Null controllability of the 1D heat equation with interior inverse square potential

Pierre Lissy[*] Tanguy Lourme [†]

May 7, 2025

Abstract

This paper aims to answer an open problem posed by Morancey in 2015 concerning the null controllability of the heat equation on $(-1, 1)$ with an internal inverse square potential located at $x = 0$. For the range of singularity under study, after having introduced a suitable self-adjoint extension that enables to transmit information from one side of the singularity to another, we prove null-controllability in arbitrary small time, firstly with an internal control supported in an arbitrary measurable set of positive measure, secondly with a boundary control acting on one side of the boundary. Our proof is mainly based on a precise spectral study of the singular operator together with some recent refinements of the moment method of Fattorini and Russell. This notably requires to use some fine (and sometimes new) properties for Bessel functions and their zeros.

Keywords. Singular parabolic equation, self-adjoint extensions, singular Sturm-Liouville operators, controllability, Bessel functions.

MSC 2020. 35K67, 35P20, 93B05, 34B24, 33C10.

Contents

[*]CERMICS, ENPC, Institut Polytechnique de Paris, Marne-la-Vallée, France. (pierre.lissy@enpc.fr)
[†]CERMICS, ENPC, Institut Polytechnique de Paris, Marne-la-Vallée, France. (tanguy.lourme@gmail.com)

1 Introduction

1.1 Statement of the problem and main result

This article is mainly dedicated to investigating the null-controllability properties of the following singular parabolic equation with inverse square potential

$$\begin{cases} \partial_t f(t,x) - \partial_{xx}^2 f(t,x) + \dfrac{c}{x^2} f(t,x) = u(t,x)\mathbb{1}_\omega(x), & (t,x) \in (0,T) \times (-1,1), \\ f(t,-1) = f(t,1) = 0, & t \in (0,T), \\ f(0,x) = f^0(x), & x \in (-1,1), \end{cases} \tag{1.1}$$

where $c \in \mathbb{R}$, $T > 0$ is a horizon time, ω is a measurable subset of $(-1,1)$ assumed to have positive Lebesgue measure, $\mathbb{1}_\omega$ denotes the characteristic function on ω, $f^0 \in L^2(-1,1)$ and $u \in L^2((0,T)\times(-1,1))$. This kind of potentials naturally appear in the context of combustion theory ([7]), but also in quantum mechanics ([5]).

It is proved in [4] that (1.1) is ill-posed in $L^2(0,1)$ (with Dirichlet boundary conditions in 0 and 1) if $c < -\frac{1}{4}$ (see also [34]). On $L^2(-1,1)$, we can prove that (1.1) is ill-posed, in the sense that there does not exist any selfadjoint extension of $\partial_{xx}^2 - \frac{c}{x^2}\mathrm{Id}$, posed on $C_0^\infty((-1,1)\setminus\{0\})$, for which this operator generates a C^0-semigroup, if $c < -\frac{1}{4}$. The proof of this fact is postponed to Appendix A.

Moreover, from [26, Chapter X], we know that for $c \geqslant \frac{3}{4}$, the operator $-\partial_{xx}^2 + \frac{c}{x^2}\mathrm{Id}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}\setminus\{0\})$, meaning that no information is passing through the singularity $x = 0$ (see *e.g.* [10, pp. 1744-1745] for an interesting discussion on this subject). Notably, in this context, it is not possible to control with a measurable set ω that lies only on one part of the singularity $x = 0$.

For all these reasons, from now on, we will restrict to coefficients c that lie in $\left(-\frac{1}{4}, +\frac{3}{4}\right)$ (the case $c = \frac{1}{4}$ cannot be directly treated with the techniques developed in this article, see Section 5 for more explanations). Throughout this paper, for the sake of consistency with the existing literature, the coefficient of the singular potential is parameterized as $c = c_\nu$, where

$$c_\nu := \nu^2 - \frac{1}{4}, \quad \forall \nu \in (0,1). \tag{1.2}$$

Our main result is the following null-controllability result in arbitrary small time.

THEOREM 1.1. *For any $f^0 \in L^2(-1,1)$ and $T > 0$, there exists a control $u \in L^2((0,T)\times(-1,1))$ such that the corresponding solution f to* (1.1) *verifies $f(T) = 0$ in $L^2(\Omega)$.*

We also have a version of this Theorem with a boundary control on one side of the boundary, which requires some nontrivial intermediate results and more notations. For the sake of simplicity, we chose to postpone it at the end of the paper (see Theorem 4.13).

REMARK 1.2. *In the previous Theorem, the solutions of* (1.1) *have to be understood in a sense explained in Section 1.2. Notably, even in the case where $c_\nu = 0$, we obtain a new result for a non-standard definition of solutions of the usual heat equation (corresponding to a particular self-adjoint extension of the Laplace operator on $C_0^\infty((-1,0)\times(0,1))$).*

REMARK 1.3. *Here, ω can be any measurable set of positive measure, whereas in most of the literature, one considers ω that is an open set. As we will see during the proof, passing to this more general case is not totally obvious (see Remark 4.7).*

The controllability properties of heat equations with inverse square potentials has been a wide subject of studies during the last fifteen years. In dimension larger than three, the case of an internal inverse square singularity with inverse square potential and internal null control was first treated in [33] in a particular geometry and then extended in [15] (see also [36] for the study of an optimal control problem in this setting). The case of a variable diffusion has been treated in [25]. The case of approximate boundary control has been studied in [28], still in dimension larger than three. The case of an inverse square singularity located at the boundary and internal control was first treated in any space dimension in [11]. A related result is also [9], involving the distance to the boundary in dimension larger that three.

Concerning specifically the case of the one-dimensional heat equation with inverse square potential at one point of the boundary, the question of the cost of controllability with a boundary control located at the other point of the boundary has been studied in [21], whereas [8] proved a result where the boundary control is located at the boundary. The case of mixed degenerate diffusion and singular potentials is treated in [17, 32], and a model with memory is studied in [2].

Remark that in all the literature cited above, in the 1D case, the case where the singularity is located *inside* the domain has not been addressed, except in [22, Theorem 1.5], where the author proved an approximate controllability result in this context. In [22], the author raised the open question of obtaining a null-controllability result in the context of equation (1.1). The goal of the present article is to give a positive answer to this question.

That this question has not been investigated yet comes from the fact that this situation is much more difficult from a theoretical point of view. Indeed, the domain $(-1,1)$ is separated in two subintervals by the inverse square potential, and notably, if one considers the usual domain of the Laplace operator $H_0^1(-1,1)\cap H^2(-1,1)$, it will not be possible to control (1.1) with ω located in one side of the boundary. In fact, we need to consider another self-adjoint extension of A_ν on $C_0^\infty((-1,1)\setminus\{0\})$. Of course, what is important is to specify what to prescribe at the boundary. As we will see in Section 1.2, we are able to introduce an appropriate self-adjoint extension, with appropriate transmission conditions, that enables to pass information through the singularity. The main difficulty is then to give an appropriate spectral decomposition of this self-adjoint extension. Notably, we will give some very precise estimates on the eigenvalues of A_ν and also on the eigenfunctions of A_ν.

The plan of the paper is as follows. In Subsection 1.2, we investigate the well-posedness of (1.1). In Subsection 1.3, we give some already-known results on Bessel functions, together with new technical results that are needed for our study. In Section 2, we study the spectral decomposition of the singular elliptic operator under consideration. In Section 3, we give some precise results on the asymptotic behaviour of the eigenvalues of our singular operator. In Section 4, we prove our main Theorem 1.1 and give an extension to the case of a boundary control. To conclude, in Section 5, we present several open problems related to the present study.

1.2 Well-posedness

We consider the following homogeneous problem:

$$\begin{cases} \partial_t f - \partial_{xx}^2 f + \dfrac{c_\nu}{x^2} f = 0, & (t,x)\in(0,T)\times(-1,1),\\ f(t,-1) = f(t,1) = 0, & t\in(0,T),\\ f(0,x) = f^0(x), & x\in(-1,1). \end{cases} \tag{1.3}$$

We begin by addressing the well-posedness of (1.3). The elliptic differential operator under consideration is

$$A_\nu f := -\partial_{xx}^2 f + \frac{c_\nu}{x^2} f.$$

It is obvious that A_ν is well-defined on $C_0^\infty((-1,0)\cup(0,1))$. We now specify the self-adjoint extension to be employed.

As in [22, Section 2.1], we introduce:

$$\tilde{H}_0^2 := \left\{ f \in H^2(-1,1),\ f(0) = f'(0) = 0 \right\},$$

$$\mathcal{F}_s^\nu := \left\{ f \in L^2(-1,1),\ \exists c_1^+, c_1^-, c_2^+, c_2^- \in \mathbb{R}, \quad f(x) = \begin{cases} c_1^-|x|^{\nu+\frac{1}{2}} + c_2^-|x|^{-\nu+\frac{1}{2}} & \text{on } (-1,0), \\ c_1^+|x|^{\nu+\frac{1}{2}} + c_2^+|x|^{-\nu+\frac{1}{2}} & \text{on } (0,1). \end{cases} \right\}.$$

Notice that for any $f_s \in \mathcal{F}_s^\nu$, $f_s \in C^\infty((-1,0)\cup(0,1))$ and

$$(-\partial_{xx}^2 + \frac{c_\nu}{x^2}) f_s(x) = 0, \quad \forall x \in (-1,0)\cup(0,1). \tag{1.4}$$

The domain of A_ν is defined as

$$\begin{aligned} D(A_\nu) := \Big\{ & f = f_r + f_s;\ f_r \in \tilde{H}_0^2(-1,1),\ f_s \in \mathcal{F}_s^\nu \text{ such that } f(-1) = f(1) = 0, \\ & c_1^- + c_2^- + c_1^+ + c_2^+ = 0, \text{ and } \left(\nu + \frac{1}{2}\right) c_1^- + \left(-\nu + \frac{1}{2}\right) c_2^- = \left(\nu + \frac{1}{2}\right) c_1^+ + \left(-\nu + \frac{1}{2}\right) c_2^+ \Big\}, \end{aligned} \tag{1.5}$$

which is a linear subspace of $L^2(-1,1)$. By [22, Remark 2.3], we have $A_\nu(D(A_\nu)) \subset L^2(-1,1)$, confirming that $(A_\nu, D(A_\nu))$ is indeed an unbounded operator on $L^2(-1,1)$, which is moreover densely defined since

$$\left\{ \phi \in C_0^\infty((-1,0)\cup(0,1)),\ \text{extended at 0 by } \phi(0) = 0 \right\} \subset \tilde{H}_0^2(-1,1) \subset D(A_\nu).$$

To conclude, using [22, Proposition 2.2], $(A_\nu, D(A_\nu))$ is self-adjoint on $L^2(-1,1)$, and for any $f \in D(A_\nu)$, the following inequality holds:

$$\langle A_\nu f, f \rangle \geqslant \min\{1, 4\nu^2\} \int_{-1}^1 \partial_x f_r(x)^2 \,\mathrm{d}x. \tag{1.6}$$

REMARK 1.4. *For $\nu \in (0,1)$, it is easy to verify that we have a direct sum $\tilde{H}_0^2(-1,1) \oplus \mathcal{F}_s^\nu$. Since $D(A_\nu) \subset \tilde{H}_0^2(-1,1) \oplus \mathcal{F}_s^\nu$, the unique decomposition of functions in $D(A_\nu)$ given by $f = f_r + f_s$, where $f \in D(A_\nu)$, $f_r \in \tilde{H}_0^2(-1,1)$, $f_s \in \mathcal{F}_s^\nu$ will be referred to as the decomposition into the regular part f_r and the singular part f_s. The conditions imposed on the coefficients of the singular part in (1.5) will be referred to as transmission conditions.*

REMARK 1.5. *On $\mathcal{F}_s^\nu$, the operator A_ν acts independently on $(0,1)$ and $(-1,0)$. Thus, the symbol ∂_{xx}^2 should not be interpreted as a derivative in the distributional sense over $\Omega = (-1,1)$. Instead, it operates separately on $(-1,0)$ and $(0,1)$. In other words, ∂_{xx}^2 has to be understood as a distributional derivative on $\Omega = (-1,0)\cup(0,1)$. Therefore, from (1.4), we have $A_\nu f_s = 0$.*

We have already remarked that $(A_\nu, D(A_\nu))$ is a densely defined self-adjoint and monotone operator (by (1.6)). From [12, Corollary 2.4.8], we deduce that $-A_\nu$ is a m-dissipative operator, so that applying the Hille-Yosida theorem for self-adjoint monotone operators (see, for instance, [12, Theorem 3.2.1]), we obtain that $-A_\nu$ generates a strongly continuous semigroup, denoted by $(e^{-A_\nu t})_{t\geqslant 0}$. In other words, the homogeneous problem (1.3) is well-posed: for any $f^0 \in L^2(-1,1)$, there exists a unique

$$f \in C^0([0,+\infty), L^2(-1,1)) \cap C^0((0,+\infty), D(A_\nu)) \cap C^1((0,+\infty), L^2(-1,1))$$

such that f verifies (1.3).
Our next goal is to define the notion of a solution to the non-homogeneous problem (1.1).
Let us introduce the operator

$$B \ :\ L^2(-1,1) \to L^2(-1,1), \quad g \mapsto \mathbb{1}_\omega g.$$

Since $B \in \mathcal{L}_c(L^2(-1,1))$, the non-homogeneous problem (1.1) is well-posed. In other words, there exists a unique weak solution (defined for in the sense of transpositions as in [13, Section 2.3] for instance) to (1.1) in $C^0([0,T], L^2(-1,1))$, that is "explicitely" given by

$$f = e^{-A_\nu t} f^0 + \int_0^t e^{-A_\nu(t-s)} \mathbb{1}_\omega u(s) \,\mathrm{d}s, \quad \forall t \in [0,T].$$

1.3 Generalities on Bessel functions

Let us consider some parameter $\nu \in \mathbb{R}$. The Bessel functions of the first kind, denoted by J_ν, are solutions to the differential equation given by

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0. \tag{1.7}$$

The Bessel function of the first kind $J_\nu(x)$ can be defined for any $x > 0$ by its power series expansion

$$J_\nu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}, \tag{1.8}$$

where $\Gamma(z)$ is the Euler Gamma function (by convention, $\frac{1}{\Gamma(x)} = 0$ if x is a non-positive integer). For all $\nu \in \mathbb{R}$ and all $n \in \mathbb{N} \setminus \{0\}$, we denote by $j_{\nu,n}$ the n-th positive real zero of J_ν. We refer the reader to [35, 20] for more details on Bessel functions.
Now, we state several results on Bessel functions that will be useful later.

The first result is the following inequality, that we did not find in the literature and might be interesting by itself.

LEMMA 1.6. *Let $\nu \in (0,1)$. Then,*

$$J_\nu(x)J_{-\nu}(x) < \frac{\sin(\nu\pi)}{\nu\pi}, \quad \forall x > 0.$$

Proof of Lemma 1.6. Let $x > 0$. We introduce

$$\varphi_x \ : \ (0,1] \longrightarrow \mathbb{R}, \quad \nu \mapsto \frac{\sin(\nu\pi)}{\nu\pi} - J_\nu(x)J_{-\nu}(x).$$

From [35, p. 150, (1)], we know that

$$J_\nu(z)J_{-\nu}(z) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} J_0(2z\cos(\theta))\cos(2\nu\theta)\mathrm{d}\theta, \quad \forall \nu \in \mathbb{R},\ \forall z > 0.$$

Therefore, φ_x is differentiable on $(0,1]$ and

$$\varphi_x{}'(\nu) = \frac{\cos(\nu\pi)}{\nu} - \frac{\sin(\nu\pi)}{\pi\nu^2} + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} J_0(2x\cos(\theta))2\theta\sin(2\nu\theta)\mathrm{d}\theta, \quad \forall \nu \in (0,1].$$

From the integral representation given in [35, p. 176, (4)], we have

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin(\theta))d\theta.$$

We deduce that $|J_0(x)| < 1$ for $x > 0$. Hence, we have

$$\left|\frac{2}{\pi}\int_0^{\frac{\pi}{2}} J_0(2x\cos(\theta))2\theta\sin(2\nu\theta)\mathrm{d}\theta\right| \leqslant \frac{2}{\pi}\int_0^{\frac{\pi}{2}} |J_0(2x\cos(\theta))|\, 2\theta\sin(2\nu\theta)\mathrm{d}\theta < \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 2\theta|\sin(2\nu\theta)|\mathrm{d}\theta,$$

and since $\nu \in (0,1)$, we have $\sin(2\nu\theta) > 0$ for all $\theta \in (0, \frac{\pi}{2})$. Furthermore,

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} 2\theta\sin(2\nu\theta)\mathrm{d}\theta = \frac{2}{\pi}\left[-\frac{\theta\cos(2\nu\theta)}{\nu} + \frac{\sin(2\nu\theta)}{2\nu^2}\right]_0^{\frac{\pi}{2}} = -\frac{\cos(\nu\pi)}{\nu} + \frac{\sin(\nu\pi)}{\pi\nu^2}.$$

Gathering the previous computations, we deduce that ${\varphi_x}'(\nu) < 0$ for any $\nu \in (0,1]$, so φ_x is decreasing on $(0,1]$. Hence,

$$\forall \nu \in (0,1), \quad \varphi_x(\nu) > \varphi_x(1) = -J_1(x)J_{-1}(x).$$

From [35, (2), p.15], we have $J_{-1}(x) = -J_1(x)$. We deduce that

$$\forall \nu \in (0,1), \quad \varphi_x(\nu) > \varphi_x(1) = J_1(x)^2 \geqslant 0,$$

which concludes the proof. □

LEMMA 1.7. *[35, Subsection 3.12]*
Let $\nu \in \mathbb{R} \setminus \mathbb{Z}$. The Wronskian of J_ν and $J_{-\nu}$, denoted as $W(J_\nu, J_{-\nu})$, satisfies

$$W(J_\nu, J_{-\nu})(x) = -\frac{2\sin(\nu\pi)}{\pi x}, \quad \forall x > 0.$$

Notably, for $\nu \in \mathbb{R} \setminus \mathbb{Z}$, J_ν and $J_{-\nu}$ form a basis of solution for (1.7)

LEMMA 1.8. *Let $\nu \in (0,1)$. The real positive zeros of J_ν and $J_{-\nu}$ are strictly interlaced, and*

$$j_{-\nu,1} < j_{\nu,1} < j_{-\nu,2} < j_{\nu,2} < \dots < j_{-\nu,n} < j_{\nu,n} < \dots$$

Proof of Lemma 1.8. Since $\nu \in (0,1)$, we have $\nu > -1$ and $-\nu > -1$, and we know from [24, Theorem 3 and Remarks] that the positive real zeros of J_ν are interlaced. Thus, it suffices to show that $j_{-\nu,1} < j_{\nu,1}$. To prove this, we will analyze the function $g : x \mapsto \frac{J_\nu(x)}{J_{-\nu}(x)}$ and show that g is increasing on $(0, j_{-\nu,1})$, and that $\lim_{x\to 0^+} g(x) = 0$. Consequently, we will have $g(x) > 0$ on $(0, j_{-\nu,1})$, which implies that $j_{-\nu,1} < j_{\nu,1}$.

$$g'(x) = \frac{{J_\nu}'(x)J_{-\nu}(x) - J_\nu(x){J_{-\nu}}'(x)}{{J_{-\nu}}^2(x)} = \frac{-W(J_\nu, J_{-\nu})(x)}{{J_{-\nu}}^2(x)} = \frac{2\sin(\nu\pi)}{\pi x {J_{-\nu}}^2(x)} > 0, \quad \forall x \in (0, j_{-\nu,1}).$$

We have shown that g is increasing on $(0, j_{-\nu,1})$. Finally, since $\lim_{x\to 0^+} J_\nu(x) = 0$ and $\lim_{x\to 0^+} J_{-\nu}(x) = +\infty$, it follows that $\lim_{x\to 0^+} g(x) = 0$. We can conclude as stated above. □

LEMMA 1.9. *[35, (2), p. 82]*

$$\forall \nu \in \mathbb{R}, \quad {J_\nu}' = \frac{1}{2}\left(J_{\nu-1} - J_{\nu+1}\right).$$

LEMMA 1.10. *[35, (1), p.199]*

$$J_\nu(x) \underset{x\to+\infty}{=} \sqrt{\frac{2}{\pi x}}\left(\cos\omega \sum_{k=0}^{n} (-1)^k \frac{a_{2k}(\nu)}{x^{2k}} - \sin\omega \sum_{k=0}^{n} (-1)^k \frac{a_{2k+1}(\nu)}{x^{2k+1}} + \mathcal{O}\left(\frac{1}{x^{2n+2}}\right)\right), \quad \forall \nu \in \mathbb{R},\ \forall n \in \mathbb{N},$$

where $\omega = x - \nu\frac{\pi}{2} - \frac{\pi}{4}$, and $a_0 = 1$, $a_k = \dfrac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\dots(4\nu^2 - (2k-1)^2)}{k! 8^k}$ for $k \in \mathbb{N} \setminus \{0\}$.

COROLLARY 1.11. *Let $\nu \in \mathbb{R}$.*

$$J_\nu(x) \underset{x\to+\infty}{=} \mathcal{O}\left(\frac{1}{x^{1/2}}\right), \quad {J_\nu}'(x) \underset{x\to+\infty}{=} \mathcal{O}\left(\frac{1}{x^{1/2}}\right).$$

Proof of Corollary 1.11. The first asymptotic behaviour is directly obtained from Lemma 1.10. The second asymptotic behavior follows immediately from Lemma 1.9 and the first asymptotic behavior. □

LEMMA 1.12. *We have*

$$j_{\nu,n} \underset{n\to+\infty}{=} \pi\left(n+\frac{\nu}{2}-\frac{1}{4}\right)+\mathcal{O}\left(\frac{1}{n}\right), \quad j_{-\nu,n} \underset{n\to+\infty}{=} \pi\left(n-\frac{\nu}{2}-\frac{1}{4}\right)+\mathcal{O}\left(\frac{1}{n}\right).$$

Proof of Lemma 1.12. Since $\nu\notin\mathbb{Z}$, we have $J_{-\nu}=J_\nu\cos(\nu\pi)-Y_\nu\sin(\nu\pi)$, with $Y_\nu=\dfrac{J_\nu\cos(\nu\pi)-J_{-\nu}}{\sin(\nu\pi)}$ the Bessel function of the second kind. Thus, using [35, end of p. 506], we obtain the desired asymptotic expansions for $j_{\nu,n}$ and $j_{-\nu,n}$ directly. $\square$

LEMMA 1.13. *Let* $\nu\in(0,1)$.

$$\int_0^1 xJ_{\pm\nu}(ax)^2\mathrm{d}x=\frac{1}{2}\left(\left(1-\frac{\nu^2}{a^2}\right)J_{\pm\nu}(a)^2+{J_{\pm\nu}}'(a)^2\right), \qquad \forall a>0, \tag{1.9.a}$$

$$\int_0^1 xJ_\nu(ax)J_{-\nu}(ax)\mathrm{d}x=\frac{1}{2}\left(\left(1-\frac{\nu^2}{a^2}\right)J_\nu(a)J_{-\nu}(a)+{J_\nu}'(a){J_{-\nu}}'(a)\right)+\frac{\nu\sin(\nu\pi)}{\pi a^2}, \qquad \forall a>0, \tag{1.9.b}$$

$$\begin{aligned}\int_\alpha^\beta xJ_{\pm\nu}(ax)^2\mathrm{d}x=&\frac{1}{2}\beta^2\left(\left(1-\frac{\nu^2}{a^2\beta^2}\right)J_{\pm\nu}(a\beta)^2+{J_{\pm\nu}}'(a\beta)^2\right) \qquad \forall\alpha,\beta>0,\ \forall a>0,\\ &-\frac{1}{2}\alpha^2\left(\left(1-\frac{\nu^2}{a^2\alpha^2}\right)J_{\pm\nu}(a\alpha)^2+{J_{\pm\nu}}'(a\alpha)^2\right),\end{aligned} \tag{1.9.c}$$

$$\begin{aligned}\int_\alpha^\beta xJ_\nu(ax)J_{-\nu}(ax)\mathrm{d}x=&\frac{1}{2}\beta^2\left(\left(1-\frac{\nu^2}{a^2\beta^2}\right)J_\nu(a\beta)J_{-\nu}(a\beta)+{J_\nu}'(a\beta){J_{-\nu}}'(a\beta)\right) \qquad \forall\alpha,\beta>0,\ \forall a>0,\\ &-\frac{1}{2}\alpha^2\left(\left(1-\frac{\nu^2}{a^2\alpha^2}\right)J_\nu(a\alpha)J_{-\nu}(a\alpha)+{J_\nu}'(a\alpha){J_{-\nu}}'(a\alpha)\right).\end{aligned} \tag{1.9.d}$$

Proof of Lemma 1.13. **Proof of (1.9.a)** [20, (4), p. 255] states that for any $a>0$,

$$\int_0^1 xJ_{\pm\nu}(ax)^2\mathrm{d}x=\frac{1}{2}\left(\left(1-\frac{\nu^2}{a^2}\right)J_{\pm\nu}(a)^2+{J_{\pm\nu}}'(a)^2\right)-\lim_{x\to0^+}\frac{1}{2}x^2\left(\left(1-\frac{\nu^2}{a^2x^2}\right)J_{\pm\nu}(ax)^2+{J_{\pm\nu}}'(ax)^2\right).$$

Moreover, by (1.8),

$$J_{\pm\nu}(ax)\underset{x\to0^+}{\sim}\frac{1}{\Gamma(\pm\nu+1)}\left(\frac{ax}{2}\right)^{\pm\nu},$$

so

$$xJ_{\pm\nu}(ax)\underset{x\to0^+}{\sim}\frac{1}{\Gamma(\pm\nu+1)}\left(\frac{a}{2}\right)^{\pm\nu}x^{\pm\nu+1}\underset{x\to0^+}{\longrightarrow}0 \quad \text{because } \nu\in(0,1).$$

Therefore, $\lim_{x\to0^+}x^2J_{\pm\nu}(ax)^2=0$, and we have

$$\lim_{x\to0^+}\frac{1}{2}x^2\left(\left(1-\frac{\nu^2}{a^2x^2}\right)J_{\pm\nu}(ax)^2+{J_{\pm\nu}}'(ax)^2\right)=\lim_{x\to0^+}\frac{1}{2a^2}\left(-\nu^2J_{\pm\nu}(ax)^2+(ax)^2{J_{\pm\nu}}'(ax)^2\right).$$

Therefore, to conclude the proof of (1.9.a), we only need to prove that

$$\lim_{x\to0^+}x^2{J_{\pm\nu}}'(x)^2-\nu^2J_{\pm\nu}(x)^2=0.$$

By examining the series expansion of ${J_{\pm\nu}}'$ issued from (1.8), we have

$$ {J_{\pm\nu}}'(x) \underset{x\to 0^+}{=} \frac{\pm\nu}{\Gamma(\pm\nu+1)}\frac{1}{2^{\pm\nu}}x^{\pm\nu-1} + \mathcal{O}(x^{1\pm\nu}). $$

Therefore, we have

$$ x^2 {J_{\pm\nu}}'(x)^2 \underset{x\to 0^+}{=} \left[\frac{\nu}{\Gamma(\pm\nu+1)}\frac{1}{2^{\pm\nu}}\right]^2 x^{\pm 2\nu} + \mathcal{O}(x^{2\pm 2\nu}) \quad \text{because } \nu\in(0,1), \tag{1.10} $$

and by examining the series expansion of $J_{\pm\nu}$ given in (1.8), we have

$$ \nu J_{\pm\nu}(x) \underset{x\to 0^+}{=} \frac{\nu}{\Gamma(\pm\nu+1)}\frac{1}{2^{\pm\nu}}x^{\pm\nu} + \mathcal{O}(x^{2\pm\nu}). $$

so

$$ \nu^2 J_{\pm\nu}(x)^2 \underset{x\to 0^+}{=} \left[\frac{\nu}{\Gamma(\pm\nu+1)}\frac{1}{2^{\pm\nu}}\right]^2 x^{\pm 2\nu} + \mathcal{O}(x^{2\pm 2\nu}) \quad \text{because } \nu\in(0,1). \tag{1.11} $$

Finally, from (1.10) and (1.11), we obtain

$$ x^2 {J_{\pm\nu}}'(x)^2 - \nu^2 J_{\pm\nu}(x)^2 \underset{x\to 0^+}{=} \mathcal{O}(x^{2\pm 2\nu}) \underset{x\to 0^+}{\longrightarrow} 0 \quad \text{because } \nu\in(0,1), $$

which concludes the proof of (1.9.a).

Proof of (1.9.b) Let $A,B,C,D\in\mathbb{R}$. We define $C_\nu := AJ_\nu + BY_\nu$ and $D_\nu := CJ_\nu + DY_\nu$, with $Y_\nu = \dfrac{J_\nu\cos(\nu\pi) - J_{-\nu}}{\sin(\nu\pi)}$ the second kind Bessel function. [20, (3), p. 254] states that for any $a>0$,

$$ \begin{aligned} \int_0^1 xC_\nu(ax)D_\nu(ax)\mathrm{d}x =& \frac{1}{4}\left(2C_\nu(a)D_\nu(a) - C_{\nu-1}(a)D_{\nu+1}(a) - C_{\nu+1}(a)D_{\nu-1}(a)\right) \\ & - \lim_{x\to 0^+}\frac{1}{4}x^2\left(2C_\nu(ax)D_\nu(ax) - C_{\nu-1}(ax)D_{\nu+1}(ax) - C_{\nu+1}(ax)D_{\nu-1}(ax)\right). \end{aligned} $$

We choose $A=1$, $B=0$, $C=\cos(\nu\pi)$, and $D=-\sin(\nu\pi)$. Thus, we have $C_\nu = J_\nu$ and $D_\nu = J_{-\nu}$. One needs to be careful when computing $D_{\nu+1}$ and $D_{\nu-1}$. We have

$$ D_{\nu+1} = CJ_{\nu+1} + D\frac{J_{\nu+1}\cos((\nu+1)\pi) - J_{-\nu-1}}{\sin((\nu+1)\pi)} = \cos(\nu\pi)J_{\nu+1} - \sin(\nu\pi)\frac{-J_{\nu+1}\cos(\nu\pi) - J_{-\nu-1}}{-\sin(\nu\pi)} = -J_{-\nu-1}. $$

Following the same kind of computations on $D_{\nu-1}$ gives $D_{\nu-1} = -J_{-\nu+1}$. Therefore, we obtain

$$ \begin{aligned} \int_0^1 xJ_\nu(ax)J_{-\nu}(ax)\mathrm{d}x =& \frac{1}{4}\left(2J_\nu(a)J_{-\nu}(a) + J_{\nu-1}(a)J_{-\nu-1}(a) + J_{\nu+1}(a)J_{-\nu+1}(a)\right) \\ & - \lim_{x\to 0^+}\frac{1}{4}x^2\left(2J_\nu(ax)J_{-\nu}(ax) + J_{\nu-1}(ax)J_{-\nu-1}(ax) + J_{\nu+1}(ax)J_{-\nu+1}(ax)\right). \end{aligned} $$

First, we compute the limit term. By examining the series expansion of J_ν, $J_{-\nu}$, $J_{\nu-1}$, $J_{-\nu+1}$, $J_{\nu+1}$, and $J_{-\nu-1}$ given in (1.8), we find that

$$ \begin{aligned} x^2 J_\nu(ax)J_{-\nu}(ax) &\underset{x\to 0^+}{\sim} x^2\frac{1}{\Gamma(\nu+1)}\left(\frac{ax}{2}\right)^{\nu}\frac{1}{\Gamma(-\nu+1)}\left(\frac{ax}{2}\right)^{-\nu} = \frac{1}{\Gamma(\nu+1)}\frac{1}{\Gamma(-\nu+1)}x^2 \underset{x\to 0^+}{\longrightarrow} 0, \\ x^2 J_{\nu+1}(ax)J_{-\nu+1}(ax) &\underset{x\to 0^+}{\sim} x^2\frac{1}{\Gamma(\nu+2)}\left(\frac{ax}{2}\right)^{\nu+1}\frac{1}{\Gamma(-\nu+2)}\left(\frac{ax}{2}\right)^{-\nu+1} = \frac{(ax/2)^2}{\Gamma(\nu+2)\Gamma(-\nu+2)}x^2 \underset{x\to 0^+}{\longrightarrow} 0, \\ x^2 J_{\nu-1}(ax)J_{-\nu-1}(ax) &\underset{x\to 0^+}{\sim} x^2\frac{1}{\Gamma(\nu)}\left(\frac{ax}{2}\right)^{\nu-1}\frac{1}{\Gamma(-\nu)}\left(\frac{ax}{2}\right)^{-\nu-1} = \frac{1}{\Gamma(\nu)\Gamma(-\nu)}\frac{4}{a^2} = \frac{-\nu}{\Gamma(\nu)\Gamma(1-\nu)}\frac{4}{a^2}, \end{aligned} $$

where the argument in the Euler Gamma function is always valid because $\nu \in (0,1)$. We recall that the complement formula for the Euler Gamma function, which is valid since $\nu \in (0,1)$, gives

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin(\nu\pi)}.$$

Therefore, we conclude that

$$\lim_{x\to 0^+} \frac{1}{4}x^2 \left(2J_\nu(ax)J_{-\nu}(ax) + J_{\nu-1}(ax)J_{-\nu-1}(ax) + J_{\nu+1}(ax)J_{-\nu+1}(ax)\right) = -\frac{\nu\sin(\nu\pi)}{\pi a^2}.$$

Thus, to conclude the proof, we need to show that

$$\frac{1}{4}\left(2J_\nu(a)J_{-\nu}(a) + J_{\nu-1}(a)J_{-\nu-1}(a) + J_{\nu+1}(a)J_{-\nu+1}(a)\right) = \frac{1}{2}\left(\left(1-\frac{\nu^2}{a^2}\right)J_\nu(a)J_{-\nu}(a) + {J_\nu}'(a){J_{-\nu}}'(a)\right). \tag{1.12}$$

To prove this, we need to use the following identities

$$\begin{aligned} \forall \nu \in (0,1), \forall x > 0, \quad J_{\nu+1}(x) &= \frac{\nu}{x}J_\nu(x) - {J_\nu}'(x), \quad & J_{-\nu+1}(x) &= \frac{-\nu}{x}J_{-\nu}(x) - {J_{-\nu}}'(x), \\ J_{\nu-1}(x) &= \frac{\nu}{x}J_\nu(x) + {J_\nu}'(x), \quad & J_{-\nu-1}(x) &= \frac{-\nu}{x}J_{-\nu}(x) + {J_{-\nu}}'(x), \end{aligned}$$

which are stated in [35, Section 3.2, (3) and (4), p45]. Substituting $J_{\nu+1}(a)$, $J_{-\nu+1}(a)$, $J_{\nu-1}(a)$, and $J_{-\nu-1}(a)$ into $\frac{1}{4}\left(2J_\nu(a)J_{-\nu}(a) - J_{\nu-1}(a)J_{-\nu+1}(a) - J_{\nu+1}(a)J_{-\nu-1}(a)\right)$ directly yields (1.12) and thus concludes the proof of (1.9.b).

Proof of (1.9.c) and (1.9.d). The result in (1.9.c) is directly obtained from [20, Section 11.2, (4)]. The result in (1.9.d) is obtained from [20, Section 11.2, (3)], using a similar approach as in the proof of (1.9.b) to substitute $J_{\nu+1}$, $J_{-\nu+1}$, $J_{\nu-1}$, and $J_{-\nu-1}$, which gives the desired result. □

2 Diagonalization

From now on, we will always assume that $\nu \in (0,1)$. Remind that c_ν is defined by (1.2). We know that A_ν is self-adjoint and non-negative by (1.6). Therefore, we know that the spectrum of A_ν is a subset of $\mathbb{R}^+$, $\sigma(A_\nu) \subset [0,+\infty)$. Our goal here is to prove that A_ν has a Hilbert basis of eigenfunctions, and to give a rather precise description of the spectrum, together with some useful estimates on the eigenfunctions.

2.1 Complete inventory of eigenvalues and eigenfunctions

First of all, let us investigate the kernel of A_ν.

PROPOSITION 2.1.

$$\begin{aligned} \mathrm{Ker}(A_\nu) &= \mathcal{F}_s^\nu \cap D(A_\nu) \\ &= \mathrm{Span}\left(x \mapsto |x|^{\nu+\frac{1}{2}} - |x|^{-\nu+\frac{1}{2}}\right). \end{aligned}$$

$\lambda_0 = 0$ is therefore the smallest eigenvalue of A_ν, of multiplicity 1.

Proof of Proposition 2.1. First, thanks to (1.4), we directly obtain $\mathcal{F}_s^\nu \cap D(A_\nu) \subset \mathrm{Ker}(A_\nu)$.
Now, let $f \in D(A_\nu)$ be such that $A_\nu f = 0$. Using (1.6), we obtain $||\partial_x f_r||_{L^2(-1,1)} = 0$.
Since $\partial_x f_r \in H^1(-1,1) \hookrightarrow C^0(-1,1)$ and $f_r'(0) = 0$, we obtain $f_r = 0$. Therefore, $f = f_s \in \mathcal{F}_s^\nu \cap D(A_\nu)$.
Thus, $\mathrm{Ker}(A_\nu) = \mathcal{F}_s^\nu \cap D(A_\nu)$.
Finally, it is easy to check that $\mathcal{F}_s^\nu \cap D(A_\nu) = \mathrm{Span}\left(x \mapsto |x|^{\nu+\frac{1}{2}} - |x|^{-\nu+\frac{1}{2}}\right)$, using the transmission conditions in the definition of A_ν together with the Dirichlet boundary conditions at ± 1. □

Let $E > 0$. Now, using Bessel functions, we want to find $f \in D(A_\nu)$ such that $A_\nu f = Ef$. Let us first investigate the differential equation $A_\nu f = f$, without taking into account the transmission and boundary conditions.

PROPOSITION 2.2. *Let $E > 0$. A_ν, let $f \in C^\infty((-1,0) \cup (0,1))$. Then, the function f satisfies*

$$\forall x \in (-1,0) \cup (0,1), \quad \left(-\frac{d^2}{dx^2} + \frac{\nu^2 - \frac{1}{4}}{x^2}\right) f(x) = Ef(x), \tag{2.1}$$

if and only if f is of the form

$$f(x) = \begin{cases} a_\nu^- \sqrt{-x} J_\nu(-\sqrt{E}x) + a_{-\nu}^- \sqrt{-x} J_{-\nu}(-\sqrt{E}x), & \forall x \in (-1,0), \\ a_\nu^+ \sqrt{x} J_\nu(\sqrt{E}x) + a_{-\nu}^+ \sqrt{x} J_{-\nu}(\sqrt{E}x), & \forall x \in (0,1), \end{cases} \tag{2.2}$$

with $a_\nu^-, a_{-\nu}^-, a_\nu^+, a_{-\nu}^+ \in \mathbb{R}$.

Therefore, a function f is in the eigenspace associated with the eigenvalue E if and only if $f \in D(A_\nu)$ and f is of the form (2.2).

Proof of Proposition 2.2.
On (0, 1). We consider the function $\psi_{\nu,E}^+(x) = \sqrt{x} J_\nu(\sqrt{E}x)$, defined on $(0,1)$. Taking the second derivative, we have

$$\psi_{\nu,E}^{+''}(x) = -\frac{1}{4x^{3/2}} J_\nu(\sqrt{E}x) + \frac{1}{\sqrt{x}} \sqrt{E} J_\nu'(\sqrt{E}x) + E\sqrt{x} J_\nu''(\sqrt{E}x).$$

Next, we use the Bessel equation (1.7) satisfied by J_ν at $\sqrt{E}x > 0$, which gives

$$Ex^2 J_\nu''(\sqrt{E}x) + x\sqrt{E} J_\nu'(\sqrt{E}x) + Ex^2 J_\nu(\sqrt{E}x) = \nu^2 J_\nu(\sqrt{E}x).$$

Therefore, we have, for all $x \in (0,1)$,

$$\begin{aligned} \left(-\frac{d^2}{dx^2} + \frac{\nu^2 - \frac{1}{4}}{x^2}\right) \psi_{\nu,E}^+(x) &= \frac{1}{4x^{3/2}} J_\nu(\sqrt{E}x) - \frac{1}{\sqrt{x}} \sqrt{E} J_\nu'(\sqrt{E}x) - E\sqrt{x} J_\nu''(\sqrt{E}x) \\ &\quad - \frac{1}{4x^2} \sqrt{x} J_\nu(\sqrt{E}x) + \frac{\sqrt{x}}{x^2} (Ex^2 J_\nu''(\sqrt{E}x) + x\sqrt{E} J_\nu'(\sqrt{E}x) + Ex^2 J_\nu(\sqrt{E}x)) \\ &= E\psi_{\nu,E}^+(x). \end{aligned}$$

Since $\nu \in \mathbb{R} \setminus \mathbb{Z}$, we have that $(J_\nu, J_{-\nu})$ is a basis of solutions of (1.7) by Lemma 1.7. Therefore, by posing $\psi_{-\nu,E}^+(x) = \sqrt{x} J_{-\nu}(\sqrt{E}x)$ on $(0,1)$, we also have that for all $x \in (0,1)$,

$$\left(-\frac{d^2}{dx^2} + \frac{\nu^2 - \frac{1}{4}}{x^2}\right) \psi_{-\nu,E}^+(x) = E\psi_{-\nu,E}^+(x).$$

On (-1, 0). We follow the same computations as on $(0,1)$ and we observe that they are the same.

We proved that any function of the form (2.2) satisfies (2.1). We have found all the solutions to (2.1) since the total solution space is of dimension 4, consisting of two independent solutions on $(0,1)$ and two independent solutions on $(-1,0)$. This matches the expected structure, as the equation is second-order and has a singularity at $x = 0$, which separates the two intervals. $\square$

Now, we want to find conditions on $a_{-\nu}^-$, a_ν^-, $a_{-\nu}^+$, and a_ν^+ to ensure that the function f defined in (2.2) is in $D(A_\nu)$.

PROPOSITION 2.3. *Let $\nu \in (0,1)$. Any function of the form (2.2) is in $\tilde{H}_0^2(-1,1) \oplus \mathcal{F}_s^\nu$.*

Proof of Proposition 2.3. By examining the power series expansion (1.8) of the Bessel functions J_ν and isolating the term for $k=0$, we have

$$\sqrt{x}J_\nu(\sqrt{E}x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^\nu |x|^{\nu+\frac{1}{2}} + \sum_{k=1}^\infty \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2k+\nu} x^{2k+\nu+\frac{1}{2}}, \quad \forall x\in(0,1),$$

$$\sqrt{x}J_{-\nu}(\sqrt{E}x) = \frac{1}{\Gamma(-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{-\nu} |x|^{-\nu+\frac{1}{2}} + \sum_{k=1}^\infty \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2k-\nu} x^{2k-\nu+\frac{1}{2}}, \quad \forall x\in(0,1).$$

We introduce

$$f_r(x) := \begin{cases} a_\nu^- \sum_{k=1}^\infty \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2k+\nu} (-x)^{2k+\nu+\frac{1}{2}} \\ +a_{-\nu}^- \sum_{k=1}^\infty \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2k-\nu} (-x)^{2k-\nu+\frac{1}{2}}, \quad \forall x\in(-1,0), \\ a_\nu^+ \sum_{k=1}^\infty \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2k+\nu} x^{2k+\nu+\frac{1}{2}} \\ +a_{-\nu}^+ \sum_{k=1}^\infty \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2k-\nu} x^{2k-\nu+\frac{1}{2}}, \qquad \forall x\in(0,1), \end{cases}$$

$$f_s(x) := \begin{cases} a_\nu^- \frac{1}{\Gamma(\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^\nu |x|^{\nu+\frac{1}{2}} + a_{-\nu}^- \frac{1}{\Gamma(-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{-\nu} |x|^{-\nu+\frac{1}{2}}, \qquad \forall x\in(-1,0), \\ a_\nu^+ \frac{1}{\Gamma(\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^\nu |x|^{\nu+\frac{1}{2}} + a_{-\nu}^+ \frac{1}{\Gamma(-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{-\nu} |x|^{-\nu+\frac{1}{2}}, \qquad \forall x\in(0,1). \end{cases} \tag{2.3}$$

Thus, we have $f=f_s+f_r$. Since $\nu\in(0,1)$, it is clear that $f_s\in L^2(-1,1)$. By examining (2.3), we see that f_s has the right form with

$$\begin{aligned} c_1^+ &= a_\nu^+ \frac{1}{\Gamma(\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^\nu, \quad c_2^+ = a_{-\nu}^+ \frac{1}{\Gamma(-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{-\nu}, \\ c_1^- &= a_\nu^- \frac{1}{\Gamma(\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^\nu, \quad c_2^- = a_{-\nu}^- \frac{1}{\Gamma(-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{-\nu}. \end{aligned} \tag{2.4}$$

Therefore, $f_s\in\mathcal{F}_s^\nu$. Finally, we need to verify that $f_r\in\tilde{H}_0^2(-1,1)$. We introduce

$$g_r(x) = \begin{cases} a_\nu^- \sum_{k=1}^\infty \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2k+\nu} (-x)^{2k+\nu+\frac{1}{2}} \\ +a_{-\nu}^- \sum_{k=2}^\infty \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2k-\nu} (-x)^{2k-\nu+\frac{1}{2}}, \quad \forall x\in(-1,0), \\ a_\nu^+ \sum_{k=1}^\infty \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2k+\nu} x^{2k+\nu+\frac{1}{2}} \\ +a_{-\nu}^+ \sum_{k=2}^\infty \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2k-\nu} x^{2k-\nu+\frac{1}{2}}, \qquad \forall x\in(0,1), \end{cases}$$

$$h_r(x) = \begin{cases} a_{-\nu}^- \frac{-1}{\Gamma(1-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2-\nu} (-x)^{2-\nu+\frac{1}{2}}, \quad \forall x\in(-1,0), \\ a_{-\nu}^+ \frac{-1}{\Gamma(1-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{2-\nu} x^{2-\nu+\frac{1}{2}}, \qquad \forall x\in(0,1). \end{cases}$$

Thus, $f_r = g_r + h_r$. Since $\nu\in(0,1)$, we have $g_r\in C^2([-1,1])\subset H^2(-1,1)$ and $g_r(0)=g_r'(0)=0$, therefore, $g_r\in\tilde{H}_0^2(-1,1)$. We also have $h_r\in C^1([-1,1])$ and $h_r(0)=h_r'(0)=0$. We only need to check that $h_r'' \in L^2(-1,1)$. Since for $\nu > \frac{1}{2}$, h_r' is only differentiable on $(-1,0)\cup(0,1)$ and not at 0, we need to compute h_r'' using the derivative in the distributional sense. Let $\phi \in C_c^\infty(-1,1)$,

$$\begin{aligned}\langle h_r'', \phi\rangle &= -\int_{-1}^{1} h_r'\phi' = -\int_{-1}^{0} h_r'\phi' - \int_{0}^{1} h_r'\phi' = \int_{-1}^{0} h_r''\phi - [h_r'\phi]_{-1}^{0} + \int_{0}^{1} h_r''\phi - [h_r'\phi]_{0}^{1} \\ &= \int_{-1}^{0} h_r''\phi + \int_{0}^{1} h_r''\phi \quad \text{since } h_r'(0) = 0.\end{aligned}$$

Therefore, h_r'' is obtained by examining the derivative of h_r' on $(-1,0)$ and $(0,1)$. We find

$$h_r''(x) = \begin{cases} a_{-\nu}^{-} \frac{-1}{\Gamma(1-\nu+1)} \left(\frac{\sqrt{E}}{2}\right)^{2-\nu} (2-\nu+\frac{1}{2})(1-\nu+\frac{1}{2})(-x)^{-\nu+\frac{1}{2}}, & \forall x \in (-1,0), \\ a_{-\nu}^{+} \frac{-1}{\Gamma(1-\nu+1)} \left(\frac{\sqrt{E}}{2}\right)^{2-\nu} (2-\nu+\frac{1}{2})(1-\nu+\frac{1}{2})x^{-\nu+\frac{1}{2}}, & \forall x \in (0,1), \end{cases}$$

which is in $L^2(-1,1)$ since $\nu \in (0,1)$. Thus, $h_r \in \tilde{H}_0^2(-1,1)$ and $g_r \in \tilde{H}_0^2(-1,1)$. Therefore, $f_r = g_r + h_r \in \tilde{H}_0^2(-1,1)$. We have shown that $f = f_s + f_r \in \tilde{H}_0^2(-1,1) \oplus \mathcal{F}_s^\nu$. ☐

PROPOSITION 2.4. *Let $\nu \in (0,1)$, let $E > 0$. The value E is a positive eigenvalue of A_ν if and only if*

$$J_\nu(\sqrt{E}) \left(\frac{\sqrt{E}}{2}\right)^{-2\nu} \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} = J_{-\nu}(\sqrt{E}), \tag{2.5}$$

or

$$J_\nu(\sqrt{E}) \left(\frac{\sqrt{E}}{2}\right)^{-2\nu} \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \frac{1-2\nu}{1+2\nu} = J_{-\nu}(\sqrt{E}). \tag{2.6}$$

Furthermore, $J_\nu(\sqrt{E}) \neq 0$, and

- *if (2.5) is satisfied, the eigenspace associated with the eigenvalue E is*

$$\operatorname{Span}\left(x \mapsto \begin{cases} -\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \left(\frac{\sqrt{E}}{2}\right)^{-2\nu} \sqrt{-x} J_\nu(-\sqrt{E}x) + \sqrt{-x} J_{-\nu}(-\sqrt{E}x) & \text{on } (-1,0) \\ -\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \left(\frac{\sqrt{E}}{2}\right)^{-2\nu} \sqrt{x} J_\nu(\sqrt{E}x) + \sqrt{x} J_{-\nu}(\sqrt{E}x) & \text{on } (0,1) \end{cases} \right), \tag{2.7}$$

- *if (2.6) is satisfied, the eigenspace associated with the eigenvalue E is*

$$\operatorname{Span}\left(x \mapsto \begin{cases} \frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \left(\frac{\sqrt{E}}{2}\right)^{-2\nu} \sqrt{-x} J_\nu(-\sqrt{E}x) - \sqrt{-x} J_{-\nu}(-\sqrt{E}x) & \text{on } (-1,0) \\ -\frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \left(\frac{\sqrt{E}}{2}\right)^{-2\nu} \sqrt{x} J_\nu(\sqrt{E}x) + \sqrt{x} J_{-\nu}(\sqrt{E}x) & \text{on } (0,1) \end{cases} \right), \tag{2.8}$$

which are both of dimension 1.

Proof of Proposition 2.4. We want to find conditions that allow candidate eigenfunctions written as (2.2) to be in $D(A_\nu)$. Referring to the definition of $D(A_\nu)$ in (1.5), Proposition 2.3, and (2.4), the function f defined in (2.2) is in $D(A_\nu)$ if and only if

$$(a_\nu^+ + a_\nu^-)\frac{1}{\Gamma(\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{\nu} + (a_{-\nu}^+ + a_{-\nu}^-)\frac{1}{\Gamma(-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{-\nu} = 0, \tag{2.9}$$

$$(a_\nu^+ - a_\nu^-)(\nu + \frac{1}{2})\frac{1}{\Gamma(\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{\nu} + (a_{-\nu}^+ - a_{-\nu}^-)(-\nu + \frac{1}{2})\frac{1}{\Gamma(-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{-\nu} = 0, \tag{2.10}$$

$$f(1) = a_\nu^+ J_\nu(\sqrt{E}) + a_{-\nu}^+ J_{-\nu}(\sqrt{E}) = 0, \tag{2.11}$$

$$f(-1) = a_\nu^- J_\nu(\sqrt{E}) + a_{-\nu}^- J_{-\nu}(\sqrt{E}) = 0. \tag{2.12}$$

We introduce the useful quantity

$$D_\nu := \frac{1}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\frac{\sqrt{E}}{2}\right)^{-2\nu}. \tag{2.13}$$

$\frac{1}{2}\left[\Gamma(\nu+1)\left(\frac{\sqrt{E}}{2}\right)^{-\nu} \times (2.9) + \frac{\Gamma(\nu+1)}{\nu+\frac{1}{2}}\left(\frac{\sqrt{E}}{2}\right)^{-\nu} \times (2.10)\right]$ gives

$$a_\nu^+ = -D_\nu(a_{-\nu}^+ + 2\nu a_{-\nu}^-), \tag{2.14}$$

and $\frac{1}{2}\left[\Gamma(\nu+1)\left(\frac{\sqrt{E}}{2}\right)^{-\nu} \times (2.9) - \frac{\Gamma(\nu+1)}{\nu+\frac{1}{2}}\left(\frac{\sqrt{E}}{2}\right)^{-\nu} \times (2.10)\right]$ gives

$$a_\nu^- = -D_\nu(a_{-\nu}^- + 2\nu a_{-\nu}^+). \tag{2.15}$$

Reason by contradiction and assume that $J_\nu(\sqrt{E}) = 0$. Then, (2.11) and (2.12) become $a_{-\nu}^+ J_{-\nu}(\sqrt{E}) = 0$ and $a_{-\nu}^- J_{-\nu}(\sqrt{E}) = 0$. Assume one more time by contradiction that $J_{-\nu}(\sqrt{E}) \neq 0$. Then, we must have $a_{-\nu}^+ = a_{-\nu}^- = 0$. Using (2.14) and (2.15), we also must have $a_\nu^+ = a_\nu^- = 0$. Thus, $f = 0$. This is not possible since we are looking for eigenfunctions. Therefore, we must have $J_{-\nu}(\sqrt{E}) = 0$, which is not possible since Lemma 1.8 shows that the positive zeros of J_ν and $J_{-\nu}$ are strictly interlaced.
Therefore, $J_\nu(\sqrt{E}) \neq 0$. We substitute a_ν^- in (2.12) using (2.15). This gives

$$a_{-\nu}^+ J_\nu(\sqrt{E}) 2\nu D_\nu = a_{-\nu}^- \left(J_{-\nu}(\sqrt{E}) - J_\nu(\sqrt{E})D_\nu\right).$$

Since $J_\nu(\sqrt{E}) \neq 0$, we have

$$a_{-\nu}^+ = a_{-\nu}^- \frac{J_{-\nu}(\sqrt{E}) - J_\nu(\sqrt{E})D_\nu}{J_\nu(\sqrt{E})2\nu D_\nu}. \tag{2.16}$$

Substituting a_ν^+ in (2.11) using (2.14) gives

$$a_{-\nu}^- J_\nu(\sqrt{E}) 2\nu D_\nu = a_{-\nu}^+ \left(J_{-\nu}(\sqrt{E}) - J_\nu(\sqrt{E})D_\nu\right).$$

We now substitute $a_{-\nu}^+$ in the previous equality, using (2.16). This gives

$$a_{-\nu}^- J_\nu(\sqrt{E}) 2\nu D_\nu = a_{-\nu}^- \frac{J_{-\nu}(\sqrt{E}) - J_\nu(\sqrt{E})D_\nu}{J_\nu(\sqrt{E})2\nu D_\nu}\left(J_{-\nu}(\sqrt{E}) - J_\nu(\sqrt{E})D_\nu\right),$$

which can be rewritten as

$$a_{-\nu}^- \left(J_\nu(\sqrt{E})2\nu D_\nu\right)^2 = a_{-\nu}^- \left(J_{-\nu}(\sqrt{E}) - J_\nu(\sqrt{E})D_\nu\right)^2.$$

We cannot have $a_{-\nu}^- = 0$ because it would lead to $a_{-\nu}^+ = 0$ using (2.16), and then $a_\nu^- = a_\nu^+ = 0$ using (2.14) and (2.15). Therefore, we must have

$$\left(J_\nu(\sqrt{E})2\nu D_\nu\right)^2 = \left(J_{-\nu}(\sqrt{E}) - J_\nu(\sqrt{E})D_\nu\right)^2. \tag{2.17}$$

To summarize, we proved that if E is an eigenvalue of A_ν for a function f of the form (2.2), then we must have (2.17), and we must have $a^+_{-\nu}$ satisfying (2.16), while a^+_ν and a^-_ν must satisfy (2.14) and (2.15) respectively, with $a^-_{-\nu}$ chosen freely in $\mathbb{R}$.
On the other hand, if (2.17) is satisfied, we can choose any function from the set

$$\left\{\begin{matrix} f \text{ of the form (2.2) such that : } \quad a^-_{-\nu} \in \mathbb{R}, \quad a^+_{-\nu} = a^-_{-\nu} \dfrac{J_{-\nu}(\sqrt{E}) - J_\nu(\sqrt{E})D_\nu}{J_\nu(\sqrt{E})2\nu D_\nu}, \\ a^+_\nu = -D_\nu(a^+_{-\nu} + 2\nu a^-_{-\nu}), \quad a^-_\nu = -D_\nu(a^-_{-\nu} + 2\nu a^+_{-\nu}) \end{matrix}\right\} \tag{2.18}$$

and we indeed have $A_\nu f = Ef$ and $f \in D(A_\nu)$. Therefore, E is an eigenvalue and the eigenspace associated with the eigenvalue E is given by (2.18).
Now, we will distinguish cases on the condition (2.17). If (2.17) is true, then we have either

$$J_\nu(\sqrt{E})2\nu D_\nu = J_{-\nu}(\sqrt{E}) - J_\nu(\sqrt{E})D_\nu \tag{2.19}$$

or

$$-J_\nu(\sqrt{E})2\nu D_\nu = J_{-\nu}(\sqrt{E}) - J_\nu(\sqrt{E})D_\nu. \tag{2.20}$$

We notice that condition (2.19) is equivalent to condition (2.5), and condition (2.20) is equivalent to condition (2.6). Therefore, what remains to be proved is that the eigenspace (2.18) takes the form stated in the theorem, whether (2.19) or (2.20) holds.
Let us suppose that (2.19) is true. It is straightforward that the eigenspace (2.18) can be rewritten as

$$\left\{\begin{matrix} f \text{ of the form (2.2) such that : } \quad a^-_{-\nu} \in \mathbb{R}, \quad a^+_{-\nu} = a^-_{-\nu}, \\ a^+_\nu = -D_\nu(1+2\nu)a^-_{-\nu}, \quad a^-_\nu = -D_\nu(1+2\nu)a^-_{-\nu} \end{matrix}\right\}.$$

By looking at the expression of D_ν in (2.13) and at the functions of the form (2.2), it is clear that this space is exactly (2.7).
Now, let us suppose that (2.20) is true. It is straightforward that the eigenspace (2.18) can be rewritten as

$$\left\{\begin{matrix} f \text{ of the form (2.2) such that : } \quad a^-_{-\nu} \in \mathbb{R}, \quad a^+_{-\nu} = -a^-_{-\nu}, \\ a^+_\nu = D_\nu(1-2\nu)a^-_{-\nu}, \quad a^-_\nu = -D_\nu(1-2\nu)a^-_{-\nu} \end{matrix}\right\}.$$

By looking at the expression of D_ν in (2.13) and at the functions of the form (2.2), it is clear that this space is exactly (2.8). This concludes the proof of Proposition 2.4. $\square$

REMARK 2.5. *It is crucial to emphasize that we have found all the eigenvalues and eigenfunctions of our operator A_ν, for any $\nu \in (0,1)$. Indeed, we know that A_ν is self-adjoint and non-negative by (1.6). Therefore, $\sigma_{point}(A_\nu) \subset \mathbb{R}_+$. Proposition 2.1 shows that 0 is an eigenvalue of multiplicity 1 and provides the kernel. Proposition 2.4 indicates that the other eigenvalues are given by the implicit equations (2.5) and (2.6), and provides the associated eigenspaces, that are also of dimension one.*

2.2 Distribution structure of the eigenvalues

Now that we have found the conditions (2.5) and (2.6) for our eigenvalues, we would like to determine which values of E satisfy one of this condition, and how these values are distributed along the positive real axis. This is the subject of this subsection, which is concluded by a proposition that characterizes the structure of the distribution of our eigenvalues.

PROPOSITION 2.6. *Let $\nu \in (0,1)$, let $n \in \mathbb{N} \setminus \{0\}$. The function $f : x \mapsto \frac{J_\nu(x)}{J_{-\nu}(x)} x^{-2\nu}$ is a well-defined, increasing, and continuous bijection from $(j_{-\nu,n}, j_{-\nu,n+1})$ to $\mathbb{R}$. Furthermore, f is also a well-defined, increasing, and continuous bijection from $(0, j_{-\nu,1})$ to $(\frac{\Gamma(-\nu+1)}{\Gamma(\nu+1)}2^{-2\nu}, +\infty)$.*

Proof of Proposition 2.6. Let $n \in \mathbb{N}\setminus\{0\}$. First, thanks to Lemma 1.8, we know that $J_{-\nu}$ does not vanish between $(j_{-\nu,n}, j_{-\nu,n+1})$ or between $(0, j_{-\nu,1})$. Therefore, f is well-defined on $(j_{-\nu,n}, j_{-\nu,n+1})$ or $(0, j_{-\nu,1})$. The function f is also differentiable on these intervals, so to show that f is increasing, we only need to examine its derivative. Let $x \in (j_{-\nu,n}, j_{-\nu,n+1})$ or $x \in (0, j_{-\nu,1})$. Then,

$$\begin{aligned} f'(x) &= \frac{\left({J_\nu}'(x)x^{-2\nu} - 2\nu J_\nu(x)x^{-2\nu-1}\right) J_{-\nu}(x) - J_\nu(x)x^{-2\nu}{J_{-\nu}}'(x)}{{J_{-\nu}}^2(x)} \\ &= \frac{-x^{-2\nu}W\left(J_\nu, J_{-\nu}\right)(x) - 2\nu x^{-2\nu-1}J_\nu(x)J_{-\nu}(x)}{{J_{-\nu}}^2(x)} \text{ with } W\left(J_\nu, J_{-\nu}\right) \text{ defined in Lemma 1.7,} \\ &= \frac{2\nu x^{-2\nu-1}\left(\frac{\sin(\nu\pi)}{\nu\pi} - J_\nu(x)J_{-\nu}(x)\right)}{{J_{-\nu}}^2(x)} > 0 \quad \text{thanks to Lemma 1.6.} \end{aligned}$$

Therefore, f is increasing on $(j_{-\nu,n}, j_{-\nu,n+1})$, and on $(0, j_{-\nu,1})$. Furthermore, $J_{-\nu}(j_{-\nu,n}) = J_{-\nu}(j_{-\nu,n+1}) = 0$, and $J_\nu(j_{-\nu,n}) \neq 0$, $J_\nu(j_{-\nu,n+1}) \neq 0$ because the positive real zeros of J_ν and $J_{-\nu}$ are strictly interlaced as shown in Lemma 1.8. Therefore, we have

$$\lim_{x\to j_{-\nu,n}^+} f(x) = -\infty, \qquad \lim_{x\to j_{-\nu,n+1}^-} f(x) = +\infty, \text{ and } \lim_{x\to j_{-\nu,1}^-} f(x) = +\infty.$$

Finally, from the asymptotic expansion (1.8), we have

$$J_\nu(x) \sim_{0^+} \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \text{ and } J_{-\nu}(x) \sim_{0^+} \frac{1}{\Gamma(-\nu+1)}\left(\frac{x}{2}\right)^{-\nu}.$$

We deduce that $\lim_{x\to 0^+} f(x) = \frac{\Gamma(-\nu+1)}{\Gamma(\nu+1)}2^{-2\nu}$, which concludes the proof. $\square$

Thanks to the above property, we are able to give a result on the distributions of the eigenvalues of A_ν.

PROPOSITION 2.7. .
Let $\nu \in (0,1)$***,***

- $\nexists E \in (0, j_{-\nu,1}^2)$, *E satisfies (2.5).*
- $\forall n \in \mathbb{N}\setminus\{0\}$, $\nexists E \in [j_{-\nu,n}^2, j_{\nu,n}^2]$, *E satisfies (2.5).*
- $\forall n \in \mathbb{N}\setminus\{0\}$, $\exists! E \in (j_{\nu,n}^2, j_{-\nu,n+1}^2)$, *E satisfies (2.5). We denote it by* λ_{2n}.

Let $\nu \in (0, \frac{1}{2})$***,***

- $\exists! E \in (0, j_{-\nu,1}^2)$, *E satisfies (2.6). We denote it by* λ_1.
- $\forall n \in \mathbb{N}\setminus\{0\}$, $\nexists E \in [j_{-\nu,n}^2, j_{\nu,n}^2]$, *E satisfies (2.6).*
- $\forall n \in \mathbb{N}\setminus\{0\}$, $\exists! E \in (j_{\nu,n}^2, j_{-\nu,n+1}^2)$, *E satisfies (2.6). We denote it by* λ_{2n+1}.

Let $\nu \in (\frac{1}{2}, 1)$***,***

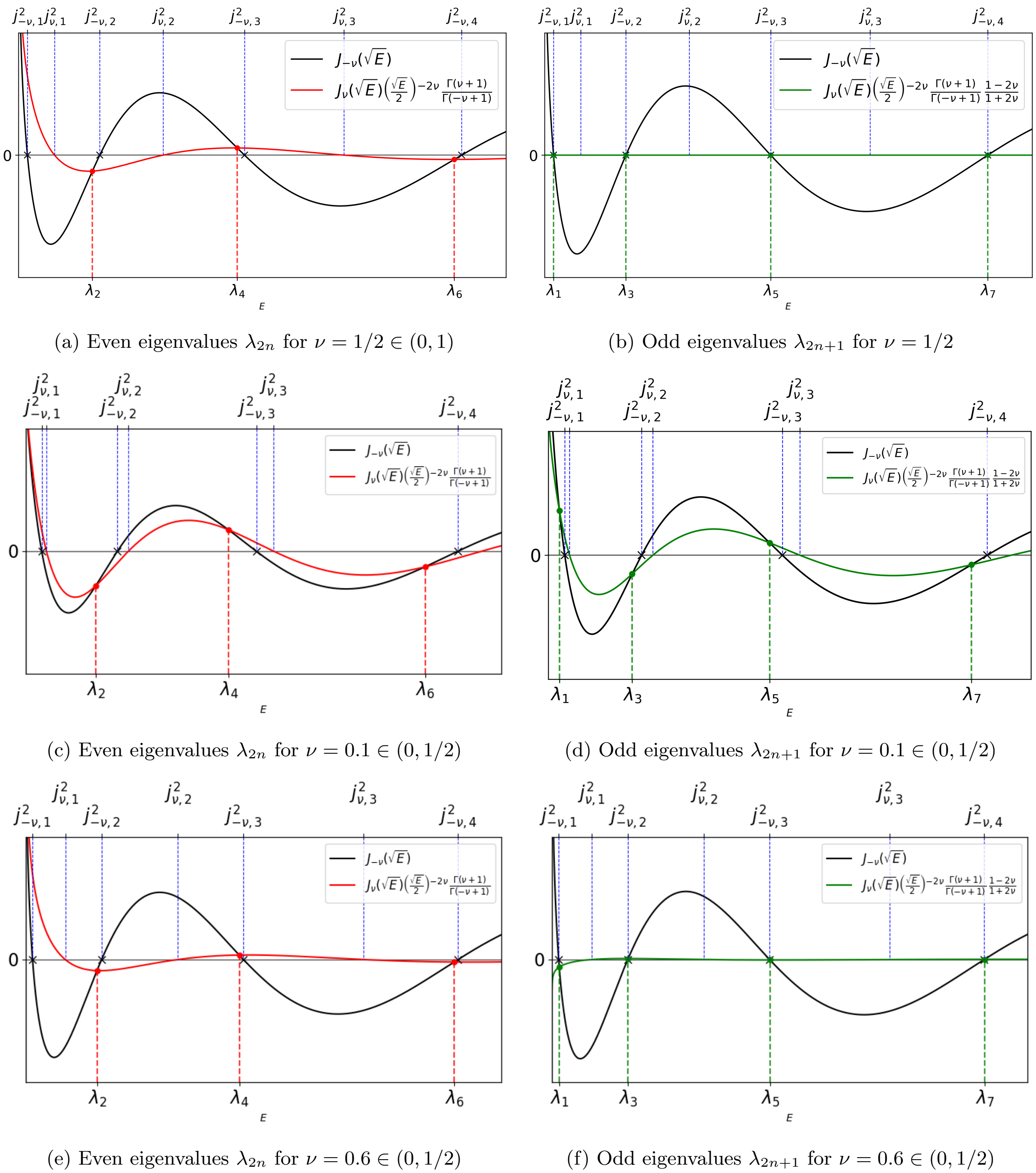
- $\nexists E \in (0, j_{-\nu,1}^2]$, *E satisfies (2.6).*
- $\forall n \in \mathbb{N}\setminus\{0\}$, $\exists! E \in (j_{-\nu,n}^2, j_{\nu,n}^2)$, *E satisfies (2.6). We denote it by* λ_{2n-1}.
- $\forall n \in \mathbb{N}\setminus\{0\}$, $\nexists E \in [j_{\nu,n}^2, j_{-\nu,n+1}^2]$, *E satisfies (2.6).*

Let $\nu = \frac{1}{2}$***,***

- $\forall n \in \mathbb{N}\setminus\{0\}$, $E = j_{-\nu,n}^2$ *satisfies (2.6). We denote it by* λ_{2n-1}.
- $\nexists E \in \mathbb{R}^+ \setminus \{0\} \bigcup_{n\geq 1}\{j_{-\nu,n}^2\}$, *E satisfies (2.6).*

We always have, regardless of $\nu \in (0,1)$

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{2n} < \lambda_{2n+1} < \ldots \tag{2.21}$$



(a) Even eigenvalues λ_{2n} for $\nu = 1/2 \in (0,1)$

(b) Odd eigenvalues λ_{2n+1} for $\nu = 1/2$

(c) Even eigenvalues λ_{2n} for $\nu = 0.1 \in (0,1/2)$

(d) Odd eigenvalues λ_{2n+1} for $\nu = 0.1 \in (0,1/2)$

(e) Even eigenvalues λ_{2n} for $\nu = 0.6 \in (0,1/2)$

(f) Odd eigenvalues λ_{2n+1} for $\nu = 0.6 \in (0,1/2)$

Figure 1: Illustration of the evolution of eigenvalues as a function of ν.

REMARK 2.8. .

- *The reader should note that the proposition is stated in such a way that for any fixed $\nu \in (0,1)$, all values of E that are solutions of (2.5) or (2.6) are listed. We ensure that we provide a complete inventory. Therefore, the point spectrum of A_ν is $\sigma_{point} = \{\lambda_n, n \in \mathbb{N}\}$.*

- *Lemma 1.8 is crucial here to better understand why the proposition is formulated this way. Lemma 1.8 ensures that we do not assign different names to the same E.*

- *Proposition 2.7 implies that $\lim_{n\to+\infty} \lambda_n = +\infty$, since $\lim_{n\to+\infty} j_{-\nu,n} = +\infty$.*

REMARK 2.9. *Proposition 2.7 implies that*

- *if $\nu \in (0, \frac{1}{2})$, $\lambda_{2n} < \lambda_{2n+1} < j^2_{-\nu,n+1}, \quad \forall n \in \mathbb{N} \setminus \{0\}$.*

- *if $\nu \in (\frac{1}{2}, 1)$, $\lambda_{2n} < j^2_{-\nu,n+1} < \lambda_{2n+1}, \quad \forall n \in \mathbb{N} \setminus \{0\}$.*

- *if $\nu = \frac{1}{2}$, $\lambda_{2n} < \lambda_{2n+1} = j^2_{-\nu,n+1}, \quad \forall n \in \mathbb{N} \setminus \{0\}$.*

Figure 1 illustrates the behavior of the eigenvalues λ_n for different values of the parameter $\nu \in (0,1)$, distinguishing even and odd eigenvalues. These plots visually reinforce Proposition 2.7, by explicitly showing where the solutions of the conditions (2.5) and (2.6) lie relative to the squared Bessel zeros $j^2_{\pm\nu,n}$.

The graphical layout emphasizes that the eigenvalues are interlaced with the Bessel zeros. For instance, one observes that between each pair of consecutive Bessel zeros, precisely one eigenvalue may exist, depending on the parity and the value of ν. The figures also show the transitions described in Remark 2.9: when ν increases from below 1/2 to above it, the location of the odd eigenvalues λ_{2n+1} shifts from being to the left of $j^2_{-\nu,n+1}$ to being to its right.

This transition is especially noticeable when comparing Figures 1b, 1d, and 1f. However, in the case $\nu = 0.6$, corresponding to Figure 1f, the values of λ_{2n+1} and $j^2_{-\nu,n+1}$ are extremely close, and the main plot does not allow for a precise visual comparison.

To address this, Figure 2 provides a magnified view centered around λ_5 and $j^2_{-\nu,3}$. This zoomed-in inset clearly confirms that $\lambda_5 > j^2_{-\nu,3}$, as given by Proposition 2.7 for $\nu \in (1/2, 1)$.

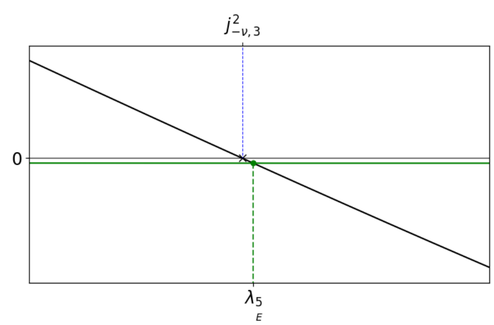


Figure 2: Zoom on the eigenvalue λ_5 and the squared Bessel zero $j^2_{-\nu,3}$ for $\nu = 0.6 \in (1/2, 1)$

Proof of Proposition 2.7. There are four clear blocks in the proposition. We will go through them one by one in Parts 1, 2, 3, and 4 of the proof. First, we should underline that, thanks to Lemma 1.8, we know that the family

$$(0, j^2_{-\nu,1}) \bigcup_{n\geq 1} \{j^2_{-\nu,n}\} \bigcup_{n\geq 1} \{j^2_{\nu,n}\} \bigcup_{n\geq 1} (j^2_{-\nu,n}, j^2_{\nu,n}) \bigcup_{n\geq 1} (j^2_{\nu,n}, j^2_{-\nu,n+1})$$

is a partition of $\mathbb{R}_+ \setminus \{0\}$. Therefore, the proposition provides a complete inventory of the solutions $E > 0$ of (2.5) or (2.6). We will use the function f defined in Proposition 2.6.

Part 1. Let us introduce h_1, which is well-defined on $\mathbb{R}_+ \setminus \{0\} \bigcup_{n\geq 1}\{j_{-\nu,n}^2\}$:

$$h_1 : x \mapsto f(\sqrt{x})\frac{1}{2^{-2\nu}}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}.$$

Since $x \mapsto \sqrt{x}$ is increasing on $\mathbb{R}_+ \setminus \{0\}$, and $\frac{1}{2^{-2\nu}}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} > 0$, we have, thanks to Proposition 2.6, that for any $n \in \mathbb{N} \setminus \{0\}$, h_1 is a continuous increasing bijection from $(j_{-\nu,n}^2, j_{-\nu,n+1}^2)$ to $\mathbb{R}$, and from $(0, j_{-\nu,1}^2)$ to $(\frac{1}{2^{-2\nu}}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \times \frac{\Gamma(-\nu+1)}{\Gamma(\nu+1)}2^{-2\nu}, +\infty) = (1, +\infty)$.
Let $E > 0$ satisfying (2.5). Then $E \notin \bigcup_{n\geq 1}\{j_{-\nu,n}^2\}$ because otherwise it would imply that $J_\nu(\sqrt{E}) = J_{-\nu}(\sqrt{E}) = 0$, which is not possible according to Lemma 1.8.
Therefore, $E > 0$ satisfies (2.5) if and only if

$$E \notin \bigcup_{n\geq 1}\{j_{-\nu,n}^2\} \quad \text{and} \quad h_1(E) = 1. \tag{2.22}$$

Using the intermediate value theorem on h_1, we directly prove the first three items of the proposition:

- $\nexists E \in (0, j_{-\nu,1}^2)$, *$E$ satisfies (2.5).* Indeed, we cannot have $h_1(E) = 1$ on $(0, j_{-\nu,1}^2)$ since we have $h_1(0) = 1$.
- $\forall n \in \mathbb{N} \setminus \{0\}$, $\nexists E \in [j_{-\nu,n}^2, j_{\nu,n}^2]$, *$E$ satisfies (2.5).* Indeed, for any $n \in \mathbb{N} \setminus \{0\}$, $h_1(j_{\nu,n}^2) = 0$, so $h_{1|_{(j_{-\nu,n}^2, j_{\nu,n}^2]}}$ is a bijection from $(j_{-\nu,n}^2, j_{\nu,n}^2]$ to $(-\infty, 0]$. Therefore, if E satisfies (2.5), then $E \notin (j_{-\nu,n}^2, j_{\nu,n}^2]$. Additionally, $E \neq j_{-\nu,n}^2$ according to (2.22), so $E \notin [j_{-\nu,n}^2, j_{\nu,n}^2]$.
- $\forall n \in \mathbb{N} \setminus \{0\}$, $\exists! E \in (j_{\nu,n}^2, j_{-\nu,n+1}^2)$ *such that E satisfies (2.5).* Indeed, $h_{1|_{(j_{\nu,n}^2, j_{-\nu,n+1}^2)}}$ is a bijection from $(j_{\nu,n}^2, j_{-\nu,n+1}^2)$ to $(0, +\infty)$.

Part 2. Let $\nu \in (0, \frac{1}{2})$. We introduce h_2, which is well-defined on $\mathbb{R}_+ \setminus \{0\} \bigcup_{n\geq 1}\{j_{-\nu,n}^2\}$:

$$h_2 : x \mapsto f(\sqrt{x})\frac{1}{2^{-2\nu}}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\frac{1-2\nu}{1+2\nu}.$$

Since $x \mapsto \sqrt{x}$ is increasing on $\mathbb{R}_+ \setminus \{0\}$, and $\frac{1}{2^{-2\nu}}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\frac{1-2\nu}{1+2\nu} > 0$, thanks to Proposition 2.6, we have that for any $n \in \mathbb{N} \setminus \{0\}$, h_2 is a continuous increasing bijection from $(j_{-\nu,n}^2, j_{-\nu,n+1}^2)$ to $\mathbb{R}$ and from $(0, j_{-\nu,1}^2)$ to $(\frac{1}{2^{-2\nu}}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\frac{1-2\nu}{1+2\nu} \times \frac{\Gamma(-\nu+1)}{\Gamma(\nu+1)}2^{-2\nu}, +\infty) = (\frac{1-2\nu}{1+2\nu}, +\infty)$.
Using the exact same arguments as in Part 1 of the proof, we find that $E > 0$ satisfies (2.6) if and only if

$$E \notin \bigcup_{n\geq 1}\{j_{-\nu,n}^2\} \quad \text{and} \quad h_2(E) = 1.$$

Using the intermediate value theorem on h_2, we directly prove the three needed items, as done in Part 1. Finally, using Lemma 1.8, the only thing left to prove to have (2.21) for $\nu \in (0, \frac{1}{2})$ is that for all $n \in \mathbb{N} \setminus \{0\}$, $\lambda_{2n} < \lambda_{2n+1}$. This is straightforward since λ_{2n} and λ_{2n+1} are both in $(j_{\nu,n}^2, j_{-\nu,n+1}^2)$, $h_2 = \frac{1-2\nu}{1+2\nu}h_1$ so $h_2 < h_1$ on $(j_{\nu,n}^2, j_{-\nu,n+1}^2)$, and $h_1(\lambda_{2n}) = h_2(\lambda_{2n+1}) = 1$.

Part 3. Let $\nu \in \left(\frac{1}{2}, 1\right)$. We follow the exact same strategy as in Part 2, except that here, $\frac{1-2\nu}{1+2\nu} < 0$, so the growth behavior of h_2 is inverted. The increasing parts are now turned into decreasing ones, and vice versa. Using the intermediate value theorem on h_2, we directly prove the three needed items. Finally, using Lemma 1.8, we have nothing left to prove in order to establish (2.21) for $\nu \in \left(\frac{1}{2}, 1\right)$.

Part 4. Let $\nu = \frac{1}{2}$. For $\nu = \frac{1}{2}$, we directly find that the fact that $E > 0$ satisfies (2.6) is equivalent to $J_{-\nu}(\sqrt{E}) = 0$. Therefore, the two items of this part are directly obtained. Finally, using Lemma 1.8, we have nothing left to prove in order to establish (2.21) for $\nu = \frac{1}{2}$. $\square$

2.3 Hilbert Basis of eigenfunctions

In order to prove that the spectrum of A_ν is restricted to the point spectrum, we prove the following result.

PROPOSITION 2.10. $(A_\nu, D(A_\nu))$ *has compact resolvent.*

Proof of proposition 2.10. Since $(A_\nu, D(A_\nu))$ is self-adjoint, it is automatically closed. Hence, for any $\lambda \in \mathbb{R}$ such that $A_\nu - \lambda \mathrm{Id}$ is invertible, the closed graph theorem ensures that $(A_\nu - \lambda \mathrm{Id})^{-1} \in \mathcal{L}_c(L^2(-1,1), D(A_\nu))$, where $D(A_\nu)$ is equipped with the usual graph norm. So, by composition, $(A_\nu, D(A_\nu))$ has a compact resolvent as soon as the injection $(D(A_\nu), ||\cdot||_{D(A_\nu)}) \hookrightarrow (L^2(-1,1), ||\cdot||_{L^2(-1,1)})$ is compact.
Let $(f_n = f_r^n + f_s^n)_{n\in\mathbb{N}} \in D(A_\nu)^{\mathbb{N}}$ be a bounded sequence for the graph norm $||\cdot||_{D(A_\nu)}$, where for all $n \in \mathbb{N}$, $f_r^n \in \tilde{H}_0^2(-1,1)$ and $f_s^n \in \mathcal{F}_s^\nu$. We have that

$$\exists C_1 > 0,\ \forall n \in \mathbb{N},\ ||f_n||_{L^2(-1,1)} \leq C_1, \quad \text{and} \quad \exists C_2 > 0,\ \forall n \in \mathbb{N},\ ||Af_n||_{L^2(-1,1)} \leq C_2.$$

Therefore, we obtain

$$\forall n \in \mathbb{N}, \quad \langle Af_n, f_n\rangle_{L^2(-1,1)} \leq ||f_n||_{L^2(-1,1)}||Af_n||_{L^2(-1,1)} \leq C_1C_2.$$

Using (1.6), we have

$$\forall n \in \mathbb{N}, \quad \min\{1, 4\nu^2\} \int_{-1}^{1} (\nabla f_r^n)^2 \leq C_1C_2.$$

Thus, we conclude that $(\nabla f_r^n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(-1,1)$. Combining this with Poincaré's inequality on (-1,1) (which valid since $f_r^n \in \tilde{H}_0^2(-1,1)$ for any $n \in \mathbb{N}$), we deduce that $(f_r^n)_{n\in\mathbb{N}}$ is also a bounded sequence in $L^2(-1,1)$. This implies that $(f_r^n)_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(-1,1)$, so that Rellich's theorem states that the injection $H^1(-1,1) \hookrightarrow L^2(-1,1)$ is compact. Therefore, there exists a subsequence $(f_r^{\varphi_1(n)})_{n\in\mathbb{N}}$ that converges strongly in $L^2(-1,1)$ to some $f_r \in L^2(-1,1)$. Furthermore,

$$\forall n \in \mathbb{N}, \quad ||f_s^n||_{L^2(-1,1)} = ||f_s^n + f_r^n - f_r^n||_{L^2(-1,1)} \leq ||f^n||_{L^2(-1,1)} + ||f_r^n||_{L^2(-1,1)}.$$

Since $(f^n)_{n\in\mathbb{N}}$ is bounded in $L^2(-1,1)$, and we have shown that $(f_r^n)_{n\in\mathbb{N}}$ is bounded in $L^2(-1,1)$, we conclude that $(f_s^n)_{n\in\mathbb{N}}$ is also bounded in $L^2(-1,1)$, so $(f_s^{\varphi_1(n)})_{n\in\mathbb{N}}$ is also bounded. Moreover, $(f_s^{\varphi_1(n)})_{n\in\mathbb{N}} \in (\mathcal{F}_s^\nu)^{\mathbb{N}}$, and since $\mathcal{F}_s^\nu$ is a finite-dimensional vector subspace of $L^2(-1,1)$, there exists a subsequence $(f_s^{\varphi_2\circ\varphi_1(n)})_{n\in\mathbb{N}}$ that converges strongly in $L^2(-1,1)$ to some $f_s \in \mathcal{F}_s^\nu \subset L^2(-1,1)$. Thus, we have proven that

$$\forall n \in \mathbb{N}, \quad f_{\varphi_2\circ\varphi_1(n)} = f_r^{\varphi_2\circ\varphi_1(n)} + f_s^{\varphi_2\circ\varphi_1(n)} \underset{n\to+\infty}{\longrightarrow} f_r + f_s \in L^2(-1,1).$$

Therefore, $(D(A_\nu), ||\cdot||_{D(A_\nu)}) \hookrightarrow (L^2(-1,1), ||\cdot||_{L^2(-1,1)})$ is compact, and we can conclude as explained at the beginning of the proof. $\square$

We are now ready to give a complete description of the spectral theory of A_ν.

THEOREM 2.11. *We introduce*

$$\psi_0 := |x|^{\nu+\frac{1}{2}} - |x|^{-\nu+\frac{1}{2}} \text{ on } (-1,1), \quad \textit{and} \quad a_0 := ||\psi_0||_{L^2(-1,1)}.$$

Let $n \in \mathbb{N}\setminus\{0\}$, *we introduce*

$$\psi_{2n} := \begin{cases} -\dfrac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\dfrac{\sqrt{\lambda_{2n}}}{2}\right)^{-2\nu} \sqrt{-x}J_\nu\left(-\sqrt{\lambda_{2n}}x\right) + \sqrt{-x}J_{-\nu}\left(-\sqrt{\lambda_{2n}}x\right), & \text{on } (-1,0), \\ -\dfrac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\dfrac{\sqrt{\lambda_{2n}}}{2}\right)^{-2\nu} \sqrt{x}J_\nu\left(\sqrt{\lambda_{2n}}x\right) + \sqrt{x}J_{-\nu}\left(\sqrt{\lambda_{2n}}x\right), & \text{on } (0,1), \end{cases}$$

$$
\begin{aligned}
a_{2n}^2 &:= ||\psi_{2n}||^2_{L^2(-1,1)} = \left(1-\frac{\nu^2}{\lambda_{2n}}\right) J_{-\nu}\left(\sqrt{\lambda_{2n}}\right)^2 + J_{-\nu}{}'\left(\sqrt{\lambda_{2n}}\right)^2 \\
&-2\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\frac{\sqrt{\lambda_{2n}}}{2}\right)^{-2\nu}\left[\left(1-\frac{\nu^2}{\lambda_{2n}}\right) J_{\nu}\left(\sqrt{\lambda_{2n}}\right) J_{-\nu}\left(\sqrt{\lambda_{2n}}\right)\right. \\
&\qquad\qquad \left. + J_{\nu}{}'\left(\sqrt{\lambda_{2n}}\right) J_{-\nu}{}'\left(\sqrt{\lambda_{2n}}\right) + \frac{2\nu\sin(\nu\pi)}{\pi\lambda_{2n}}\right] \\
&+\left(\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\frac{\sqrt{\lambda_{2n}}}{2}\right)^{-2\nu}\right)^2\left[\left(1-\frac{\nu^2}{\lambda_{2n}}\right) J_{\nu}\left(\sqrt{\lambda_{2n}}\right)^2 + J_{\nu}{}'\left(\sqrt{\lambda_{2n}}\right)^2\right] > 0
\end{aligned}
\tag{2.23}
$$

Let $n \in \mathbb{N}$, we introduce

$$
\psi_{2n+1} := \begin{cases} \dfrac{1-2\nu}{1+2\nu}\dfrac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\dfrac{\sqrt{\lambda_{2n+1}}}{2}\right)^{-2\nu}\sqrt{-x}J_{\nu}\left(-\sqrt{\lambda_{2n+1}}x\right) - \sqrt{-x}J_{-\nu}\left(-\sqrt{\lambda_{2n+1}}x\right), & \text{on}(-1,0), \\ -\dfrac{1-2\nu}{1+2\nu}\dfrac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\dfrac{\sqrt{\lambda_{2n+1}}}{2}\right)^{-2\nu}\sqrt{x}J_{\nu}\left(\sqrt{\lambda_{2n+1}}x\right) + \sqrt{x}J_{-\nu}\left(\sqrt{\lambda_{2n+1}}x\right), & \text{on}(0,1), \end{cases}
$$

$$
\begin{aligned}
a_{2n+1}^2 &:= ||\psi_{2n+1}||^2_{L^2(-1,1)} = \left(1-\frac{\nu^2}{\lambda_{2n+1}}\right) J_{-\nu}\left(\sqrt{\lambda_{2n+1}}\right)^2 + J_{-\nu}{}'\left(\sqrt{\lambda_{2n+1}}\right)^2 \\
&-2\frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\frac{\sqrt{\lambda_{2n+1}}}{2}\right)^{-2\nu}\left[\left(1-\frac{\nu^2}{\lambda_{2n+1}}\right) J_{\nu}\left(\sqrt{\lambda_{2n+1}}\right) J_{-\nu}\left(\sqrt{\lambda_{2n+1}}\right).\right. \\
&\qquad\qquad \left. + J_{\nu}{}'\left(\sqrt{\lambda_{2n+1}}\right) J_{-\nu}{}'\left(\sqrt{\lambda_{2n+1}}\right) + \frac{2\nu\sin(\nu\pi)}{\pi\lambda_{2n+1}}\right] \\
&+\left(\frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\frac{\sqrt{\lambda_{2n+1}}}{2}\right)^{-2\nu}\right)^2\left[\left(1-\frac{\nu^2}{\lambda_{2n+1}}\right) J_{\nu}\left(\sqrt{\lambda_{2n+1}}\right)^2 + J_{\nu}{}'\left(\sqrt{\lambda_{2n+1}}\right)^2\right] > 0.
\end{aligned}
\tag{2.24}
$$

Let $n \in \mathbb{N}$, we define

$$
\phi_n := \frac{\psi_n}{a_n}.
$$

Then, $(\phi_n)_{n\in\mathbb{N}}$ is a Hilbert basis of $L^2(-1,1)$ consisting of eigenfunctions of A_ν.

Proof of Theorem 2.11. Thanks to Proposition 2.1 and Corollary 2.4, we know that $\sigma_{point}(A_\nu) = \{\lambda_n,\ n \in \mathbb{N}\}$, and that all these eigenvalues are distinct and have multiplicity 1. Moreover, for any $n \in \mathbb{N}$, $\mathrm{Ker}(A_\nu - \lambda_n \,\mathrm{Id}) = \mathrm{Span}(\psi_n) = \mathrm{Span}(\phi_n)$. Furthermore, using (1.9.a) and (1.9.b), we directly obtain (2.23) and (2.24). Since A_ν is self-adjoint with compact resolvent, we deduce that $\sigma(A_\nu) = \{\lambda_n,\ n \in \mathbb{N}\}$ and that the sequence $(\phi_n)_{n\in\mathbb{N}}$ forms a Hilbert basis of $L^2(-1,1)$. □

3 Asymptotic behaviour of the eigenvalues

3.1 Case $\nu \in \left(0, \frac{1}{2}\right)$

During this whole subsection, we focus on the case $\nu \in (0, \frac{1}{2})$.

LEMMA 3.1. *Let* $\nu \in \left(0, \frac{1}{2}\right)$,

$$\sqrt{\frac{\pi x}{2}}\left(J_\nu(x)\left(\frac{x}{2}\right)^{-2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} - J_{-\nu}(x)\right) \underset{x\to+\infty}{=} -\cos\left(x+\nu\frac{\pi}{2}-\frac{\pi}{4}\right) + \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\left(\frac{2}{x}\right)^{2\nu} + \mathcal{O}\left(\frac{1}{x}\right), \tag{3.1}$$

$$\sqrt{\frac{\pi x}{2}}\left(J_\nu(x)\left(\frac{x}{2}\right)^{-2\nu}\frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} - J_{-\nu}(x)\right) \underset{x\to+\infty}{=} -\cos\left(x+\nu\frac{\pi}{2}-\frac{\pi}{4}\right) + \frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\left(\frac{2}{x}\right)^{2\nu} + \mathcal{O}\left(\frac{1}{x}\right). \tag{3.2}$$

Proof of Lemma 3.1. Let $\nu \in \left(0, \frac{1}{2}\right)$, using Lemma 1.10, we have

$$\sqrt{\frac{\pi x}{2}}J_{-\nu}(x) \underset{x\to+\infty}{=} \cos\left(x+\nu\frac{\pi}{2}-\frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{x}\right), \tag{3.3}$$

$$\sqrt{\frac{\pi x}{2}}J_{\nu}(x)\left(\frac{x}{2}\right)^{-2\nu} \underset{x\to+\infty}{=} \cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\left(\frac{2}{x}\right)^{2\nu} + \mathcal{O}\left(\frac{1}{x^{1+2\nu}}\right). \tag{3.4}$$

$\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\times$(3.4)−(3.3)=(3.1), and $\frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\times$(3.4)−(3.3)=(3.2). □

PROPOSITION 3.2. *Let* $\nu \in \left(0, \frac{1}{2}\right)$,

$$\sqrt{\lambda_{2(n-1)}} \underset{n\to+\infty}{=} \pi\left(n-\frac{\nu}{2}-\frac{1}{4}\right) - \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\sin(\nu\pi)\left(\frac{2}{\pi n}\right)^{2\nu} + \mathcal{O}\left(\frac{1}{n^{\min(1,4\nu)}}\right), \tag{3.5}$$

$$\sqrt{\lambda_{2(n-1)+1}} \underset{n\to+\infty}{=} \pi\left(n-\frac{\nu}{2}-\frac{1}{4}\right) - \frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\sin(\nu\pi)\left(\frac{2}{\pi n}\right)^{2\nu} + \mathcal{O}\left(\frac{1}{n^{\min(1,4\nu)}}\right). \tag{3.6}$$

Proof of Proposition 3.2. Let $\nu \in \left(0, \frac{1}{2}\right)$.
Proof of (3.5).
Step 1. Thanks to (3.1), we know that

$$\sqrt{\frac{\pi x}{2}}\left(J_\nu(x)\left(\frac{x}{2}\right)^{-2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} - J_{-\nu}(x)\right) \underset{x\to+\infty}{=} -\cos\left(x+\nu\frac{\pi}{2}-\frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{x^{2\nu}}\right).$$

We also know that we need to have for $n>1$

$$\sqrt{\frac{\pi\sqrt{\lambda_{2(n-1)}}}{2}}\left(J_\nu\left(\sqrt{\lambda_{2(n-1)}}\right)\left(\frac{\sqrt{\lambda_{2(n-1)}}}{2}\right)^{-2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} - J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}\right)\right) = 0. \tag{3.7}$$

Therefore, we must have

$$0 \underset{n\to+\infty}{=} -\cos\left(\sqrt{\lambda_{2(n-1)}} + \nu\frac{\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda_{2(n-1)}}}\right), \tag{3.8}$$

because $\sqrt{\lambda_{2(n-1)}} \underset{n\to+\infty}{\longrightarrow} +\infty$ as explained in Remark 2.8. Thus, we have

$$\cos\left(\sqrt{\lambda_{2(n-1)}} + \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \underset{n\to+\infty}{=} o(1).$$

That implies that there must exist a sequence of integers $(k_n)_{n\in\mathbb{N}\setminus\{0\}} \in \mathbb{Z}^{\mathbb{N}\setminus\{0\}}$ such that

$$\sqrt{\lambda_{2(n-1)}} \underset{n\to+\infty}{=} \pi\left(k_n - \frac{\nu}{2} - \frac{1}{4}\right) + o(1),$$

and thanks to Proposition 2.7 we know that for $n > 1$, $\sqrt{\lambda_{2(n-1)}} \in (j_{\nu,n-1}, j_{-\nu,n})$. Using Lemma 1.12, we obtain, as $n \to +\infty$,

$$\sqrt{\lambda_{2(n-1)}} \in \left(\pi\left(n - 1 + \frac{\nu}{2} - \frac{1}{4}\right) + \mathcal{O}\left(\frac{1}{n}\right), \quad \pi\left(n - \frac{\nu}{2} - \frac{1}{4}\right) + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Therefore, since $\nu \in (0,1)$, for n large enough, we must have $k_n = n$. This proves that

$$\sqrt{\lambda_{2(n-1)}} \underset{n\to+\infty}{=} \pi\left(n - \frac{\nu}{2} - \frac{1}{4}\right) + \varepsilon_{2\nu}(n) \quad \text{with } \varepsilon_{2\nu}(n) = o(1). \tag{3.9}$$

Step 2. Now, we want to prove that $\varepsilon_{2\nu}(n) \underset{n\to+\infty}{=} \mathcal{O}\left(\frac{1}{n^{2\nu}}\right)$. We will repeat the same process. Using (3.8) and (3.9), we obtain

$$\cos\left(\pi n + \varepsilon_{2\nu}(n) - \frac{\pi}{2}\right) \underset{n\to+\infty}{=} \mathcal{O}\left(\frac{1}{n^{2\nu}}\right).$$

Therefore,

$$\sin\left(\varepsilon_{2\nu}(n)\right) \underset{n\to+\infty}{=} \mathcal{O}\left(\frac{1}{n^{2\nu}}\right), \quad \text{which gives} \qquad \varepsilon_{2\nu}(n) + \mathcal{O}\left(\varepsilon_{2\nu}(n)^3\right) \underset{n\to+\infty}{=} \mathcal{O}\left(\frac{1}{n^{2\nu}}\right).$$

Thus,

$$\varepsilon_{2\nu}(n)\left(1 + \mathcal{O}\left(\varepsilon_{2\nu}(n)^2\right)\right) \underset{n\to+\infty}{=} \mathcal{O}\left(\frac{1}{n^{2\nu}}\right),$$

which implies that Thus,

$$\varepsilon_{2\nu}(n)\left(1 + o(1)\right) \underset{n\to+\infty}{=} \mathcal{O}\left(\frac{1}{n^{2\nu}}\right).$$

This implies that $\varepsilon_{2\nu}(n) \underset{n\to+\infty}{=} \mathcal{O}\left(\frac{1}{n^{2\nu}}\right)$.

Step 3. Now, we want to obtain an even more precise expression of $\varepsilon_{2\nu}$. We will use the same strategy. Using (3.1), we know that

$$\sqrt{\frac{\pi x}{2}}\left(J_\nu(x)\left(\frac{x}{2}\right)^{-2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} - J_{-\nu}(x)\right) \underset{x\to+\infty}{=} -\cos\left(x + \nu\frac{\pi}{2} - \frac{\pi}{4}\right) + \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\cos\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)\left(\frac{2}{x}\right)^{2\nu} + \mathcal{O}\left(\frac{1}{x}\right).$$

(3.7) is still true. Therefore, we must have

$$
\begin{aligned}
&-\cos\left(\sqrt{\lambda_{2(n-1)}}+\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\cos\left(\sqrt{\lambda_{2(n-1)}}-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\left(\frac{2}{\sqrt{\lambda_{2(n-1)}}}\right)^{2\nu}\\
&\underset{n\to+\infty}{=}\mathcal{O}\left(\frac{1}{\sqrt{\lambda_{2(n-1)}}}\right).
\end{aligned}
$$

Using (3.9), we obtain

$$
(-1)^{n+1}\sin\left(\varepsilon_{2\nu}(n)\right)\ +\ \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}(-1)^{n}\sin\left(-\nu\pi+\varepsilon_{2\nu}(n)\right)\left(\frac{2}{\pi n}\right)^{2\nu}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)^{-2\nu}\underset{n\to+\infty}{=}\mathcal{O}\left(\frac{1}{n}\right).
$$

Now, we use the asymptotic expansions of $\sin(x)$, $(1+x)^{-2\nu}$, and $\sin(-\nu\pi+x)$ around 0. We need to remember that we already know that $\varepsilon_{2\nu}(n)\underset{n\to+\infty}{=}\mathcal{O}\left(\frac{1}{n^{2\nu}}\right)$ by the previous step, and that since $\nu\in\left(0,\frac{1}{2}\right)$, $2\nu\in(0,1)$. We obtain

$$
\begin{aligned}
&\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}(-1)^{n}\left(\sin(-\nu\pi)+\mathcal{O}\left(\frac{1}{n^{2\nu}}\right)\right)\left(\frac{2}{\pi n}\right)^{2\nu}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\\
&+(-1)^{n+1}\left(\varepsilon_{2\nu}(n)+\mathcal{O}\left(\frac{1}{n^{6\nu}}\right)\right)\underset{n\to+\infty}{=}\mathcal{O}\left(\frac{1}{n}\right),
\end{aligned}
$$

which gives

$$
\varepsilon_{2\nu}(n)\underset{x\to+\infty}{=}-\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\sin(\nu\pi)\left(\frac{2}{\pi n}\right)^{2\nu}+\mathcal{O}\left(\frac{1}{n^{\min(1,4\nu)}}\right),
$$

and concludes the proof of (3.5).

Proof of (3.6). It is crucial to note that the asymptotic expansion (3.2) is simply the asymptotic expansion of (3.1) but with $\frac{\Gamma(\nu+1)}{\Gamma(\nu-1)}$ replaced by $\frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(\nu-1)}$. Therefore, if we could prove that

$$
\sqrt{\lambda_{2(n-1)+1}}\underset{n\to+\infty}{=}\pi\left(n-\frac{\nu}{2}-\frac{1}{4}\right)+o(1), \tag{3.10}
$$

then we would follow the exact same proof as for (3.5) except that $\frac{\Gamma(\nu+1)}{\Gamma(\nu-1)}$ would always be replaced by $\frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(\nu-1)}$. Therefore, we would have proven (3.6).
Using (3.2) instead of (3.1) and for $n\in\mathbb{N}\setminus\{0\}$, the condition that $\sqrt{\lambda_{2(n-1)+1}}$ satisfies is now (2.6) instead of (2.5), we can follow the same computations as in Step 1 to obtain that there must exist a sequence of integers $(k_n)_{n\in\mathbb{N}\setminus\{0\}}\in\mathbb{Z}^{\mathbb{N}\setminus\{0\}}$ such that

$$
\sqrt{\lambda_{2(n-1)+1}}\underset{n\to+\infty}{=}\pi\left(k_n-\frac{\nu}{2}-\frac{1}{4}\right)+o(1).
$$

Since $\nu\in\left(0,\frac{1}{2}\right)$, Proposition 2.7 gives that for $n>1$, $\sqrt{\lambda_{2(n-1)+1}}\in(j_{\nu,n-1},j_{-\nu,n})$. Thus, by following the same computations as in Step 1, we obtain (3.10). As explained before, this is enough to conclude the proof of (3.6). $\square$

Taking the square of (3.5) and (3.6) and keeping only the terms of the right order of precision leads to the following result.

COROLLARY 3.3. *Let $\nu \in \left(0, \frac{1}{2}\right)$,*

$$\lambda_{2(n-1)} \underset{n\to+\infty}{=} \pi^2 \left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2 - 4\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \sin(\nu\pi) \left(\frac{2}{\pi n}\right)^{2\nu-1} + \mathcal{O}\left(\frac{1}{n^{\min(0,4\nu-1)}}\right), \tag{3.11}$$

$$\lambda_{2(n-1)+1} \underset{n\to+\infty}{=} \pi^2 \left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2 - 4\frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \sin(\nu\pi) \left(\frac{2}{\pi n}\right)^{2\nu-1} + \mathcal{O}\left(\frac{1}{n^{\min(0,4\nu-1)}}\right), \tag{3.12}$$

$$\lambda_{2(n-1)+1} - \lambda_{2(n-1)} \underset{n\to+\infty}{=} \frac{16\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \sin(\nu\pi) \left(\frac{2}{\pi n}\right)^{2\nu-1} + \mathcal{O}\left(\frac{1}{n^{\min(0,4\nu-1)}}\right). \tag{3.13}$$

3.2 Case $\nu \in \left[\frac{1}{2}, 1\right)$

During this whole subsection, we will focus only on the case $\nu \in [\frac{1}{2}, 1)$.

LEMMA 3.4. *Let $\nu \in \left[\frac{1}{2}, 1\right)$,*

$$\begin{aligned}\sqrt{\frac{\pi x}{2}} \left(J_\nu(x) \left(\frac{x}{2}\right)^{-2\nu} \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} - J_{-\nu}(x)\right) \underset{x\to+\infty}{=} & -\cos\left(x + \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \\ & + \sin\left(x + \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \frac{4\nu^2-1}{8} \times \frac{1}{x} \\ & + \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \cos\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \left(\frac{2}{x}\right)^{2\nu} \\ & + \mathcal{O}\left(\frac{1}{x^2}\right).\end{aligned} \tag{3.14}$$

Proof of Lemma 3.4. Let $\nu \in \left[\frac{1}{2}, 1\right)$, using Lemma 1.10, we have

$$\sqrt{\frac{\pi x}{2}} J_{-\nu}(x) \underset{x\to+\infty}{=} \cos\left(x + \nu\frac{\pi}{2} - \frac{\pi}{4}\right) - \sin\left(x + \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \frac{4\nu^2-1}{8} \times \frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{3.15}$$

and

$$\sqrt{\frac{\pi x}{2}} J_\nu(x) \left(\frac{x}{2}\right)^{-2\nu} \underset{x\to+\infty}{=} \cos\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \left(\frac{2}{x}\right)^{2\nu} + \mathcal{O}\left(\frac{1}{x^{1+2\nu}}\right). \tag{3.16}$$

$\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}$(3.16)−(3.15)=(3.14), and we keep $\mathcal{O}\left(\frac{1}{x^2}\right)$ instead of $\mathcal{O}\left(\frac{1}{x^{1+2\nu}}\right)$ because $\nu \in \left[\frac{1}{2}, 1\right)$. □

As soon as we have proved this asymptotic expansion, it is not difficult to figure out that one can follow exactly the proof of Proposition 3.2 and obtain the following result.

PROPOSITION 3.5. *Let $\nu \in \left[\frac{1}{2}, 1\right)$,*

$$\sqrt{\lambda_{2(n-1)}} \underset{n\to+\infty}{=} +\pi\left(n - \frac{\nu}{2} - \frac{1}{4}\right) - \frac{4\nu^2-1}{8\pi n} - \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \sin(\nu\pi) \left(\frac{2}{\pi n}\right)^{2\nu} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{3.17}$$

$$\sqrt{\lambda_{2(n-1)+1}} \underset{n\to+\infty}{=} +\pi\left(n - \frac{\nu}{2} - \frac{1}{4}\right) - \frac{4\nu^2-1}{8\pi n} - \frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \sin(\nu\pi) \left(\frac{2}{\pi n}\right)^{2\nu} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{3.18}$$

Taking the square of (3.17) and (3.18) leads to the following result.

COROLLARY 3.6. *Let* $\nu \in \left[\frac{1}{2}, 1\right)$,

$$\lambda_{2(n-1)} \underset{n\to+\infty}{=} \pi^2 \left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2 - 2\frac{4\nu^2-1}{8} - 4\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \sin(\nu\pi) \left(\frac{2}{\pi n}\right)^{2\nu-1} + \mathcal{O}\left(\frac{1}{n}\right), \tag{3.19}$$

$$\lambda_{2(n-1)+1} \underset{n\to+\infty}{=} \pi^2 \left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2 - 2\frac{4\nu^2-1}{8} - 4\frac{1-2\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \sin(\nu\pi) \left(\frac{2}{\pi n}\right)^{2\nu-1} + \mathcal{O}\left(\frac{1}{n}\right), \tag{3.20}$$

$$\lambda_{2(n-1)+1} - \lambda_{2(n-1)} \underset{n\to+\infty}{=} \frac{16\nu}{1+2\nu}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \sin(\nu\pi) \left(\frac{2}{\pi n}\right)^{2\nu-1} + \mathcal{O}\left(\frac{1}{n}\right). \tag{3.21}$$

We can also deduce the following less precise result, true for any $\nu \in (0,1)$.

COROLLARY 3.7.

$$\sqrt{\lambda_{2(n-1)}} \underset{n\to+\infty}{=} \pi \left(n - \frac{\nu}{2} - \frac{1}{4}\right) + \mathcal{O}\left(\frac{1}{n^{\min(1,2\nu)}}\right)$$

and

$$\sqrt{\lambda_{2(n-1)+1}} \underset{n\to+\infty}{=} \pi \left(n - \frac{\nu}{2} - \frac{1}{4}\right) + \mathcal{O}\left(\frac{1}{n^{\min(1,2\nu)}}\right).$$

Proof of Corollary 3.7. This is directly obtained from (3.5), (3.6), (3.17) and (3.18), with a case distinction on ν. □

4 Null Controllability

4.1 Internal Control

To prove the null controllability property stated in Theorem 1.1, we will follow the moment method, that was introduced in [16] for the study of the boundary null controllability of the 1D heat equation, in the spirit of the work [18], which explains how to treat the case of an internal control. The moment method is presented in [30, Section 5.3.3] for instance. To be able to find a control $u \in L^2((0,T)\times\omega)$ that is well-defined and that brings the final state to 0, we will have to

- find a biorthogonal family $(q_n)_{n\in\mathbb{N}}$ to the family of exponentials $(t \mapsto e^{-\lambda_n t})_{n\in\mathbb{N}}$ in $L^2(0,T)$, *i.e.* verifying
$$\int_0^T q_k(t)e^{-\lambda_j t}dt = \delta_{k,j}, \quad \forall k, j \in \mathbb{N},$$
with $\delta_{k,j}$ the Kronecker symbol;
- obtain an estimate on the L^2-norm of the $(q_n)_{n\in\mathbb{N}}$;
- prove that $\inf_{n\in\mathbb{N}} \int_\omega \phi_n^2 > 0$, with $(\phi_n)_{n\in\mathbb{N}}$ the Hilbert basis of eigenfunctions defined in Theorem 2.11.

The first point is easily obtained. Moreover, in order to find the estimate on the L^2-norm, we will need to introduce the notion of condensation index, following the work of [3, Section 3]. Then, using results from [3, Section 4], we will be able to obtain our second point. These two steps are stated in Proposition 4.5. Finally, to obtain the third point, we will need several asymptotic behaviours that are stated in Lemma 4.8. Then, we prove it on a control domain of the form $\omega = (\alpha, \beta) \subset (-1,1)$ in Proposition 4.10. Afterwards, we generalize the result to the measurable case in Corollary 4.6. With all those intermediate results, we will finally prove Theorem 1.1.

DEFINITION 4.1. *[3, Definition 3.1 and Remark 3.10]*
Let $\Lambda = (\mu_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be a positive and increasing sequence such that $\sum_{n\in\mathbb{N}} \dfrac{1}{\mu_n} < +\infty$. The condensation index of the sequence Λ is defined by

$$c(\Lambda) := \limsup_{n\to+\infty} -\frac{1}{\mu_n} \ln |C'(\mu_n)| \in [0, +\infty], \tag{4.1}$$

with

$$C(\mu) = \prod_{n=0}^{+\infty} \left(1 - \frac{\mu^2}{\mu_n^2}\right), \quad \forall \mu > 0. \tag{4.2}$$

REMARK 4.2. *In fact, to define the condensation index, we only need to assume that $(\lambda_n)_{n\in\mathbb{N}}$ is positive for large enough n. Indeed, one easily proves that removing a finite number of terms in the sequence Λ does not change the value of $c(\Lambda)$, in the sense that for any $N \in \mathbb{N}$, we have*

$$c(\Lambda) = \limsup_{n\to+\infty} -\frac{1}{\mu_n} \ln |C_N'(\mu_n)| \in [0, +\infty],$$

with

$$C_N(\mu) = \prod_{n=N}^{+\infty} \left(1 - \frac{\mu^2}{\mu_n^2}\right), \quad \forall \mu > 0.$$

REMARK 4.3. *We study the condensation index since it provides a measure of the separation of the elements λ_n of the sequence Λ. It will be useful to obtain an essential estimate on the L^2-norm of a biorthogonal family in Proposition 4.5.*

PROPOSITION 4.4. *We introduce $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$, with $\{\lambda_n, n \in \mathbb{N}\}$ the eigenvalues of A_ν defined in Propositions 2.1 and 2.7. Then, the condensation index $c(\Lambda)$ is 0.*

Proof of Proposition 4.4. This proof takes some inspiration from the proof of [23, Lemma 4.1]. First, from (2.21), Corollary 3.3, and Corollary 3.6, we obtain that $(\lambda_n)_{n\in\mathbb{N}\setminus\{0\}}$ is a positive increasing sequence and $\sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{\lambda_n} < +\infty$. Therefore, the notion of condensation index is well-defined. By examining (4.2) (where we start the summation at $n = 1$, see Remark 4.2), we notice that

$$|C'(\lambda_n)| = \frac{2}{\lambda_n} \prod_{\substack{j=1\\ j\neq n}}^{+\infty} \left|1 - \frac{\lambda_n^2}{\lambda_j^2}\right|, \quad \forall n \in \mathbb{N}\setminus\{0\}.$$

Therefore, we get, for any $n \in \mathbb{N}\setminus\{0\}$,

$$\begin{aligned}\frac{1}{\lambda_n} \ln |C'(\lambda_n|) &= \frac{1}{\lambda_n}\left(\ln\frac{2}{\lambda_n} + \sum_{j=1}^{n-2} \ln\left|1 - \frac{\lambda_n^2}{\lambda_j^2}\right| + \ln\left|1 - \frac{\lambda_n^2}{\lambda_{n-1}^2}\right| + \ln\left|1 - \frac{\lambda_n^2}{\lambda_{n+1}^2}\right| + \sum_{j=n+2}^{+\infty} \ln\left|1 - \frac{\lambda_n^2}{\lambda_j^2}\right|\right)\\ &= \frac{1}{\lambda_n}\ln\frac{2}{\lambda_n} + \frac{1}{\lambda_n}\ln\left(\frac{\lambda_n^2}{\lambda_{n-1}^2} - 1\right) + \frac{1}{\lambda_n}\ln\left(1 - \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) + F_n + G_n,\end{aligned}$$

with $F_n = \dfrac{1}{\lambda_n}\displaystyle\sum_{j=1}^{n-2} \ln\left(\frac{\lambda_n^2}{\lambda_j^2} - 1\right)$, and $G_n = \dfrac{1}{\lambda_n}\displaystyle\sum_{j=n+2}^{+\infty} \ln\left(1 - \frac{\lambda_n^2}{\lambda_j^2}\right)$.
Moreover, thanks to Corollary 3.3 and Corollary 3.6, we know that the asymptotic growth speed of $(\lambda_n)_{n\in\mathbb{N}}$ is of order n^2, so it is also the asymptotic growth speed of $(\lambda_{n+1} + \lambda_n)_{n\in\mathbb{N}}$, and that the asymptotic growth

speed of $(\lambda_{2n+2}-\lambda_{2n+1})_{n\in\mathbb{N}}$ is of order n. Moreover, (3.13) and (3.21) give that the asymptotic decay speed of $(\lambda_{2n+1}-\lambda_{2n})_{n\in\mathbb{N}}$ is of order $\dfrac{1}{n^{2\nu-1}}$. Therefore, we have

$$\begin{aligned}
&\frac{1}{\lambda_n}\ln\frac{2}{\lambda_n}\underset{n\to+\infty}{\longrightarrow}0,\\
&\frac{1}{\lambda_n}\ln\left(\frac{\lambda_n^2}{\lambda_{n-1}^2}-1\right)=-\frac{1}{\lambda_n}2\ln(\lambda_{n-1})+\frac{1}{\lambda_n}\ln(\lambda_n+\lambda_{n-1})+\frac{1}{\lambda_n}\ln(\lambda_n-\lambda_{n-1})\underset{n\to+\infty}{\longrightarrow}0,\\
&\frac{1}{\lambda_n}\ln\left(1-\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)=-\frac{1}{\lambda_n}2\ln(\lambda_{n+1})+\frac{1}{\lambda_n}\ln(\lambda_{n+1}+\lambda_n)+\frac{1}{\lambda_n}\ln(\lambda_{n+1}-\lambda_n)\underset{n\to+\infty}{\longrightarrow}0.
\end{aligned}$$

Thus, if we prove that $F_n\underset{n\to+\infty}{\longrightarrow}0$ and $G_n\underset{n\to+\infty}{\longrightarrow}0$, then, according to Definition 4.1, we will have proved that $c(\Lambda)=0$.

Proof of $F_n\underset{n\to+\infty}{\longrightarrow}0$.

First, we notice that

$$|F_n|\leqslant\frac{1}{\lambda_n}\sum_{\substack{j=1\\ \sqrt{2}\lambda_j<\lambda_n}}^{n-2}\ln\left(\frac{\lambda_n^2}{\lambda_j^2}-1\right)+\frac{1}{\lambda_n}\sum_{\substack{j=1\\ \sqrt{2}\lambda_j>\lambda_n}}^{n-2}\ln\left(\frac{\lambda_j^2}{\lambda_n^2-\lambda_j^2}\right),\quad\forall n\in\mathbb{N}\setminus\{0\}.$$

Moreover, for $n\geqslant 2$,

$$0\leqslant\frac{1}{\lambda_n}\sum_{\substack{j=1\\ \sqrt{2}\lambda_j<\lambda_n}}^{n-2}\ln\left(\frac{\lambda_n^2}{\lambda_j^2}-1\right)\leqslant\frac{1}{\lambda_n}\sum_{\substack{j=1\\ \sqrt{2}\lambda_j<\lambda_n}}^{n-2}\ln\left(\frac{\lambda_n^2}{\lambda_1^2}-1\right)\leqslant\frac{n-2}{\lambda_n}\ln\left(\frac{\lambda_n^2}{\lambda_1^2}-1\right)\underset{n\to+\infty}{\longrightarrow}0,$$

since the asymptotic growth speed of $(\lambda_n)_{n\in\mathbb{N}}$ is of order n^2. Moreover, for $n\geqslant 2$,

$$0\leqslant\frac{1}{\lambda_n}\sum_{\substack{j=1\\ \sqrt{2}\lambda_j>\lambda_n}}^{n-2}\ln\left(\frac{\lambda_j^2}{\lambda_n^2-\lambda_j^2}\right)\leqslant\frac{1}{\lambda_n}\sum_{\substack{j=1\\ \sqrt{2}\lambda_j>\lambda_n}}^{n-2}\ln\left(\frac{\lambda_n^2}{\lambda_n^2-\lambda_{n-2}^2}\right)\leqslant\frac{n-2}{\lambda_n}\ln\left(\frac{\lambda_n^2}{\lambda_n^2-\lambda_{n-2}^2}\right)\underset{n\to+\infty}{\longrightarrow}0,$$

thanks to the already mentioned information about the asymptotic expansion of the eigenvalues and their differences.

Therefore, we proved that $F_n\underset{n\to+\infty}{\longrightarrow}0$.

Proof of $G_n\underset{n\to+\infty}{\longrightarrow}0$. First, we notice that

$$|G_n|\leqslant\frac{1}{\lambda_n}\sum_{j=n+2}^{+\infty}\ln\left(\frac{\lambda_j^2}{\lambda_j^2-\lambda_n^2}\right)=\frac{1}{\lambda_n}\sum_{j=n+2}^{+\infty}\ln\left(1+\frac{\lambda_n^2}{\lambda_j^2-\lambda_n^2}\right),\quad\forall n\in\mathbb{N}\setminus\{0\}.$$

Moreover, we recall that for any $x>0$, we have $\ln(1+x)\leqslant x$. Therefore,

$$|G_n|\leqslant\frac{1}{\lambda_n}\sum_{j=n+2}^{+\infty}\frac{\lambda_n^2}{\lambda_j^2-\lambda_n^2}=\sum_{j=n+2}^{+\infty}\frac{\lambda_n}{(\lambda_j+\lambda_n)(\lambda_j-\lambda_n)}\leqslant\sum_{j=n+2}^{+\infty}\frac{1}{\lambda_j-\lambda_n}.$$

We notice that the series on the right-hand side of this inequality can be written as

$$\sum_{j=n+2}^{+\infty}\frac{1}{\lambda_j-\lambda_n}=\sum_{k\ \text{ such that } 2(k-1)\geqslant n+2}^{+\infty}\frac{1}{\lambda_{2(k-1)}-\lambda_n}+\sum_{k\ \text{ such that } 2(k-1)+1\geqslant n+2}^{+\infty}\frac{1}{\lambda_{2(k-1)+1}-\lambda_n},\quad\forall n>0.$$

First, we want to prove that $G_{2(n-1)} \underset{n\to+\infty}{\longrightarrow} 0$. We have

$$\left|G_{2(n-1)}\right| \leqslant \sum_{k=n+1}^{+\infty} \frac{1}{\lambda_{2(k-1)} - \lambda_{2(n-1)}} + \sum_{k=n+1}^{+\infty} \frac{1}{\lambda_{2(k-1)+1} - \lambda_{2(n-1)}}, \quad \forall n > 1. \tag{4.3}$$

If$\nu \in \left(0, \frac{1}{2}\right)$, Corollary 3.3 gives that

$$\lambda_{2(n-1)} \underset{n\to+\infty}{=} \pi^2 \left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2 + \varepsilon_{2\nu-1}^{even}(n), \quad \lambda_{2(n-1)+1} \underset{n\to+\infty}{=} \pi^2 \left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2 + \varepsilon_{2\nu-1}^{odd}(n),$$

with $\varepsilon_{2\nu-1}^{even}(n), \varepsilon_{2\nu-1}^{odd}(n) = \mathcal{O}\left(\frac{1}{n^{2\nu-1}}\right)$. From there, it is not difficult to infer that

$$\begin{aligned} \exists N \in \mathbb{N}, \forall n > N, \forall k > n, \quad \lambda_{2(k-1)} - \lambda_{2(n-1)} &\geqslant (\pi^2 - 1)\left(k - \frac{\nu}{2} - \frac{1}{4}\right)^2 - (\pi^2 - 1)\left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2, \\ \text{and } \lambda_{2(k-1)+1} - \lambda_{2(n-1)} &\geqslant (\pi^2 - 1)\left(k - \frac{\nu}{2} - \frac{1}{4}\right)^2 - (\pi^2 - 1)\left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2. \end{aligned} \tag{4.4}$$

If $\nu \in [\frac{1}{2}, 1)$, Corollary 3.6 gives that

$$\begin{aligned} \lambda_{2(n-1)} &\underset{n\to+\infty}{=} \pi^2 \left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2 - 2\frac{4\nu^2 - 1}{8} + \varepsilon_{2\nu-1}^{even}(n), \\ \lambda_{2(n-1)+1} &\underset{n\to+\infty}{=} \pi^2 \left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2 - 2\frac{4\nu^2 - 1}{8} + \varepsilon_{2\nu-1}^{odd}(n), \end{aligned}$$

with $\varepsilon_{2\nu-1}^{even}(n), \varepsilon_{2\nu-1}^{odd}(n) = \mathcal{O}\left(\frac{1}{n^{2\nu-1}}\right)$. Therefore, it is very easy to obtain the same statement (4.4).
Let $\nu \in (0, 1)$. From (4.3) and (4.4), we obtain that for any $n > N$,

$$\begin{aligned} \left|G_{2(n-1)}\right| &\leqslant \frac{2}{\pi^2 - 1} \sum_{k=n+1}^{+\infty} \frac{1}{\left(k - \frac{\nu}{2} - \frac{1}{4}\right)^2 - \left(n - \frac{\nu}{2} - \frac{1}{4}\right)^2} = \frac{2}{\pi^2 - 1} \sum_{k=1}^{+\infty} \frac{1}{\left(k - \frac{\nu}{2} - \frac{1}{4}\right)^2 + 2n\left(k - \frac{\nu}{2} - \frac{1}{4}\right)} \\ &\leqslant \frac{2}{\pi^2 - 1}\left(\frac{1}{\left(1 - \frac{\nu}{2} - \frac{1}{4}\right)^2 + 2n\left(1 - \frac{\nu}{2} - \frac{1}{4}\right)} + \sum_{k=2}^{+\infty} \int_{k-1}^{k} \frac{1}{\left(x - \frac{\nu}{2} - \frac{1}{4}\right)^2 + 2n\left(x - \frac{\nu}{2} - \frac{1}{4}\right)} \mathrm{d}x\right) \\ &= \frac{2}{\pi^2 - 1}\left(\frac{1}{\left(1 - \frac{\nu}{2} - \frac{1}{4}\right)^2 + 2n\left(1 - \frac{\nu}{2} - \frac{1}{4}\right)} + \left[\frac{1}{2n}\ln\left(\frac{x - \frac{\nu}{2} - \frac{1}{4}}{x - \frac{\nu}{2} - \frac{1}{4} + 2n}\right)\right]_1^{+\infty}\right) \underset{n\to+\infty}{\longrightarrow} 0. \end{aligned}$$

We can reiterate the proof by following the same steps to obtain that we have as well $G_{2(n-1)+1} \underset{n\to+\infty}{\longrightarrow} 0$. As a consequence, we have that the condensation index $c(\Lambda)$ is 0, which concludes the proof. □

From the previous Proposition, we easily deduce the existence of biorthogonal families at any time $T > 0$.

PROPOSITION 4.5. *Let $T > 0$. We consider $(t \mapsto e^{-\lambda_n t})_{n\in\mathbb{N}} \in L^2(0,T)^{\mathbb{N}}$, with $\{\lambda_n, n \in \mathbb{N}\}$ the eigenvalues of A_ν defined in Proposition 2.7 and 2.1. Then, there exists a biorthogonal family $(q_n)_{n\in\mathbb{N}} \in L^2(0,T)^{\mathbb{N}}$ to $(t \mapsto e^{-\lambda_n t})_{n\in\mathbb{N}}$ such that*

$$\forall \varepsilon > 0, \ \exists C_\varepsilon(T) > 0, \ \forall n \in \mathbb{N}, \quad ||q_n||_{L^2(0,T)} \leqslant C_\varepsilon(T) e^{\varepsilon \lambda_n}. \tag{4.5}$$

Proof of Proposition 4.5. Let $T > 0$. From (2.21), Corollary 3.3, and Corollary 3.6, we obtain that $(\lambda_n)_{n\in\mathbb{N}}$ is a non-negative increasing sequence and $\sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{\lambda_n} < +\infty$. Therefore, we can use [3, Theorem 4.1] (which can be easily adapted to include the case where 0 is a simple eigenvalue by an appropriate translation argument in time), which gives the existence of the biorthogonal family $(q_n)_{n\in\mathbb{N}}$. Proposition 4.4 gives that the condensation index is equal to 0. Therefore, [3, Remark 4.3] gives (4.5), which concludes the proof. $\square$

Now, we want to prove the following Proposition.

PROPOSITION 4.6. *Let ω be a measurable set of $(-1,1)$ of positive Lebesgue measure. Then,*

$$\inf_{n\in\mathbb{N}} \int_\omega \phi_n^2 > 0,$$

with $(\phi_n)_{n\in\mathbb{N}}$ the Hilbert basis of eigenfunctions defined in Theorem 2.11.

REMARK 4.7. *Let us emphasize here that obtaining this proposition is not straightforward, and relies on the explicit form of our eigenfunctions. This comes from the fact that we are working here with singular potentials and non-conventional self-adjoint extensions, so that many tools from perturbation theory are not available. Indeed, for the usual eigenfunctions of the Dirichlet-Laplace operator on $(0,1)$ given by $\sqrt{2}\sin(n\pi x)$, we have, for any measurable set $\omega \subset (0,1)$ of positive measure, by the Riemann-Lebesgue Lemma,*

$$\lim_{n\to+\infty} \int_\omega 2\sin(n\pi x)^2 dx = \lim_{n\to+\infty} \int_\omega (1-\cos(2n\pi x)dx) = |\omega|,$$

where $|\cdot|$ denotes the Lebesgue measure. So, we easily infer that

$$\inf_{n\in\mathbb{N}\setminus\{0\}} \int_\omega 2\sin(n\pi x)^2 dx > 0.$$

For the Dirichlet-Laplace operator with bounded potential, it is easy to prove by a perturbation argument that the same property holds (see e.g. [19, Appendix A]). However, this reasoning is ineffective in our case, because our eigenfunctions cannot be written as adequate perturbations of sin *or* cos *functions.*

In order to prove Proposition 4.6, we need several intermediate steps. First, we state several asymptotic results that will be useful later on.

LEMMA 4.8. *Let $\beta_0 \in (0,1)$.*

$$J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}\beta\right) \underset{n\to+\infty}{=} \frac{1}{\sqrt{\pi}}\left(\frac{2}{\pi\beta n}\right)^{1/2}\left(\sin\left(\sqrt{\lambda_{2(n-1)}}\beta+\nu\frac{\pi}{2}+\frac{\pi}{4}\right)+\mathcal{O}_{\beta_0}\left(\frac{1}{n}\right)\right), \forall 1>\beta\geqslant\beta_0, \tag{4.6.a}$$

$$J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}\right) \underset{n\to+\infty}{=} \mathcal{O}\left(\frac{1}{n^{1/2+\min(1,2\nu)}}\right), \tag{4.6.b}$$

$$J_{-\nu}{}'\left(\sqrt{\lambda_{2(n-1)}}\beta\right) \underset{n\to+\infty}{=} \frac{1}{\sqrt{\pi}}\left(\frac{2}{\pi\beta n}\right)^{1/2}\left(\cos\left(\sqrt{\lambda_{2(n-1)}}\beta+\nu\frac{\pi}{2}+\frac{\pi}{4}\right)+\mathcal{O}_{\beta_0}\left(\frac{1}{n}\right)\right), \forall 1>\beta\geqslant\beta_0, \tag{4.6.c}$$

$$J_{-\nu}{}'\left(\sqrt{\lambda_{2(n-1)}}\right) \underset{n\to+\infty}{=} \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{2}{\pi n}\right)^{1/2}\left(1+\mathcal{O}\left(\frac{1}{n^{\min(1,4\nu)}}\right)\right), \tag{4.6.d}$$

$$a_{2(n-1)}^2 \underset{n\to+\infty}{=} \frac{1}{\pi}\frac{2}{\pi n}\left(1+\mathcal{O}\left(\frac{1}{n^{\min(1,2\nu)}}\right)\right), \tag{4.6.e}$$

where $a_{2(n-1)}$ is defined in (2.23), and $\mathcal{O}_{\beta_0}(\,\cdot\,)$ means: there exists $C(\beta_0)>0$ and $n_0\in\mathbb{N}\setminus\{0\}$, that might depend on β_0, such that for any $n\geqslant n_0$ and any $\beta\geqslant\beta_0$,

$$|\mathcal{O}_{\beta_0}(\,\cdot\,)| \leqslant C(\beta_0)\,|\cdot|\,.$$

Proof of Lemma 4.8. Let $\beta_0 \in (0,1)$.
Proof of (4.6.a) and (4.6.b). From Lemma 1.10, we obtain

$$J_{-\nu}(x) \underset{x\to+\infty}{=} \sqrt{\frac{2}{\pi x}}\left(\cos\left(x+\nu\frac{\pi}{2}-\frac{\pi}{4}\right)+\mathcal{O}\left(\frac{1}{x}\right)\right).$$

Let $\beta \geqslant \beta_0$. We know that $\sqrt{\lambda_{2(n-1)}} \underset{n\to+\infty}{\longrightarrow} +\infty$. Therefore,

$$\begin{aligned}J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}\beta\right)&=\sqrt{\frac{2}{\pi\sqrt{\lambda_{2(n-1)}}\beta}}\left(\cos\left(\sqrt{\lambda_{2(n-1)}}\beta+\nu\frac{\pi}{2}-\frac{\pi}{4}\right)+\mathcal{O}_{\beta_0}\left(\frac{1}{\sqrt{\lambda_{2(n-1)}}}\right)\right)\\&=\frac{1}{\sqrt{\pi}}\left(\frac{2}{\pi\beta n}\right)^{1/2}\left(1+\mathcal{O}_{\beta_0}\left(\frac{1}{n}\right)\right)^{-1/2}\left(\sin\left(\sqrt{\lambda_{2(n-1)}}\beta+\nu\frac{\pi}{2}+\frac{\pi}{4}\right)+\mathcal{O}_{\beta_0}\left(\frac{1}{n}\right)\right),\end{aligned}$$

where we used Corollary 3.7 for the asymptotic expansion of $\sqrt{\lambda_{2(n-1)}}$. This concludes the proof of (4.6.a). Now, let us take $\beta=1$. Using the previous computations, we obtain

$$\begin{aligned}J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}\right)&\underset{n\to+\infty}{=}\mathcal{O}\left(\frac{1}{n^{1/2}}\right)\left[\sin\left(\pi\left(n-\frac{\nu}{2}-\frac{1}{4}\right)+\mathcal{O}\left(\frac{1}{n^{\min(1,2\nu)}}\right)+\nu\frac{\pi}{2}+\frac{\pi}{4}\right)+\mathcal{O}\left(\frac{1}{n}\right)\right]\\&\underset{n\to+\infty}{=}\mathcal{O}\left(\frac{1}{n^{1/2}}\right)\left((-1)^n\mathcal{O}\left(\frac{1}{n^{\min(1,2\nu)}}\right)+\mathcal{O}\left(\frac{1}{n}\right)\right)\underset{n\to+\infty}{=}\mathcal{O}\left(\frac{1}{n^{1/2+\min(1,2\nu)}}\right),\end{aligned}$$

which concludes the proof of (4.6.b).

Proof of (4.6.c) and (4.6.d). Lemma 1.9 states that ${J_{-\nu}}' = \frac{1}{2}\left(J_{-\nu-1}-J_{-\nu+1}\right)$. Therefore, by using Lemma 1.10 on $J_{-\nu-1}$ and $J_{-\nu+1}$, we obtain that

$$\begin{aligned}{J_{-\nu}}'(x)&\underset{x\to+\infty}{=}\frac{1}{2}\sqrt{\frac{2}{\pi x}}\left(\cos\left(x-(-\nu-1)\frac{\pi}{2}-\frac{\pi}{4}\right)-\cos\left(x-(-\nu+1)\frac{\pi}{2}-\frac{\pi}{4}\right)+\mathcal{O}\left(\frac{1}{x}\right)\right)\\&\underset{x\to+\infty}{=}\frac{1}{2}\sqrt{\frac{2}{\pi x}}\left(\cos\left(x+\nu\frac{\pi}{2}+\frac{\pi}{4}\right)-\cos\left(x+\nu\frac{\pi}{2}-3\frac{\pi}{4}\right)+\mathcal{O}\left(\frac{1}{x}\right)\right)\\&\underset{x\to+\infty}{=}\sqrt{\frac{2}{\pi x}}\left(\cos\left(x+\nu\frac{\pi}{2}+\frac{\pi}{4}\right)+\mathcal{O}\left(\frac{1}{x}\right)\right).\end{aligned}$$

Let $\beta \geqslant \beta_0$. Since $\sqrt{\lambda_{2(n-1)}} \underset{n\to+\infty}{\longrightarrow} +\infty$, we have

$$\begin{aligned}{J_{-\nu}}'\left(\sqrt{\lambda_{2(n-1)}}\beta\right)&\underset{n\to+\infty}{=}\sqrt{\frac{2}{\pi\sqrt{\lambda_{2(n-1)}}\beta}}\left(\cos\left(\sqrt{\lambda_{2(n-1)}}\beta+\nu\frac{\pi}{2}+\frac{\pi}{4}\right)+\mathcal{O}\left(\frac{1}{\sqrt{\lambda_{2(n-1)}}\beta}\right)\right)\\&\underset{n\to+\infty}{=}\frac{1}{\sqrt{\pi}}\left(\frac{2}{\pi\beta n}\right)^{1/2}\left(1+\mathcal{O}_{\beta_0}\left(\frac{1}{n}\right)\right)^{-1/2}\left(\cos\left(\sqrt{\lambda_{2(n-1)}}\beta+\nu\frac{\pi}{2}+\frac{\pi}{4}\right)+\mathcal{O}_{\beta_0}\left(\frac{1}{n}\right)\right)\end{aligned}$$

which concludes the proof of (4.6.c). Now, we fix $\beta=1$. Substituting $\sqrt{\lambda_{2(n-1)}}$ in (4.6.c) with Corollary 3.7 easily gives (4.6.d).

Proof of (4.6.e). Let $n>1$. We recall that the expression of $a_{2(n-1)}$ is given in (2.23).

Since $\sqrt{\lambda_{2(n-1)}} \underset{n\to+\infty}{=} \mathcal{O}(n)$, we obtain from Corollary 1.11 that

$$\begin{aligned}
a_{2(n-1)}^2 &= \left(1+\mathcal{O}\left(\frac{1}{n^2}\right)\right) J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}\right)^2 + {J_{-\nu}}'\left(\sqrt{\lambda_{2(n-1)}}\right)^2 \\
&+\mathcal{O}\left(\frac{1}{n^{2\nu}}\right)\left[\left(1+\mathcal{O}\left(\frac{1}{n^2}\right)\right)\mathcal{O}\left(\frac{1}{n^{1/2}}\right)\mathcal{O}\left(\frac{1}{n^{1/2}}\right)+\mathcal{O}\left(\frac{1}{n^{1/2}}\right)\mathcal{O}\left(\frac{1}{n^{1/2}}\right)+\mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&+\mathcal{O}\left(\frac{1}{n^{4\nu}}\right)\left[\left(1+\mathcal{O}\left(\frac{1}{n^2}\right)\right)\mathcal{O}\left(\frac{1}{n}\right)+\mathcal{O}\left(\frac{1}{n}\right)\right] \\
&= \left(1+\mathcal{O}\left(\frac{1}{n^2}\right)\right) J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}\right)^2 + {J_{-\nu}}'\left(\sqrt{\lambda_{2(n-1)}}\right)^2 + \mathcal{O}\left(\frac{1}{n^{1+2\nu}}\right).
\end{aligned}$$

Using (4.6.b) and (4.6.d), we obtain

$$\begin{aligned}
a_{2(n-1)}^2 &\underset{n\to+\infty}{=} \left(1+\mathcal{O}\left(\frac{1}{n^2}\right)\right)\mathcal{O}\left(\frac{1}{n^{1+\min(2,4\nu)}}\right)+\frac{2}{\pi^2 n}\left(1+\mathcal{O}\left(\frac{1}{n^{\min(1,4\nu)}}\right)\right)+\mathcal{O}\left(\frac{1}{n^{1+2\nu}}\right) \\
&\underset{n\to+\infty}{=} \frac{2}{\pi^2 n}\left(1+\mathcal{O}\left(\frac{1}{n^{\min(1,4\nu)}}\right)+\mathcal{O}\left(\frac{1}{n^{\min(2,4\nu)}}\right)+\mathcal{O}\left(\frac{1}{n^{2\nu}}\right)\right), \quad \text{which gives (4.6.e).}
\end{aligned}$$

□

REMARK 4.9. *The reader can convince themselves that all the asymptotic behaviours that we obtained so far in this section are still valid when the index* $2(n-1)$ *is replaced by* $2(n-1)+1$ *and* $\dfrac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}$ *is replaced by* $\dfrac{1-2\nu}{1+2\nu}\dfrac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}$. *Notably, Lemma 4.8 also holds for even indexes* $2(n-1)+1$.

The previous asymptotic result is useful to prove the following non-concentration property of our eigenfunctions on any non-trivial interval that is far from 0.

PROPOSITION 4.10. *Let* $\alpha_0 \in (0,1)$. *There exists* $C>0$ *and* $n_0 \in \mathbb{N}\setminus\{0\}$ *(that might depend on* α_0 *and* ν*) such that*

$$\int_\alpha^\beta \phi_n^2 \geqslant C(\beta-\alpha), \quad \alpha_0 \leqslant \alpha < \beta < 1, \quad \forall n \geqslant n_0.$$

Let $\beta_0 \in (0,1)$. *There exists* $C>0$ *and* $n_0 \in \mathbb{N}\setminus\{0\}$ *(that might depend on* β_0 *and* ν*) such that*

$$\int_\alpha^\beta \phi_n^2 \geqslant C(\beta-\alpha), \quad -1 < \alpha < \beta \leqslant -\beta_0, \quad \forall n \geqslant n_0,$$

with $(\phi_n)_{n\in\mathbb{N}}$ *the Hilbert basis of* $L^2(-1,1)$ *of eigenfunctions defined in Theorem 2.11.*

Proof of Proposition 4.10. Since $\phi_{2(n-1)}^2$ and $\phi_{2(n-1)+1}^2$ are even, there is no loss of generality by assuming that we are in the case $\alpha_0 \leqslant \alpha < \beta < 1$. First, let us start with the even indexes $2(n-1)$. We have, for all $n>1$,

$$\begin{aligned}
\int_\alpha^\beta \psi_{2(n-1)}^2 &= \int_\alpha^\beta x J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}x\right)^2 \mathrm{d}x \\
&\quad - 2\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\frac{\sqrt{\lambda_{2(n-1)}}}{2}\right)^{-2\nu}\int_\alpha^\beta x J_\nu\left(\sqrt{\lambda_{2(n-1)}}x\right)J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}x\right)\mathrm{d}x \qquad (4.7) \\
&\quad + \left(\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\frac{\sqrt{\lambda_{2(n-1)}}}{2}\right)^{-2\nu}\right)^2 \int_\alpha^\beta x J_\nu\left(\sqrt{\lambda_{2(n-1)}}x\right)^2 \mathrm{d}x.
\end{aligned}$$

From the mean value Theorem for integrals, there exists $r_n \in (\alpha, \beta)$ such that

$$\int_\alpha^\beta xJ_\nu\left(\sqrt{\lambda_{2(n-1)}}x\right) J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}x\right) \mathrm{d}x = (\beta-\alpha) r_n J_\nu\left(\sqrt{\lambda_{2(n-1)}}r_n\right) J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}r_n\right).$$

From Corollary 1.11, we deduce that there exists $C > 0$ such that for any n large enough,

$$\left| r_n J_\nu\left(\sqrt{\lambda_{2(n-1)}}r_n\right) J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}r_n\right) \mathrm{d}x \right| \leqslant \frac{C}{\lambda_{2(n-1)}}.$$

Using Corollary 3.7, we deduce that for some new $C > 0$,

$$\left| -2\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \left(\frac{\sqrt{\lambda_{2(n-1)}}}{2}\right)^{-2\nu} \int_\alpha^\beta xJ_\nu\left(\sqrt{\lambda_{2(n-1)}}x\right) J_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}x\right) \mathrm{d}x \right| \leqslant \frac{(\beta-\alpha)C}{n^{1+2\nu}}. \tag{4.8}$$

A similar reasoning also ensures that for n large enough,

$$\left| \left(\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \left(\frac{\sqrt{\lambda_{2(n-1)}}}{2}\right)^{-2\nu}\right)^2 \int_\alpha^\beta xJ_\nu\left(\sqrt{\lambda_{2(n-1)}}x\right)^2 \mathrm{d}x \right| \leqslant \frac{(\beta-\alpha)C}{n^{1+4\nu}}. \tag{4.9}$$

Now, let us go back to the first term of (4.7). Using (1.9.c), with (4.6.a) and (4.6.c), we obtain that

$$\begin{aligned}
\int_\alpha^\beta xJ_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}x\right)^2 &= \frac{1}{2}\beta^2 \left(\left(1 - \frac{\nu^2}{\sqrt{\lambda_{2(n-1)}}^2 \beta^2}\right) J_{-\nu}(\sqrt{\lambda_{2(n-1)}}\beta)^2 + {J_{-\nu}}'(\sqrt{\lambda_{2(n-1)}}\beta)^2 \right) \\
&\quad - \frac{1}{2}\alpha^2 \left(\left(1 - \frac{\nu^2}{\sqrt{\lambda_{2(n-1)}}^2 \alpha^2}\right) J_{-\nu}(\sqrt{\lambda_{2(n-1)}}\alpha)^2 + {J_{-\nu}}'(\sqrt{\lambda_{2(n-1)}}\alpha)^2 \right) \\
&= \frac{1}{2}\beta^2 \left[\left(1 + \mathcal{O}_{\alpha_0}\left(\frac{1}{n^2}\right)\right) \frac{2}{\pi^2 \beta n} \left(\sin^2\left(\beta\sqrt{\lambda_{2(n-1)}} + \nu\frac{\pi}{2} + \frac{\pi}{4}\right) + \mathcal{O}_{\alpha_0}\left(\frac{1}{n}\right)\right) \right. \\
&\quad \left. + \frac{2}{\pi^2 \beta n} \left(\cos^2\left(\beta\sqrt{\lambda_{2(n-1)}} + \nu\frac{\pi}{2} + \frac{\pi}{4}\right) + \mathcal{O}_{\alpha_0}\left(\frac{1}{n}\right)\right) \right] \\
&\quad - \frac{1}{2}\alpha^2 \left[\left(1 + \mathcal{O}_{\alpha_0}\left(\frac{1}{n^2}\right)\right) \frac{2}{\pi^2 \alpha n} \left(\sin^2\left(\alpha\sqrt{\lambda_{2(n-1)}} + \nu\frac{\pi}{2} + \frac{\pi}{4}\right) + \mathcal{O}_{\alpha_0}\left(\frac{1}{n}\right)\right) \right. \\
&\quad \left. + \frac{2}{\pi^2 \alpha n} \left(\cos^2\left(\alpha\sqrt{\lambda_{2(n-1)}} + \nu\frac{\pi}{2} + \frac{\pi}{4}\right) + \mathcal{O}_{\alpha_0}\left(\frac{1}{n}\right)\right) \right].
\end{aligned}$$

Since $\cos^2 + \sin^2 = 1$, we have

$$\int_\alpha^\beta xJ_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}x\right)^2 \mathrm{d}x = \frac{1}{\pi^2 n}(\beta-\alpha)\left(1 + \mathcal{O}_{\alpha_0}\left(\frac{1}{n}\right)\right).$$

Hence, we deduce that there exists $C > 0$, that might depend on ν and α_0, and that might from now on change from line to line, there exists $\tilde{n}_0 \in \mathbb{N} \setminus \{0\}$ (depending also on ν and α_0) such that for any $n \geqslant \tilde{n}_0$,

$$\int_\alpha^\beta xJ_{-\nu}\left(\sqrt{\lambda_{2(n-1)}}x\right)^2 \mathrm{d}x \geqslant \frac{C}{n}(\beta-\alpha).$$

Coming back to (4.7) and using (4.8) and (4.9), we deduce that there exists some $n_0 \geqslant \tilde{n}_0$ such that for $n \geqslant n_0$, we have

$$\int_\alpha^\beta \psi_{2(n-1)}^2 \geqslant \frac{C}{n}(\beta-\alpha).$$

Therefore, using (4.6.e), we obtain that for some (possibly larger) n_0, we have

$$\int_\alpha^\beta \phi_{2(n-1)}^2 \geqslant C(\beta-\alpha), \quad \forall n \geqslant n_0.$$

This concludes the proof for even indexes of eigenfunctions and for $\alpha_0 \leqslant \alpha, \beta < 1$.

Then, we observe that for $n > 0$, the expression of $\phi_{2(n-1)+1}^2$ is very similar to the one of $\phi_{2(n-1)}^2$ on $(0,1)$, except that the index $2(n-1)$ is replaced by $2(n-1)+1$ and that $\dfrac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}$ is replaced by $\dfrac{1-2\nu}{1+2\nu}\dfrac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}$. Therefore, using Remark 4.9, we also have (for some possibly larger n_0)

$$\int_\alpha^\beta \phi_{2(n-1)+1}^2 \geqslant C(\beta-\alpha), \quad \forall n \geqslant n_0.$$

□

By using the structure of the open sets of $\mathbb{R}$, we can deduce the following result.

COROLLARY 4.11. *Let $\alpha_0 \in (0,1)$. There exists $C > 0$ and $n_0 \in \mathbb{N} \setminus \{0\}$ (that might depend on α_0 and ν) such that for any open subset $\mathcal{O}$ that is included in $(\alpha_0, 1)$ or $(-1, -\alpha_0)$, we have*

$$\int_{\mathcal{O}} \phi_n^2 \geqslant C|\mathcal{O}|, \quad ,\forall n \geqslant n_0.$$

Proof of Corollary 4.11. As already explained, it is enough to treat the case where $\mathcal{O} \subset (\alpha_0, 1)$. Then, $\mathcal{O}$ is a disjoint union of a countable collection of open intervals $\{(a_k, b_k)\}_{k\in\mathcal{I}}$ with $\mathcal{I}$ at most countable, and for any $k \in I$, $\alpha_0 < a_k < b_k < 1$. With these notations, we have

$$\int_{\mathcal{O}} \phi_n^2 = \sum_{k\in\mathcal{I}} \int_{a_k}^{b_k} \phi_n^2.$$

Using Lemma 4.10, this gives, for some $C > 0$ and $n_0 \in \mathbb{N} \setminus \{0\}$ depending on α_0 and ν,

$$\int_{\mathcal{O}} \phi_n^2 \geqslant C \sum_{k\in\mathcal{I}} (b_k - a_k) = C|\mathcal{O}|, \quad \forall n \geqslant n_0,$$

whence the result.

□

The last step is to pass from an open set to a measurable set of positive measure.

Proof of Proposition 4.6 Now, we consider ω a measurable set of $(-1,1)$ assumed to have positive Lebesgue measure. Let $\tilde{\omega}$ be another measurable set of $(-1,1)$. We will use repeatedly the following property:

$$\tilde{\omega} \subset \omega \Rightarrow \inf_{n\in\mathbb{N}} \int_\omega \phi_n^2 \geqslant \inf_{n\in\mathbb{N}} \int_{\tilde{\omega}} \phi_n^2.$$

Hence, to prove the desired result, is is enough to prove it for a measurable subset of ω.

Assume without loss of generality that $|\omega \cap [0,1)| > 0$ (otherwise, we use a symmetry argument as before). Therefore, we can change ω into $\omega \cap [0,1)$, according to the discussion above, and we still call it ω. By the Lebesgue density Theorem, for almost every $x \in \omega$ and for $r_x > 0$ small enough, we have $|\omega \cap B_x(r_x)| > 0$. Since $\{0\}$ is of null measure, and $\omega \subset [0,1)$, there exists some $x_0 \in \omega$ with $x_0 > 0$ and some $r > 0$ such that $|\omega \cap B_{x_0}(r)| > 0$. Reducing r if necessary, we can assume that $r < x_0/2$. We call $2\alpha_0 = x_0/2$. Then

$\omega \cap B_{x_0}(r) \subset (2\alpha_0, 1)$. Hence, we can change ω into $\omega \cap B_{x_0}(r)$, so we reduce to the case where $\omega \subset (2\alpha_0, 1)$ with $\alpha_0 > 0$.

By exterior regularity of the Lebesgue measure [27, Theorem 2.20], for all $k \in \mathbb{N} \setminus \{0\}$, there exists an open set $\mathcal{O}_k$ of $(-1, 1)$ with $\omega \subseteq \mathcal{O}_k$ such that $|\mathcal{O}_k \setminus \omega| < 1/k$. Moreover, reducing $\mathcal{O}_k$ if necessary, we can assume that $\mathcal{O}_k \subset (\alpha_0, 1)$ for any $k \in \mathbb{N} \setminus \{0\}$. By the "reverse" of the dominated convergence Theorem, we know that for some extraction φ,

$$\mathbb{1}_{\mathcal{O}_{\varphi(k)} \setminus \omega} \underset{k \to +\infty}{\longrightarrow} 0, \quad \text{a.e. on } (-1, 1).$$

Since $\phi_n^2 \in L^1(-1, 1)$ for any $n \in \mathbb{N} \setminus \{0\}$, and

$$|\mathbb{1}_{\mathcal{O}_{\varphi(k)}} \phi_n^2| \leqslant |\phi_n^2|,$$

the dominated convergence Theorem ensures that

$$\int_{-1}^{1} \mathbb{1}_{\mathcal{O}_{\varphi(k)}} \phi_n^2 \underset{k \to +\infty}{\longrightarrow} \int_\omega \phi_n^2.$$

In other words,

$$\int_{\mathcal{O}_{\varphi(k)}} \phi_n^2 \underset{k \to +\infty}{\longrightarrow} \int_\omega \phi_n^2. \tag{4.10}$$

Moreover, since $\mathcal{O}_{\varphi(k)} \subset (\alpha_0, 1)$ for any $k \in \mathbb{N} \setminus \{0\}$, thanks to Proposition 4.11, there exists $C > 0$ and $n_0 \in \mathbb{N} \setminus \{0\}$, that might depend on α_0 and ν, such that for any $k \in \mathbb{N} \setminus \{0\}$, we have

$$\int_{\mathcal{O}_{\varphi(k)}} \phi_n^2 \geqslant C |\mathcal{O}_{\varphi(k)}|, \quad \forall n \geqslant n_0.$$

One can make $k \to +\infty$ in this inequality, and deduce by (4.10) that

$$\int_\omega \phi_n^2 \geqslant C|\omega|, \quad \forall n \geqslant n_0.$$

For $n < n_0$, as a consequence of the analyticity of the Bessel functions on $\mathbb{C} \setminus \{0\}$ together with the fact that every normalized eigenfunction ϕ_n for $n \in \mathbb{N} \setminus \{0\}$ is under the form

$$\phi_n(x) = \alpha_n \sqrt{x} J_\nu(\delta x) + \beta_n \sqrt{x} J_{-\nu}(\delta x),\ x > 0,$$

for some $\alpha_n, \beta_n \in \mathbb{R} \setminus \{0\}$, for some $\delta \neq 0$, and for some $\nu \in (0, 1)$, we clearly have

$$\int_\omega \phi_n^2 > 0, \quad \forall n \leqslant n_0, n > 0,$$

and from Proposition 2.1, for any normalized eigenfunction ϕ_0 associated to the eigenvalue 0, we also have

$$\int_\omega \phi_0^2 > 0.$$

Hence, we have proved our desired result. $\square$

We are now ready for the proof of our main Theorem.

Proof of Theorem 1.1. Let $T > 0$, let ω be a measurable set of $(-1, 1)$ assumed to have positive Lebesgue measure. Let $f^0 \in L^2(-1, 1)$. We know that $(\phi_n)_{n \in \mathbb{N}}$ is a Hilbert basis of $L^2(-1, 1)$, with $(\phi_n)_{n \in \mathbb{N}}$ defined in Theorem 2.11. Therefore,

$$f^0 = \sum_{j \in \mathbb{N}} \left\langle f^0, \phi_j \right\rangle_{L^2(-1,1)} \phi_j.$$

We denote the unique solution of (1.1) in $C^0\left([0,T], L^2(-1,1)\right)$ by f. We recall that by the Duhamel formula, we have

$$f(t) = e^{-A_\nu t} f^0 + \int_0^t e^{-A_\nu (t-s)} \mathbb{1}_\omega u(s) \mathrm{d}s, \quad \forall t \in [0,T]. \tag{4.11}$$

Choosing $t = T$ in (4.11), and taking the scalar product with ϕ_n gives that

$$\begin{aligned} \forall n \in \mathbb{N}, \quad \langle f(T),\ \phi_n \rangle &= \left\langle e^{-A_\nu T} f^0,\ \phi_n \right\rangle_{L^2(-1,1)} + \left\langle \int_0^T e^{-A_\nu (T-s)} \mathbb{1}_\omega u(s) \mathrm{d}s,\ \phi_n \right\rangle_{L^2(-1,1)} \\ &= e^{-\lambda_n T} \left\langle f^0,\ \phi_n \right\rangle_{L^2(-1,1)} + \int_0^T e^{-\lambda_n (T-s)} \left\langle u(s),\ \mathbb{1}_\omega \phi_n \right\rangle_{L^2(-1,1)} \mathrm{d}s. \end{aligned} \tag{4.12}$$

According to [30, Section 5.3.3] (for instance), we choose the following control

$$u(t,x) = -\sum_{n \in \mathbb{N}} \left\langle f^0,\ \phi_n \right\rangle_{L^2(-1,1)} e^{-\lambda_n T} q_n(T-t) \frac{\mathbb{1}_\omega \phi_n(x)}{||\mathbb{1}_\omega \phi_n||^2_{L^2(-1,1)}}, \tag{4.13}$$

after having checked that it makes sense.
Proposition 4.6 allows to be sure that for any $n \in \mathbb{N}$, $||\mathbb{1}_\omega \phi_n||_{L^2(-1,1)} > 0$. Now, we need to check that $u \in L^2\left((0,T) \times (-1,1)\right)$. First, we have for all $t \in [0,T]$

$$||u(t)||^2_{L^2(-1,1)} = \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \left\langle f^0, \phi_n \right\rangle \left\langle f^0, \phi_k \right\rangle e^{-\lambda_n T} e^{-\lambda_k T} \frac{q_n(T-t)}{||\mathbb{1}_\omega \phi_n||} \frac{q_k(T-t)}{||\mathbb{1}_\omega \phi_k||} \left\langle \frac{\mathbb{1}_\omega \phi_n}{||\mathbb{1}_\omega \phi_n||},\ \frac{\mathbb{1}_\omega \phi_k}{||\mathbb{1}_\omega \phi_k||} \right\rangle.$$

Moreover,

$$\begin{aligned} \forall n,k \in \mathbb{N}, \ &\left| \int_0^T \left\langle f^0,\ \phi_n \right\rangle \left\langle f^0,\ \phi_k \right\rangle \ e^{-\lambda_n T} e^{-\lambda_k T} \frac{q_n(T-t)}{||\mathbb{1}_\omega \phi_n||} \frac{q_k(T-t)}{||\mathbb{1}_\omega \phi_k||} \left\langle \frac{\mathbb{1}_\omega \phi_n}{||\mathbb{1}_\omega \phi_n||},\ \frac{\mathbb{1}_\omega \phi_k}{||\mathbb{1}_\omega \phi_k||} \right\rangle \mathrm{d}t \right| \\ &\qquad \leqslant \frac{\left| \left\langle f^0,\ \phi_n \right\rangle \left\langle f^0,\ \phi_k \right\rangle \right|}{\inf_{j \in \mathbb{N}} ||\mathbb{1}_\omega \phi_j||^2} e^{-\lambda_n T} e^{-\lambda_k T} ||q_n||_{L^2(0,T)} ||q_k||_{L^2(0,T)} \\ &\qquad \leqslant \frac{\left| \left\langle f^0,\ \phi_n \right\rangle \left\langle f^0,\ \phi_k \right\rangle \right|}{\inf_{j \in \mathbb{N}} ||\mathbb{1}_\omega \phi_j||^2} e^{-\lambda_n T} e^{-\lambda_k T} C^2_{T/2} e^{\lambda_n \frac{T}{2}} e^{\lambda_k \frac{T}{2}}, \end{aligned}$$

where we used Proposition 4.5 for $\varepsilon = \frac{T}{2}$ to obtain an upper bound to $||q_n||^2_{L^2(0,T)}$, and Proposition 4.6, which gives $\inf_{n \in \mathbb{N}} ||\mathbb{1}_\omega \phi_n||^2_{L^2(-1,1)} > 0$. Therefore, we can use Fubini's theorem to get

$$\begin{aligned} \int_0^T ||u(t)||^2 \mathrm{d}t &= \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \int_0^T \left\langle f^0,\ \phi_n \right\rangle \left\langle f^0,\ \phi_k \right\rangle e^{-\lambda_n T} e^{-\lambda_k T} \frac{q_n(T-t)}{||\mathbb{1}_\omega \phi_n||} \frac{q_k(T-t)}{||\mathbb{1}_\omega \phi_k||} \left\langle \frac{\mathbb{1}_\omega \phi_n}{||\mathbb{1}_\omega \phi_n||},\ \frac{\mathbb{1}_\omega \phi_k}{||\mathbb{1}_\omega \phi_k||} \right\rangle \mathrm{d}t \\ &\leqslant \frac{C^2_{T/2}}{\inf_{j \in \mathbb{N}} ||\mathbb{1}_\omega \phi_j||^2} \left(\sum_{n \in \mathbb{N}} \left| \left\langle f^0,\ \phi_n \right\rangle \right| e^{-\lambda_n \frac{T}{2}} \right)^2 < +\infty \text{ because } f^0 \in L^2(-1,1). \end{aligned}$$

This proves that $u \in L^2\left((0,T) \times (-1,1)\right)$. Now, we examine what (4.12) gives when we choose this control. Using the biorthogonal property of $(q_n)_{n \in \mathbb{N}}$, we have

$$\begin{aligned} \forall n \in \mathbb{N}, \quad \int_0^T e^{-\lambda_n (T-s)} \left\langle u(s),\ \mathbb{1}_\omega \phi_n \right\rangle_{L^2(-1,1)} \mathrm{d}s &= -\left\langle f^0,\ \phi_n \right\rangle e^{-\lambda_n T} \left\langle \frac{\mathbb{1}_\omega \phi_n}{||\mathbb{1}_\omega \phi_n||^2_{L^2(-1,1)}},\ \mathbb{1}_\omega \phi_n \right\rangle_{L^2(-1,1)} \\ &= -\left\langle f^0,\ \phi_n \right\rangle e^{-\lambda_n T}. \end{aligned}$$

Therefore, from (4.12), we have

$$\langle f(T),\ \phi_n\rangle_{L^2(-1,1)} = 0, \quad \forall n \in \mathbb{N},$$

which means that $f(T) = 0$. This concludes the proof of Theorem 1.1.

□

4.2 Boundary Control

Let us now explain briefly how one can deduce an appropriate boundary control result for the system

$$\begin{cases} \partial_t f(t,x) - \partial^2_{xx} f(t,x) + \dfrac{c}{x^2} f(t,x) = 0, & (t,x) \in (0,T)\times(-1,1), \\ f(t,-1) = 0,\ f(t,1) = u(t), & t \in (0,T), \\ f(0,x) = f^0(x), & x \in (-1,1), \end{cases} \tag{4.14}$$

where $u \in L^2(0,T)$ (the case where the control acts at $x=-1$ can be treated similarly). The first important point is to understand the well-posedness of (4.14), which is less obvious as for (4.14), since the control operator is now unbounded. The procedure is classical and we only give the main ingredients. Without loss of generality, we study instead

$$\begin{cases} \partial_t f(t,x) - \partial^2_{xx} f(t,x) + \dfrac{c}{x^2} f(t,x) + f(t,x) = 0, & (t,x) \in (0,T)\times(-1,1), \\ f(t,-1) = 0,\ f(t,1) = u(t), & t \in (0,T), \\ f(0,x) = f^0(x), & x \in (-1,1). \end{cases} \tag{4.15}$$

Indeed, one can pass from one equation to another by multiplying the solutions by e^{-t} (or e^t), without changing the well-posedness or controllability results. The main interest of system (4.15) is that the underlying elliptic operator $A_\nu + \mathrm{Id}$ is now (self-adjoint and) positive. The eigenvectors are unchanged and the eigenvalues are shifted by 1.

PROPOSITION 4.12. *For any $n \in \mathbb{N}\setminus\{0\}$, we have $\phi_n'(1) \neq 0$. Moreover,*

$$\phi'_{2(n-1)}(1) \underset{n\to+\infty}{=} (-1)^n \pi n \left(1 + \mathcal{O}\left(\frac{1}{n^{\min(1,2\nu)}}\right)\right), \tag{4.16}$$

$$\phi'_{2(n-1)+1}(1) \underset{n\to+\infty}{=} (-1)^n \pi n \left(1 + \mathcal{O}\left(\frac{1}{n^{\min(1,2\nu)}}\right)\right), \tag{4.17}$$

with $(\phi_n)_{n\in\mathbb{N}}$ the Hilbert basis of eigenfunctions defined in Theorem 2.11.

Proof of Proposition 4.12. The first assertion comes from the fact that $\phi_n(1) = 0$. Since ϕ_n is an eigenfunction of A_ν, it verifies a second-order ODE that is not singular in a neighborhood of $x=1$, so we cannot have simultaneously $\phi_n(1) = 0$ and $\phi_n'(1) = 0$.
Let $n \in \mathbb{N}\setminus\{0\}$, let $x \in (0,1]$, by looking at the form of $\psi_{2(n-1)}$ given in Theorem 2.11, we get

$$\begin{aligned} \psi'_{2(n-1)}(x) = {} & \frac{1}{2\sqrt{x}} J_{-\nu}(\sqrt{\lambda_{2(n-1)}}x) + \sqrt{x}\sqrt{\lambda_{2(n-1)}} J'_{-\nu}(\sqrt{\lambda_{2(n-1)}}x) \\ & - \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \left(\frac{\sqrt{\lambda_{2(n-1)}}}{2}\right)^{-2\nu} \left(\frac{1}{2\sqrt{x}} J_\nu(\sqrt{\lambda_{2(n-1)}}x) + \sqrt{x}\sqrt{\lambda_{2(n-1)}} J'_\nu(\sqrt{\lambda_{2(n-1)}}x)\right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
\psi'_{2(n-1)}(1) &= \frac{1}{2}\psi_{2(n-1)}(1) \\
&\quad + \sqrt{\lambda_{2(n-1)}}\left(J'_{-\nu}(\sqrt{\lambda_{2(n-1)}}) - \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\left(\frac{\sqrt{\lambda_{2(n-1)}}}{2}\right)^{-2\nu} J'_{\nu}(\sqrt{\lambda_{2(n-1)}})\right) \\
&\underset{n\to+\infty}{=} 0 + \pi n\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\left(J'_{-\nu}(\sqrt{\lambda_{2(n-1)}}) + \mathcal{O}\left(\frac{1}{n^{2\nu}}\right)\mathcal{O}\left(\frac{1}{n^{1/2}}\right)\right),
\end{aligned}
$$

where we used Corollary 3.7 and Lemma 1.9. Now, from Lemma 1.9 and Lemma 1.10, we get

$$
\begin{aligned}
J'_{-\nu}(x) &\underset{x\to+\infty}{=} \frac{1}{2}\left(J_{-\nu-1}(x) - J_{-\nu+1}(x)\right) \\
&\underset{x\to+\infty}{=} \frac{1}{2}\left(\frac{2}{\pi x}\right)^{1/2}\left(\cos\left(x+\nu\frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{4}\right) - \cos\left(x+\nu\frac{\pi}{2}-\frac{\pi}{2}-\frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{x}\right)\right) \\
&\underset{x\to+\infty}{=} \left(\frac{2}{\pi x}\right)^{1/2}\left(\cos\left(x+\nu\frac{\pi}{2}+\frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{x}\right)\right).
\end{aligned}
$$

We then replace x by $\sqrt{\lambda_{2(n-1)}}$ in the expression above. Thanks to Corollary 3.7, we get

$$
\begin{aligned}
J'_{-\nu}(\sqrt{\lambda_{2(n-1)}}) &\underset{n\to+\infty}{=} \frac{1}{\pi}\left(\frac{2}{n}\right)^{1/2}\left[1+\mathcal{O}\left(\frac{1}{n}\right)\right]\left[(-1)^n\cos\left(\mathcal{O}\left(\frac{1}{n^{\min(1,2\nu)}}\right)\right) + \mathcal{O}\left(\frac{1}{n}\right)\right] \\
&\underset{n\to+\infty}{=} \frac{(-1)^n}{\pi}\left(\frac{2}{n}\right)^{1/2}\left[1+\mathcal{O}\left(\frac{1}{n^{\min(1,4\nu)}}\right)\right].
\end{aligned}
$$

Thus,

$$
\psi'_{2(n-1)}(1) \underset{n\to+\infty}{=} \pi n\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\frac{(-1)^n}{\pi}\left(\frac{2}{n}\right)^{1/2}\left(1+\mathcal{O}\left(\frac{1}{n^{\min(1,2\nu)}}\right)\right).
$$

Using the asymptotic expression of $a_{2(n-1)}$ at (4.6.e), we directly obtain (4.16). We can follow the same computations to prove (4.17). ☐

From Proposition 4.12 and Corollary 3.3, we see that there exists some constant $c_1, c_2 > 0$ such that for any $n \in \mathbb{N}\setminus\{0\}$, we have

$$
c_1\sqrt{\lambda_n+1} \leqslant |\phi'_n(1)| \leqslant c_2\sqrt{\lambda_n+1}, \quad \forall n \in \mathbb{N}. \tag{4.18}
$$

From this estimate, one can obtain the following well-posedness result. Since $A_\nu + \mathrm{Id}$ is positive, one can define $(A_\nu+\mathrm{Id})^{1/2}$. Moreover, $A_\nu + \mathrm{Id}$ extends as an unbounded operator with domain $D((A_\nu+\mathrm{Id})^{1/2})$ and state space $D((A_\nu+\mathrm{Id})^{-1/2}) = D((A_\nu+\mathrm{Id})^{1/2})'$ (see [31, Corollary 3.4.6]), where for any $s \in \mathbb{R}$, $D((A_\nu+\mathrm{Id})^s)$ is equipped with the Hilbertian norm

$$
||f||_s^2 = \sum_{k=1}^{+\infty}(\lambda_k+1)^{2s}|\langle f,\phi_k\rangle_{L^2(-1,1)}|^2,
$$

which makes it a Hilbert space. Notably, we obtain a Hilbert basis $(\tilde{\phi}_n)_{n\in\mathbb{N}}$ of eigenfunctions in $D(A_\nu^{-1/2})$ as

$$
\tilde{\phi}_n = \sqrt{\lambda_n+1}\phi_n,
$$

and estimate (4.18) becomes

$$
(\lambda_n+1)c_1 \leqslant |\tilde{\phi}'_n(1)| \leqslant c_2(\lambda_n+1), \quad \forall n \in \mathbb{N}. \tag{4.19}
$$

For the sake of simplicity, from now on, we introduce

$$H_{\pm 1/2} = D((A_\nu + \mathrm{Id})^{\pm 1/2}).$$

Standard computations (see for instance [31, Section 10.7], where the usual Laplacian operator is replaced by A_ν) show that in the state space $H_{1/2}$, the first line of (4.15) can be written in an abstract way as

$$y' = (A_\nu + \mathrm{Id})y + bu,$$

where $b \in H_{-1/2}$ is a scalar control operator that can be written as $b = (A_\nu + \mathrm{Id})\mathcal{D}$, where $\mathcal{D} : \mathbb{R} \to L^2(-1,1)$ is the Dirichlet map that is given by duality by (see [31, Proposition 10.6.1])

$$\mathcal{D}^* g = -((A_\nu + \mathrm{Id})^{-1} g)'(1),\ \forall g \in L^1(-1,1).$$

Notably, for any $k \in \mathbb{N}$,

$$\begin{aligned} b_k &:= \langle b, \tilde{\phi}_k \rangle_{H_{-1/2}} \\ &= \frac{1}{\lambda_k + 1} \langle b, \tilde{\phi}_k \rangle_{L^2(-1,1)} \\ &= \frac{1}{\lambda_k + 1} \langle (A_\nu + \mathrm{Id})\mathcal{D}, \tilde{\phi}_k \rangle_{L^2(-1,1)} \\ &= \frac{1}{\lambda_k + 1} \langle 1, \mathcal{D}^*(A_\nu + \mathrm{Id})\tilde{\phi}_k \rangle_{\mathbb{R}} \\ &= -\frac{1}{1 + \lambda_k} \tilde{\phi}_k'(1). \end{aligned}$$

Hence, (4.19) becomes

$$c_1 \leqslant |b_k| \leqslant c_2,\ k \in \mathbb{N}. \tag{4.20}$$

The upper bound in (4.20) together with the fact that there exists some constant $C > 0$ such that for any $k \in \mathbb{N} \setminus \{0\}$, we have $1 \leqslant 1 + \lambda_k \leqslant Ck^2$ (by Corollary 3.3) easily imply that the sequence $(b_k)_{k \in \mathbb{N}}$ verifies the Carleson measure criterion for admissibility given in [31, Definition 5.3.1].

From [31, Theorem 5.3.2], we deduce that b is an admissible control operator, and we have the following well-posedness result: for any $f^0 \in D((A_\nu + \mathrm{Id})^{1/2})'$, there exists a unique weak solution f to (4.15) verifying $f \in C^0([0,T], D((A_\nu + \mathrm{Id})^{1/2})')$.

We are now ready to give or null boundary controllability result in arbitrary small time.

THEOREM 4.13. *For any $f^0 \in D((A_\nu + \mathrm{Id})^{1/2})'$ and any $T > 0$, there exists a control $u \in L^2(0,T)$ such that the corresponding solution f to* (4.15) *verifies $f(T) = 0$ in $D((A_\nu + \mathrm{Id})^{1/2})'$.*

Proof of Theorem 4.13. Since b is an admissible control operator, since the eigenvalues $\lambda_k + 1$ are positive for $k \in \mathbb{N}$, and $\sum_{k=0}^{+\infty} \frac{1}{\lambda_k + 1} < +\infty$ by Corollary 3.3, one can apply directly [3, Theorem 2.5] and obtain that (4.15) is null controllable for any $T > T_0$, where

$$T_0 = \limsup_{k \to +\infty} \left(\frac{\log \frac{1}{|b_k|}}{\lambda_k + 1} + c(\Lambda + 1) \right).$$

Here, $c(\Lambda + 1)$ is the condensation index of the sequence $\{\lambda_{n+1} + 1\}_{n \in \mathbb{N} \setminus \{0\}}$, as defined in Definition 4.1. From the lower bound given in (4.20), we have that

$$\frac{\log \frac{1}{|b_k|}}{\lambda_k + 1} \underset{k \to +\infty}{\to} 0.$$

Moreover, it is easy to infer that $c(\Lambda + 1) = c(\Lambda)$, where $c(\Lambda)$ is defined in Proposition 4.4 and verifies $c(\Lambda) = 0$. Hence, $T_0 = 0$, and our result follows. $\square$

5 Further comments and open problems

The case $c_\nu = -\frac{1}{4}$. We have cautiously excluded this case in our study (as in [22]), whereas this critical case does not lead necessarily to ill-posedness (the other critical case $c = \frac{3}{4}$ is impossible, on the contrary). However, the functional setting is not clear. Indeed, this is equivalent to impose $\nu = 0$, and we lose the coercivity estimate (1.6), which is notably crucial for the definition of the domain of A_ν. In the case $\nu = 0$, we need to change the definition of $D(A_\nu)$ according to [1, Proposition 3.1], and to find a suitable replacement for (1.6). In fact, (1.6) is proved by using the following Hardy inequality:

$$\int_{-1}^{1} \frac{z(x)^2}{x^2} \leqslant 4 \int_{-1}^{1} z_x(x)^2,\ \forall z \in H^{-1}(-1,1) \text{ such that } z(0) = 0,$$

that might be replaced in the case $\nu = 0$ by an improved Hardy-Poincaré type inequalities like in [34, Theorem 2.2]. This case would require an appropriate and extensive treatment that is likely to be different from the situation we presently studied, and is outside the scope of this article.

Backstepping. Since we proved null-controllability in any time $T > 0$, we know that rapid stabilization (*i.e.* exponential stabilization at any arbitrarily large rate) holds (see [29, Proposition 21 and Theorem 25]). However, it would be interesting to construct an explicit feedback thanks to the backstepping method. This would require to solve a second order PDE of wave type, with two singular potentials, one depending on space and one on time. We were not able to understand in which functional setting such an equation could be well-posed.

Other self-adjoint extensions. As highlighted before, our result highly depends on the self-adjoint extension that we use for the Laplace operator with core $C_0^\infty((-1,0) \cup (0,1))$. The natural (and "minimal") one leads to a lack of null-controllability from one side, whereas choosing whats seems to be the "best" extension (in terms of transmission conditions) leads to a positive null-controllability result. However, there are a lot of intermediate self-adjoint extensions. It would be interesting from a conceptual point of view to perform an exhaustive study in the particular case of our heat equation with inverse square potential.

Mixed singular/degenerate equations. In [32], the author proved some controllability results for parabolic equations posed on $(0,1)$, with degenerate diffusion and singular potential at the boundary $x = 0$. It would be interesting to understand if these results can be extended in our case of an interior degeneracy/singularity and a control region that is only located in one side of the boundary. It would require to develop a similar functional setting, with adequate transmission conditions, but we expect to the computations to be much heavier.

Grushin equation with a singularity. The original motivation of [22] was to study the controllability properties of the Grushin equation with inverse square potential

$$\partial_t f(t,x,y) - \partial_{xx}^2 f(t,x,y) + \frac{c}{x^2} f(t,x,y) = u(t,x,y)\mathbb{1}_\omega(x,y),\ (t,x,y) \in (0,T) \times (-1,1) \times (0,1),$$

with Dirichlet boundary condition on the boundary of $(-1,1) \times (0,1)$, and ω is an open subset on $(-1,1) \times (0,1)$. This kind of model naturally arises when looking at the Laplace operator on almost-riemannian manifolds. In [22], approximate controllability is proved. It would be interesting to understand if null-controllability can hold at least in some particular geometrical settings and in large enough time, when controlling for instance in vertical strip that is located at one side of the singularity $x = 0$, as in [6] (where there is no singular potential), by using a Fourier decomposition in the y variable and reducing the problem to a uniform observability problem, as in [6]. In view of the present study, we expect that this question may be rather difficult.

Acknowledgements

This work was funded by the french Agence Nationale de la Recherche (Grant ANR-22-CPJ1-0027-01).

The first author would like to thank Eric Cances, David Gontier and Dario Prandi for having suggested the strategy of Proposition A.1.

A Ill-posedness of (1.1) for $c < -\frac{1}{4}$

The goal of this appendix is to prove the following Proposition.

PROPOSITION A.1. *Assume we work in the state space $L^2(-1,1)$. Assume that $c < -\frac{1}{4}$. Then, there does not exist any selfadjoint extension of $\partial_{xx}^2 - \dfrac{c}{x^2}\mathrm{Id}$, posed on $C_0^\infty((-1,1)\setminus\{0\})$, for which this operator generates a C^0-semigroup.*

Proof of Proposition A.1. We reason by contradiction. Consider $(A, D(A))$ any selfadjoint extension of $(\partial_{xx}^2 - \dfrac{c}{x^2}\mathrm{Id}, C_0^\infty((-1,1)\setminus\{0\}))$, such that A generates a C^0-semigroup. Then, it is well-known (see [14, Proposition, p. 91]) that A is semibounded in the following sense: there exists $C > 0$ such that for any $f \in D(A)$, we have

$$\langle f, Af\rangle_{L^2(-1,1)} \leqslant C\|f\|^2_{L^2(-1,1)}. \tag{A.1}$$

Notably, for $f \in C_0^\infty((0,1))$ extended by 0 on $(-1,0)$, (A.1) becomes (after one integration by parts)

$$-\int_0^1 f'(x)^2dx - c\int_0^1 \frac{f(x)^2}{x^2}dx \leqslant C\int_0^1 f(x)^2dx. \tag{A.2}$$

By an easy density argument, (A.2) also holds for any $f \in H_0^1((0,1))$. Let $\varepsilon \in (0,1)$. Let $f(x) = x^{1/2+\varepsilon}(1-x)$. We have $f \in H_0^1(0,1)$. Explicit computations give

$$\int_0^1 f(x)^2dx = \frac{1}{4\varepsilon^3 + 18\varepsilon^2 + 26\varepsilon + 12},$$

$$\int_0^1 \frac{f(x)^2}{x^2}dx = \frac{1}{2\varepsilon + 6\varepsilon^2 + 4\varepsilon^3},$$

$$\int_0^1 f'(x)^2dx = \frac{2\epsilon + 1}{8\epsilon^2 + 8\epsilon}.$$

Hence, as $\varepsilon \to 0$, the right-hand side of (A.2) tends to $C/12$, whereas the left-hand side is equivalent to

$$\frac{1}{\varepsilon}\left(-\frac{1}{8} - \frac{c}{2}\right).$$

Since $c < -1/4$, this quantity goes to $+\infty$ as $\varepsilon \to 0$. This is in contradiction with (A.2), which concludes the proof. $\square$

References

[1] V. Alekseeva and A. Ananeva, *On extensions of the Bessel operator on a finite interval and the half-line*, Ukr. Mat. Visn. 9 (2012), no. 2, 147–156, 297; translation in J. Math. Sci. (N.Y.) 187 (2012), no. 1, 1–8.

[2] B. Allal, G. Fragnelli and J. Salhi, *Null controllability for a singular heat equation with a memory term*, Electron. J. Qual. Theory Differ. Equ. 2021, Paper No. 14, 24 pp.

[3] F. Ammar Khodja, A. Benabdallah, M. Gonzalez-Burgos, and L. De Teresa, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*,J. Funct. Anal. 267 (2014), no. 7, 2077–2151.

[4] P. Baras and J.A. Goldstein, *The heat equation with a singular potential*, Trans. Amer. Math. Soc. 284 (1984), no. 1, 121–139.

[5] P. Baras and J.A. Goldstein, *Remarks on the inverse square potential in quantum mechanics*, Differential equations (Birmingham, Ala., 1983), 31–35. North-Holland Math. Stud., 92, Amsterdam, 1984.

[6] K. Beauchard, K., P. Cannarsa, and R. Guglielmi, *Null controllability of Grushin-type operators in dimension two*, J. Eur. Math. Soc. (JEMS) 16 (2014), no. 1, 67–101.

[7] J. Bebernes and D. Eberly, *Mathematical problems from combustion theory*, Appl. Math. Sci., 83 Springer-Verlag, New York, 1989. viii+177 pp.

[8] U. Biccari, *Boundary controllability for a one-dimensional heat equation with a singular inverse square potential*, Math. Control Relat. Fields 9 (2019), no. 1, 191–219.

[9] U. Biccari and E. Zuazua, *Null controllability for a heat equation with a singular inverse square potential involving the distance to the boundary function*, J. Differential Equations 261 (2016), no. 5, 2809–2853.

[10] U. Boscain and C. Laurent, *The Laplace-Beltrami operator in almost-Riemannian geometry*, Ann. Inst. Fourier (Grenoble) 63 (2013), no. 5, 1739–1770.

[11] C. Cazacu, *Controllability of the heat equation with an inverse square potential localized on the boundary*, SIAM J. Control Optim. 52 (2014), no. 4, 2055–2089.

[12] T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations*, Translated from the 1990 French original by Yvan Martel and revised by the authors, Oxford Lecture Ser. Math. Appl., 13 The Clarendon Press, Oxford University Press, New York, 1998. xiv+186 pp.

[13] J.-M. Coron, *Control and Nonlinearity*, Mathematical Surveys and Monographs, Volume 136, American Mathematical Society, 2007.

[14] K. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Grad. Texts in Math., 194 Springer-Verlag, New York, 2000. xxii+586 pp.

[15] S. Ervedoza, *Control and stabilization properties for a singular heat equation with an inverse square potential*, Comm. Partial Differential Equations 33 (2008), no. 10-12, 1996–2019.

[16] H. O. Fattorini and D.L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.

[17] M. Fotouhi and L. Salimi, *Null controllability of degenerate/singular parabolic equations*, J. Dyn. Control Syst. 18 (2012), no. 4, 573–602.

[18] J. Lagnese, *Control of wave processes with distributed controls supported on a subregion*, SIAM J. Control Optim. 21 (1983), no. 1, 68–85.

[19] T. Liard, P. Lissy and Y. Privat, *Non-localization of eigenfunctions for Sturm-Liouville operators and applications*, J. Differential Equations 264 (2018), no. 4, 2449–2494.

[20] Y. L. Luke, *Integrals of Bessel functions*, McGraw-Hill Book Co., Inc., New York-Toronto-London, 1962.

[21] P. Martinez and J. Vancostenoble, *The cost of boundary controllability for a parabolic equation with inverse square potential*, Evol. Equ. Control Theory 8 (2019), no. 2, 397–422.

[22] M. Morancey, *Approximate controllability for a 2D Grushin equation with potential having an internal singularity*, Annales de l'Institut Fourier, Tome 65, no 4 (2015), pp. 1525-1556.

[23] L. Ouaili, *Minimal time of null controllability of two parabolic equations*, Math. Control Relat. Fields 10 (2020), no. 1, 89–112.

[24] T. Pálmai, *On the interlacing of cylinder functions*, Math. Inequal. Appl. 16 (2013), no. 1, 241–247.

[25] X. Qin and S. Li, *The null controllability for a singular heat equation with variable coefficients*, Appl. Anal. 101 (2022), no. 3, 1052–1076.

[26] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. xv+361 pp.

[27] W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill Book Company, International Edition, 1987.

[28] A. Shao, and B. Vergara, *Approximate boundary controllability for parabolic equations with inverse square infinite potential wells*, Nonlinear Anal. 248 (2024), Paper No. 113624, 25 pp.

[29] E. Trélat, G. Wang and Y.and Xu, *Characterization by observability inequalities of controllability and stabilization properties*, Pure Appl. Anal. 2 (2020), no. 1, 93–122.

[30] E. Trélat, *Control in Finite and Infinite Dimension*,SpringerBriefs PDEs Data Sci, Springer, Singapore, 2024. viii+138 pp.

[31] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Verlag, Basel, 2009. xii+483 pp.

[32] J. Vancostenoble, *Improved Hardy-Poincaré inequalities and sharp Carleman estimates for degenerate/singular parabolic problems*, Discrete Contin. Dyn. Syst. Ser. S 4 (2011), no. 3, 761–790.

[33] J. Vancostenoble, and E. Zuazua, *Null controllability for the heat equation with singular inverse square potentials*, J. Funct. Anal. 254 (2008), no. 7, 1864–1902.

[34] J. L. Vazquez, and E. Zuazua, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse square potential*, J. Funct. Anal. 173 (2000), no. 1, 103–153.

[35] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Dover Publications, 1944.

[36] G. Zheng, B.-Z. Guo and M. Montaz Ali, *Stability of optimal control of heat equation with singular potential*, Systems Control Lett. 74 (2014), 18–23.