)-N_R(t) \\ &= \int_s^t \frac{r}{(B_R(r))^2}\left(\int_{T_1}\frac{(v')^2}{-t}\,\mathrm{d}\gamma\int_{T_1}\frac{v^2}{-t}\,\mathrm{d}\gamma - \left(\int_{T_1}\frac{vv'}{-t}\,\mathrm{d}\gamma\right)^2\right) \geq 0, \end{aligned}$$

whenever $s \geq t$, due to the Cauchy-Schwarz intequality.

Now suppose that $N(r_1) = N(r_2)$ for $r_2 > r_1$. For N to be constant we need equality in the Cauchy-Schwartz inequality and therefore we have $C(r)v = v'$ for $r \in (r_1, r_2)$. This implies by the formula for A

$$N(r) = \frac{rC(r)}{2} = N(r_1).$$

To see that this implies the correct homogeneity let

$$u_r(x,t) = \frac{u(rx, r^2t)}{r^\alpha}.$$

Then u is parabolic homogeneous of degree α if and only if

$$\frac{\mathrm{d}u_r}{\mathrm{d}r} = \frac{1}{r^\alpha}\left(y\nabla u(rx,r^2t) + 2rtu_t(rx,r^2t) - \alpha u(rx,r^2t)/r)\right) = 0.$$

This is equivalent to $\alpha v = rv'$ which then implies $2N(r_1) = \alpha$. Therefore N is constant if and only if u is parabolic homogeneous, with $\alpha = 2N(r_1)$. □

In Lemma 4.1 in [Wei01] Weiss presented a monotonicity formula for harmonic functions. Here we need the corresponding caloric result in a special case. Nevertheless we present it in a more general form since it might be of general interest.

Lemma 6. *Let $\alpha = 1, 2, 3, ...$ and $w \in L^2_\gamma(S_1)$ be a caloric function in S_1, which is the limit of functions all having bounded heat operator and compact support. Assume further that $w(0,0) = 0$, $D_x^j w(0,0)$ vanishes for $0 \leq j \leq \alpha - 1$ and $\partial_t^i w(0,0)$ vanishes for $i = 1, ..., \lfloor(\alpha-1)/2\rfloor$. Then*

$$\int_{T_r}|\nabla w|^2\,\mathrm{d}\gamma \geq \frac{\alpha}{2}\int_{T_r}\frac{w^2}{-t}\,\mathrm{d}\gamma, \tag{3.7}$$

for $0 < r < 1$ and with equality if and only if w is parabolic homogeneous of degree α.

Proof. Assume the contrary, then there exists an $r \in (0,1]$ such that $2N(r,w) < \alpha$. Since $N(r)$ is nondecreasing $N(r) < \alpha/2$ as $r \to 0$. Let

$$w_r = \frac{w(rx, r^2t)}{\left\|\frac{w}{(-t)^{\frac{1}{2}}}\right\|_{L^2_\gamma(T_r)}}.$$

We have that $N(r_m)$ is bounded for some sequence $r_m \to 0$. Therefore,

$$\left\|\frac{w_{r_m}}{(-t)^{\frac{1}{2}}}\right\|_{L^2_\gamma(T_1)} = 1,$$

$$\|\nabla w_{r_m}\|_{L^2_\gamma(T_r)} \leq \|\nabla w_{r_m}\|_{L^2_\gamma(T_1)} \leq C \text{ for } r < 1.$$

Lemma 4 together with interior estimates now imply that we have local uniform boundedness in $C^\infty(\mathbb{R}^n \times (-4,0])$. Applying Lemma 4 again and using the fact that $\partial_t w_{r_m} = \Delta w_{r_m} \in \nabla L^2_\gamma$, we have

$$\|w\|_{L^\infty((-4,-1),L^2_{\gamma^t}(\mathbb{R}^n))} \leq C,$$
$$\|\partial_t w_{r_m}\|_{L^1((-4,-1),Y^*)} \leq C,$$

Now define $\hat{w_m}(x,t) = w_{r_m}(x\sqrt{-t},t)$. Then

$$\begin{aligned}\nabla \hat{w}_m(x,t) &= \sqrt{-t}\nabla w_{r_m}(x\sqrt{-t},t)\\ \partial_t \hat{w}_m(x,t) &= \partial_t w_{r_m}(x\sqrt{-t},t) + \frac{x}{2\sqrt{-t}}\nabla w_{r_m}(x\sqrt{-t},t)\\ &\subset \nabla L^2_{\gamma^{-1}}(T_1) + L^1_{\gamma^{-1}}(T_1) \subset L^1((-4,-1),Y^*).\end{aligned}$$

This implies that have the uniform estimates

$$\left\|\frac{\hat{w}_m}{(-t)^{\frac{1}{2}}}\right\|_{L^2_{\gamma^{-1}}} + \|\nabla \hat{w}_m\|_{L^2_{\gamma^{-1}}(T_1)} + \|\partial_t \hat{w}_m\|_{L^1((-4,-1),Y^*)} \leq C.$$

For $X_0 = H^1_{\gamma^s}(\mathbb{R}^n)$, $X = L^2_{\gamma^s}(\mathbb{R}^n)$ and $X_1 = Y^*$ we can use Lemma 3. This implies that for a subsequence $\hat{w}_m$ converges in $L^2_\gamma(T_1)$, and so does w_{r_m}. Therefore

$$w_0 = \lim_{m\to\infty} w_{r_m}$$

is a caloric function in S_1, with

$$D^j_x w_0(0,0) = 0 \text{ for } j = 0,1,2,...,\alpha-1,$$
$$\partial^i_t w_0(0,0) = 0 \text{ for } i = 1,...,\lfloor(\alpha-1)/2\rfloor$$

and

$$\left\|\frac{w_0}{(-t)^{\frac{1}{2}}}\right\|_{L^2_\gamma(T_1)} = 1.$$

Furthermore,

$$N(\beta, w_{r_m}) - N(\alpha, w_{r_m}) = N(\beta r_m, w) - N(\alpha r_m, w) \to 0$$

Therefore,

$$N(r)(w_0) = N(0^+) \text{ for } 0 < r < 1,$$

which implies that, by Lemma 5, w_0 must be parabolic homogeneous of degree $2N(0+) \in [0,\alpha)$. From the regularity at the origin we have that $2N(0^+) \in \mathbb{N}$. Since $2N(0+) < \alpha$ and

$$D_{x_j} w_0(0,0) = 0 \text{ for } j = 0,1,...,\alpha-1,$$
$$\partial^i_t w(0,0) = 0 \text{ for } i = 1,...,\lfloor(\alpha-1)/2\rfloor,$$

this implies that $w_0 = 0$, which is a contradiction.

Hence, $2N(r) \geq \alpha$ for $0 < r \leq 1$ and $2N(s) = \alpha$ implies that N is constant in $0 \leq r \leq s$ and therefore w is parabolic homogeneous of degree α in $0 \leq r \leq s$. □

4. THE OPTIMAL REGULARITY

In this section we prove the main theorem. First we obtain some estimates which we will need. The lemma below proves that under the assumptions in Theorem 1, we have quadratic growth at the level r_0.

Lemma 7. *Let* $u \in P_4(M_1, M_2)$, $\psi \in C_0^\infty(B_{\frac{3}{4}})$ *such that* $\psi = 1$ *on* $B_{\frac{1}{2}}$ *and* $v = u\psi$. *Given* $\varepsilon > 0$, *there exists* $\lambda_{\varepsilon,M_i} > 0$ *and* $C_{\varepsilon,M_i} < \infty$ *such that if for some* $0 < r_0 < \lambda_{\varepsilon,M_i}$

$$\delta(r_0/2, v) > \varepsilon \text{ and } W(r_0; v, f) \leq 2A_n - \varepsilon$$

then

$$\int_{T_{r_0}} \frac{v^2}{-t}\, \mathrm{d}\gamma \leq C_{\varepsilon,M_i} r_0^4.$$

Proof. Suppose towards a contradiction that there exists a sequence $r_m \to 0$ and a sequence of functions $u_m \in P_1(M_1, M_2)$ with $v_m = \psi u_m$, such that

$$\delta(r_m/2, v_m) > \varepsilon,$$
$$W(r_m; v_m, f) \leq W_0,$$
$$C_m = \frac{\left\| \frac{v_m}{(-t)^{\frac{1}{2}}} \right\|_{L^2_\gamma(T_{r_m})}}{r_m^2} \to \infty.$$

Now we use the following rescaling

$$w_{r_m}(x,t) = \frac{v_m(r_m x, r_m^2 t)}{C_m r_m^2}.$$

Then w_{r_m} is a solution to (1.1) with

$$f'_m = \frac{f_{u_m}(r_m x, r_m^2 t)}{C_m} \to 0 \text{ in } B_{1/2r^m} \times (-4/r_m^2, 0]. \tag{4.1}$$

We also have

$$\delta(1/2, w_{r_m}) \geq \varepsilon,$$
$$\left\| \frac{w_{r_m}}{(-t)^{\frac{1}{2}}} \right\|_{L^2_\gamma(T_1)} = 1.$$

Moreover, the energy assumption implies

$$W(1; w_{r_m}, f'_m) = \frac{W(r_m; w_{r_m}, f_{u_m})}{C_m^2} \leq \frac{W_0}{C_m^2},$$

and therefore, in particular

$$\int_{T_1} |\nabla w_{r_m}|^2\, \mathrm{d}\gamma \leq \int_{T_1} \left(\frac{w_{r_m}^2}{t} + \frac{2A_n - \varepsilon}{C_m^2} + \frac{1}{C_m} \sup_{T_{r_m}} f_{u_m} \right) \mathrm{d}\gamma, \tag{4.2}$$

where the last two terms vanish in the limit, and the right hand side is bounded. We observe that since u is locally in $C_x^{1,\alpha} \cap C_t^\alpha(Q_2^-)$ (due to local estimates for f bounded, see Theorem 4.8 in [Lie96]), $Hv \in L^\infty(S_1)$. Therefore, $Hv_m \in L^\infty(S_1)$. In what follows we argue as in Lemma 6. We have

$$\left\| \frac{w_{r_m}}{(-t)^{\frac{1}{2}}} \right\|_{L^2_\gamma(T_1)} = 1,$$
$$\|\nabla w_{r_m}\|_{L^2_\gamma(T_1)} \leq C.$$

Lemma 4 together with interior estimates implies that w_{r_m} is locally uniformly bounded in $C_t^{1,\alpha} \cap C_t^{0,\alpha}(\mathbb{R}^n \times (-4, 0])$. In addition, Lemma 4 implies

$$\|w_{r_m}\|_{L^\infty((-4,0), L^2_{\gamma t}(\mathbb{R}^n))} \leq C.$$

The equation for v_m gives

$$\partial_t v_m = \Delta v_m - H v_m \subset \nabla L^2_\gamma(T_1) + L^\infty(T_1).$$

Applying Lemma 3 as in the proof of Lemma 6, we find a subsequence converging in $L^2_\gamma(T_1)$. Thus

$$\int_{T_1} \frac{w_0^2}{-t}\,\mathrm{d}\gamma = 1, \tag{4.3}$$

where

$$w_0 = \lim_{m\to\infty} w_{r_m}.$$

If we pass (4.2) to the limit we obtain by the weak convergence

$$\int_{T_1} |\nabla w_0|^2\,\mathrm{d}\gamma \le \int_{T_1} \frac{w_0^2}{-t}\,\mathrm{d}\gamma. \tag{4.4}$$

We also have that (4.1) implies that w_0 is caloric in S_1 and that both w_0 and $|\nabla w_0|$ are zero at the origin. This together with Lemma 6 implies

$$\int_{T_1} |\nabla w_0|^2\,\mathrm{d}\gamma \ge \int_{T_1} \frac{w_0^2}{-t}\,\mathrm{d}\gamma. \tag{4.5}$$

Therefore, again by Lemma 6, (4.4) and (4.5) are possible if and only if w_0 is a parabolic homogeneous quadratic polynomial in T_1, and thus also in Q_1^- by unique continuation. Moreover, the local uniform convergence gives

$$\delta(1/2, w_0) \ge \varepsilon.$$

But we now that w_0 can (up to tilting the coordinates) be written as

$$w_0 = \sum_1^n \lambda_i x_i^2 + \alpha t.$$

So unless $\alpha = \lambda_i = 0$ for all i, then the set $\{u = |\nabla u| = 0\}$ contains only the origin. Therefore w_0 has to be identically zero, which contradicts (4.3). □

The next result is very similar to Lemma 8, but here we obtain a fix constant in the quadratic growth, and also an estimate of how close the solution is to a halfspace solution.

Lemma 8. *Let $u \in P_4(M_1, M_2)$, $\psi \in C_0^\infty(B_{\frac{3}{4}})$ such that $\psi = 1$ on $B_{\frac{1}{2}}$ and $v = u\psi$. Given $\varepsilon > 0$, $d > 0$ and $\beta_0 > 0$, there exists $\lambda = \lambda_{\varepsilon,M_i,d,\beta_0} > 0$ such that if for some $0 < r_0 < \lambda$*

$$\delta(r_0/2, v) > \varepsilon \text{ and } W(r_0; v, f) \le 2A_n - \varepsilon$$

then

$$\sup_{Q_{r_0}^-} |v| \le r_0^2$$

and

$$\|v - h\|_{L^\infty(Q^-_{r_0 d})} < \beta_0 (r_0 d)^2.$$

Here h is again a half-space solution, with $f = 1$.

Proof. As usual, we argue by contradiction. If the assertion does not hold, then there is a sequence $r_j \to 0$ such that the hypothesis hold for r_j but still either

$$\sup_{Q_{r_j}^-} |v| > r_j^2,$$

or

$$\|v - h\|_{L^\infty(Q^-_{r_j d})} > \beta_0 r_j^2 d^2,$$

for all halfspace solutions h. Define the rescaled functions

$$w_j(r,x) = w_{r_j}(x,t) = \frac{u(r_jx, r_j^2t)}{r_j^2}$$

as usual. Then, since $Hv_j \in L^\infty(\mathbb{R}^n \times \mathbb{R}^+)$ (due to the local estimates on u), we also have $Hw_j \in L^\infty(\mathbb{R}^n \times \mathbb{R}^+)$ uniformly. Moreover,

$$\sup_{Q_1^-} |w_j| > 1,$$

$$\|w_j - h\|_{L^\infty(Q_d^-)} > \beta_0 d^2,$$

for all halfspace solutions h. By Lemma 7

$$\int_{T_1} \frac{w_j^2}{-t}\,\mathrm{d}\gamma \le C_{\varepsilon,M_i}.$$

In addition, we have

$$Hw_j = f(r_jx, r_j^2t)\chi_{\Omega_j} \text{ in } B_{\frac{1}{2r_j}} \times (-4/r_j, 0),$$

and from the assumption on the Weiss energy

$$\int_{T_1} |\nabla w_j|^2\,\mathrm{d}\gamma \le C.$$

Now we proceed as in the proof Lemma 6; Lemma 4 implies

$$\int_{\mathbb{R}^n} w_j^2(x,t)\,\mathrm{d}\gamma^t \le C$$

for all $t \in (-4,0)$. Using the equation for w_j we have

$$\partial_t w_j = -Hw_j + \Delta w_j \subset L^\infty(T_1) + \nabla L^2_\gamma(T_1).$$

Therefore, by Lemma 3 applied as in Lemma 6, and by interior estimates (Theorem 6.17 combined with Theorem 4.8 in [Lie96]), up to a subsequence, w_j converges in $L^2_\gamma(T_1)$ and locally in $C_x^{1,\alpha} \cap C_t^\alpha(\mathbb{R}^n \times (-4,0])$. The limit function w_0 satisfies

$$\sup_{Q_1^-} |w_0| \ge 1, \tag{4.6}$$

or

$$\|w_0 - h\|_{L^\infty(Q_d^-)} \ge \beta_0 d^2, \tag{4.7}$$

and

$$Hw_0 = f(0,0)\chi_\Omega = \chi_\Omega \text{ in } \mathbb{R}^n \times (-4,0).$$

Moreover, due to the almost monotonicity of the Weiss' functional

$$W(s; w_0, 1) = \lim_{j\to\infty} W(sr_j; v, f) = W(0; v, f) < 2A_n - \varepsilon,$$

for all $s > 0$. Hence, w_0 is parabolic homogeneous of degree 2. This implies, by the classification of global solutions (Theorem I [CPS04]) that $w_0 = h$. This is a contradiction to both (4.6) and (4.7). □

Remark 2. *We remark that Lemma 7 and Lemma 8 will still be valid, if we have a solution with compact support and defined everywhere (so that the Weiss energy is well-defined), and if we assume*

$$W(r; v, f) \le 2A_n - \varepsilon,$$

for all $r \le r_0$*, and also that* $W(r, v, f)$ *has a limit as* $r \to 0$*.*

In the following lemma we prove that if a solution is sufficiently close to a halfspace solution, then the set $\Lambda(u)$ cannot be too small.

Lemma 9. *Fix $d > 0$ and let $u \in P_1(M_1, M_2)$. Suppose*

$$\|u - h\|_{L^\infty(Q_1^-)} \le \beta_0,$$

where h is a halfspace solution. Then, if β_0 is sufficient small (depending on M_2 and d), we will have

$$\delta(u, 1/2) > 1/2.$$

Proof. Suppose the contrary is true. Then there are halfspace solutions h_m and a subsequence of functions $u_m \in P_1(M_{1_m}, M_2)$ such that

$$\lim_{m\to\infty} \|u_m - h_m\|_{L^\infty(Q_1^-)} = 0$$

and

$$\delta(u_m, 1/2) \le 1/2.$$

If h_m converges to h_0 it follows by interior estimates (Theorem 4.8 in [Lie96]) that there is a subsequence u_m, that converges to h_0 in $C^{1,\alpha}_{x,\mathrm{loc}} \cap C^{0,\alpha}_{t,\mathrm{loc}}(Q_1^-)$. For this sequence we have

$$\lim_{m\to\infty} \delta(u_m, 1/2) = \delta(h_0, 1/2) = 1,$$

which is a contradiction. □

We are now ready to give the proof of our main theorem. The idea of the proof is very similar to the one of the proof of Theorem A1 in [PS07a].

Proof of Theorem 1. To make the proof more clear we divide into several steps. In what follows we will take $d = 1/8$.

Step 1: Bound on W. First we establish an upper bound for $W(r; v, f)$. To prove the estimate, take $u \in P_1(M_1, M_2)$, $\psi \in C_0^\infty(B_1)$, $\psi = 1$ in $B_{\frac{1}{2}}$ and let $v = u\psi$. Since in the monotonicity formula (Proposition 1) $F(r)$ is a continuous function that vanishes at the origin, there exists a positive constant r_{ε,M_1,M_2} such that

$$W(r_1; v, f) \le W(r_2; v, f) + \varepsilon/2,$$

for every $0 < r_1 \le r_2 \le r_{\varepsilon,M_1,M_2}$. Therefore it follows that if $W(r_0; v, f) \le 2A_n - \varepsilon$ for some $0 < r_0 < r_{\varepsilon,M_1,M_2}$ then

$$W(r; v, f) \le 2A_n - \varepsilon/2,$$

for every $0 < r \le r_0$.

Step 2: First bound on δ. The second step of the proof is to find a lower bound of the thickness function of a rescaled version of v.

As in Lemma 8 take $\varepsilon > 0$ and $\beta_0 > 0$ small enough (as in Lemma 9). Then to this end chose

$$\lambda = \min(\lambda_{\varepsilon, M_1/d^{2k}, M_2, \beta_0}, \lambda_{\varepsilon, 1/d^{2k}, M_2, \beta_0}),$$

where k is an integer such that

$$d^k \le r_0 \le d^{k-1}. \tag{4.8}$$

Then if for some $0 < r_0 < \lambda$

$$\delta(r_0/2, v) > \varepsilon \text{ and } W(r_0; v, f) \le 2A_n - \varepsilon/2$$

we have by Lemma 8

$$\sup_{Q_{r_0}^-} |v| \le r_0^2,$$

and

$$\|v - h\|_{L^\infty(Q_{r_0 d}^-)} \le \beta_0 r^2 d^2.$$

Then Lemma 9 implies that

$$\delta(r_0/2, v_d) = \delta(\frac{1}{2}, v_{r_0 d}) > \frac{1}{2}.$$

By the simplest possible estimate we have

$$\sup_{Q_{4d}^-} |v| \leq M_1,$$

so $v_d \in P_4(M_1/d^2, M_2)$.

Step 3: Repeat Step 1 and Step 2 $(k-1)$ times. We see that all the assumptions in Lemma 8 (see also the remark after Lemma 8) are again satisfied for v_d, except that M_1 is changed to M_1/d^2. So we iterate Step 1 and Step 2 repeatedly to v_d, v_{d^2} until v_{d^k}. Then we have all along the way that for $i \leq k$

$$v_{d^i} \in P_4(M_1/d^{2i}, M_2)$$

and

$$\delta(r_0/2, v_{d^i}) > \frac{1}{2}.$$

Step 4: The k-th iteration. Now we wish to estimate the supremum of v_{d^k}, for which we use the relation between r_0 and d for as follows

$$\sup_{Q_4^-} |v_{d^k}| = \frac{1}{d^{2k}} \sup_{Q_{4d^k}^-} |v| \leq \frac{1}{d^{2k}} \sup_{Q_{r_0}^-} |v| \leq \frac{r_0^2}{d^{2k}} \leq 64,$$

by (4.8). Hence, $v_{d^k} \in P_4(64, M_2)$ and also

$$\delta(r_0/2, v_{d^k}) > \varepsilon$$

and

$$W(r; v_{d^k}, f(d^k x, d^{2k} t)) \leq 2A_n - \varepsilon,$$

for all $r \leq r_0$, as before.

Step 5: Continue the iterations. Now we can iterate Step 1 and Step 2 to v_{d^k} k times to obtain

$$v_{d^{k+i}} \in P_4(64/d^{2i}, M_2)$$

and

$$\delta(r_0/2, v_{d^{k+i}}) > \frac{1}{2},$$

for $i \leq k$.

And when we reach d^k we use the estimate at level r_0 of v_{d^k} gotten from applying Lemma 8 to v_{d^k} which gives us again that

$$v_{d^{2k}} \in P_4(64, M_2),$$

so that the constants never blow up, they are between 64 and $64/d^{2k}$. Thus for all $j = 0, 1, 2, \ldots$ we obtain

$$d^{2j} \sup_{Q_{r_0}^-} |v_{d^j}| = \sup_{Q_{r_0 d^j}^-} |v| \leq (r_0 d^j)^2.$$

This implies that we have for all $r < r_0$ the estimate

$$\sup_{Q_r^-} |v| \leq C r^2,$$

where C is a fixed constant.

Step 5: Apply for points close the origin. Since $f \in C^{0,\beta}(Q_1^-)$, interior estimates applies away from the free boundary. Therefore, we only have to obtain regularity close to the free boundary. In order to obtain the estimate, take a free boundary point $(x_0, t_0) \in \Gamma \cap Q^-_{c_0}$ and introduce the scaling

$$w(x,t) = \frac{v(x + x_0, t + t_0)}{f(x_0, t_0)} \in P_2(2M_1, 2M_2).$$

If $c_0 = C(\varepsilon, M_1, M_2, r_0)$ is small enough we have the following for w:

$$\delta((3/4)r_0, w) \geq \varepsilon/2$$

and

$$W(r_0; w, f) \leq \frac{2A_n - \varepsilon/2}{(f(x_0))^2} \leq 2A_n - \varepsilon/4$$

This is similar to (1.2) and we can adjust the lemmata and Step 1 to Step 4 for these assumptions. Using this result we will find that v satisfies

$$|v(x,t)| \leq C \operatorname{dist}((x,t), \Gamma)^2 \text{ for } (x,t) \in Q^-_{c_0}.$$

To do the final estimates for v take $(x_1, t_1) \in Q^-_{c_0}$, let $\rho = \operatorname{dist}((x_1, t_1), \Gamma)$ and

$$v_\rho(x,t) = \frac{v(\rho x, \rho^2 t)}{\rho^2}.$$

Then $v_\rho \in L^\infty(Q^-_1)$ and

$$Hv = f \text{ in } Q^-_1.$$

By Schauder estimates (see for instance Theorem 4.9 in [Lie96]) we have

$$\|v\|_{C^{1,1}_x \cap C^{0,1}_t (Q^-_\rho(x_1,t_1))} = \|v_\rho\|_{C^{1,1}_x \cap C^{0,1}_t (Q^-_1)} \leq C,$$

which is independent of ρ. □

5. Acknowledgements

The authors would like to thank Henrik Shahgholian for suggesting the problem and also for all the ideas and helpful discussions along the development of this paper. They are also grateful to Régis Monneau for suggesting the use of Simon's lemma in order to obtain compactness.

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