A business dinner problem

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A "scheduling" problem

A problem submitted to the Junior company of the Ecole des Ponts a few years ago.

- A pool of providers and a pool of customers.
- Each provider and each customer have to meet once.
- Two providers have to meet at most once.

A series of dinners has to be planned. Tables are available, with a given capacity.

Two people meet if they take their dinner at a same table.

Objective. Minimize the number of dinners in order to achieve the constraints above.

An example

Feasible schedule in 6 dinners for an instance with

- 2 tables, 5 suppliers, 6 customers
- at most 2 suppliers per table, at most 3 customers per table

dinner 1:	table 1:	suppliers: 1,2 customers: 1,2,3	table 2:	suppliers: 3,4 customers: 4,5
dinner 2:	table 1:	suppliers: 3 customers: 2	table 2:	suppliers: 5 customers: 5
dinner 3:	table 1:	suppliers: 2 customers: 4,5	table 2:	suppliers: 4 customers: 2,6
dinner 4:	table 1:	suppliers: 3 customers: 1,3	table 2:	suppliers: 5 customers: 2,6
dinner 5:	table 1:	suppliers: 1,5 customers: 4	table 2:	suppliers: 2,3 customers: 6
dinner 6:	table 1:	suppliers: 4,5 customers: 1,3	table 2:	suppliers: 1 customers: 5,6

Parameters and Results

Parameters:

- Number of tables: t
- Number of suppliers: s
- Number of customers: c
- At most σ suppliers per table
- At most γ customers per table

Contribution:

- Feasible schedule with not too many dinners for any value of the parameters
- Lower bound for any value of the parameters
- Special cases with closed formula
- Open questions

Related works: combinatorial design

Kirkman Schoolgirl problem

15 *schoolgirls must be split into five groupe of three girls each day of the week so that no two girls are in the same group twice.*

Similarity with our problem: suppliers = schoolgirls, tables = groups, etc.

Social Golfer problem

Howell designs

Oberworlfach problem (Ringel, 1967)

Is it possible to seat an odd number n of mathematicians at m round tables of sizes $(\ell_1, \ldots, \ell_m)$ with $\sum_i \ell_i = n$ in $(n-1)/2$ dinners so that each mathematician sits next to everyone else exactly once?

Conjecture: 'yes' (except for exceptional values of the ℓ_i's)

Bipartite Analogue of the Oberwolfach problem

Oberwolfach Rectangular Table Negotiation problem

None of these are optimization problems.

Lower bounds

- Number of tables: t
- Number of suppliers: s
- Number of customers: c
- At most σ suppliers per table
- At most γ customers per table

Two suppliers at most once together; each supplier and each customer exactly once together.

Lower bounds on the number of dinners:

$$lb_1 = \left\lceil \frac{s}{\sigma} \right\rceil \qquad lb_2 = \left\lceil \frac{c}{\gamma} \right\rceil \qquad lb_3 = \left\lceil \frac{s}{t\sigma} \left\lceil \frac{c}{\gamma} \right\rceil \right\rceil$$

$$lb_4 = \left\lceil \frac{\sqrt{s}}{t\gamma} \left((c-\gamma) \max\left(\sqrt{\frac{\gamma}{c-\gamma}}, 1 \right) + \frac{\gamma}{\max\left(\sqrt{\frac{\gamma}{c-\gamma}}, 1 \right)} \right) \right\rceil$$

$$lb_5 = \max_{j \in \{2,\ldots,\sigma\}} \left\lceil \frac{s}{t} \left(\frac{2}{j} \left\lceil \frac{c}{\gamma} \right\rceil - \frac{s-1}{j(j-1)} \right) \right\rceil$$

Lower bounds lb_1 and lb_2

- Number of tables: t
- Number of suppliers: s
- Number of customers: c
- At most σ suppliers per table
- At most γ customers per table

During a dinner, any customer sits at a table with at most σ suppliers. To meet all suppliers, he needs at least $\lceil s/\sigma \rceil$ dinners.

During a dinner, any supplier sits at a table with at most γ customers. To meet all customers, he needs at least $\lceil c/\gamma \rceil$ dinners.

Lower bounds lb_3 and lb_5

x_{ij} = number of times supplier i sits at a table with j suppliers (him included)

$$
\begin{aligned}
\min \quad & \frac{1}{t}\sum_{i=1}^{s}\sum_{j=1}^{\sigma}\frac{1}{j}x_{ij} & & \\
\text{s.t.} \quad & \sum_{j=1}^{\sigma} x_{ij} \quad \geq \quad \left\lceil \frac{c}{\gamma} \right\rceil & & i \in [s] \\
& \sum_{j=1}^{\sigma}(j-1)x_{ij} \quad \leq \quad s-1 & & i \in [s] \\
& x_{ij} \in \mathbb{R}_+ & & i \in [s],\ j \in [\sigma].
\end{aligned}
$$

Optimal value equal to

$$\frac{s}{t}\max\left(\frac{1}{\sigma}\left\lceil\frac{c}{\gamma}\right\rceil, \max_{j\in\{2,\ldots,\sigma\}}\left(\frac{2}{j}\left\lceil\frac{c}{\gamma}\right\rceil - \frac{s-1}{j(j-1)}\right)\right).$$

Proof. Strong duality. Maximization of a piecewise linear concave function $\mathbb{R} \to \mathbb{R}$.

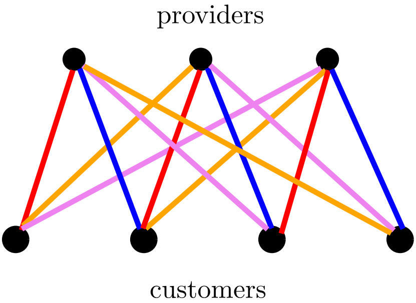
Exact solutions

If $\sigma = 1$, then the optimal number of dinners is

$$\max\left(s, \left\lceil \frac{c}{\gamma} \right\rceil, \left\lceil \frac{s}{t} \left\lceil \frac{c}{\gamma} \right\rceil \right\rceil \right).$$

Proof. If $\gamma = 1$, proper edge-coloring of $K_{s,c}$ with each color present at most t times and lb_1, lb_2, lb_3. For $\gamma > 1$, make groups.

$t = s = 3, c = 4, \sigma = \gamma = 1$



4 colors $\Rightarrow$ 4 dinners

Howell designs

A square array is a Howell design if

1. every cell is either empty or contains an unordered pair of elements (*symbols*) chosen from a set of size $2n$,
2. every symbol occurs exactly once in each row and each column,
3. every unordered pair of symbols occurs at most once in the array.

$t = 3$ tables, $s = 4$ suppliers, $\lceil c/\gamma \rceil = 3$ groups of customers (the columns), at most $\sigma = 2$ suppliers per table, in 3 dinners (the rows)

1,2	3,4	
3	1	2,4
4	2	1,3

$t = 5$ tables, $s = 6$ suppliers, $\lceil c/\gamma \rceil = 5$ groups of customers (the columns), at most $\sigma = 2$ suppliers per table, in 5 dinners (the rows)

	6	2,3	4,5	1
6	3,4	1	2	5
3,5	1,2		6	4
2,4		5	1	3,6
1	5	4,6	3	2

Exact solutions

If $\sigma = 2$, $s > c/\gamma$, $t \geq \min(c/\gamma, s/2)$ *and* $(\lceil c/\gamma \rceil, s) \neq (2, 4)$, *the optimal number of dinners is*

$$\max\left(\left\lceil \frac{c}{\gamma} \right\rceil, \left\lceil \frac{s}{2} \right\rceil\right).$$

Proof. Almost directly from Howell design and lb_1, lb_2.

If there are 4 tables, 7 suppliers, 26 customers, at most 2 suppliers per table, at most 4 customers per table, the optimal number of dinners is 7.

Exact solutions

If $t = \lceil s/2 \rceil$, $\sigma = 2$, *and* $\lceil c/\gamma \rceil \geq \frac{3}{2}s$, *then the optimal number of dinners is*

$$2\left\lceil \frac{c}{\gamma} \right\rceil - s + 1.$$

Proof. Combination of the previous two exact solutions and lb_5.

If there are 5 tables, 9 suppliers, 41 customers, at most 2 suppliers per table, at most 3 customers per table, the optimal number of dinners is 20.

Feasible schedules

$r(t, s, c, \sigma, \gamma) :=$ optimal value

$$r(t_1, s, c, \sigma, \gamma) \leq \left\lceil \frac{t_2}{t_1} \right\rceil r(t_2, s, c, \sigma, \gamma)$$

$$r(t, s, c, \sigma_1, \gamma) \leq \left\lceil \frac{\sigma_2}{\sigma_1} \right\rceil r(t, s, c, \sigma_2, \gamma)$$

$$r(t, s, c, \sigma, \gamma_1) \leq r(t, s, \lceil c/\gamma_2 \rceil, \sigma, \lceil \gamma_1/\gamma_2 \rceil) \qquad \text{if } \gamma_1 \geq \gamma_2$$

General upper bounds on $r(t, s, c, \sigma, \gamma)$

$$ub_1 = \left\lceil \frac{2}{\sigma} \right\rceil \left\lceil \frac{1}{t} \min\left(\left\lceil \frac{c}{\gamma} \right\rceil, s \right) \right\rceil \max\left(\left\lceil \frac{c}{\gamma} \right\rceil, \left\lceil \frac{s}{2} \right\rceil \right)$$

$$ub_2 = \left\lceil \frac{1}{t} \left\lceil \frac{s}{\sigma} \right\rceil \right\rceil \left(1 - \sigma + \sigma \max\left(\left\lceil \frac{c}{\gamma} \right\rceil, 2 \left\lceil \frac{s}{\sigma} \right\rceil \right) \right) \qquad \text{if } \lceil s/\sigma \rceil \leq \lceil c/\gamma \rceil$$

Inefficient when $\sigma \geq 3$.

Open question: making groups of customers

Example from the beginning:

$t = 2$ tables, $s = 5$ suppliers, $c = 6$ customers, at most $\sigma = 2$ suppliers per table, at most $\gamma = 3$ customers per table

dinner 1:	table 1:	suppliers: customers:	1,2 1,2,3	table 2:	suppliers: customers:	3,4 4,5
dinner 2:	table 1:	suppliers: customers:	3 2	table 2:	suppliers: customers:	5 5
dinner 3:	table 1:	suppliers: customers:	2 4,5	table 2:	suppliers: customers:	4 2,6
dinner 4:	table 1:	suppliers: customers:	3 1,3	table 2:	suppliers: customers:	5 2,6
dinner 5:	table 1:	suppliers: customers:	1,5 4	table 2:	suppliers: customers:	2,3 6
dinner 6:	table 1:	suppliers: customers:	4,5 1,3	table 2:	suppliers: customers:	1 5,6

Open question: making groups of customers

$t = 2$ tables, $s = 5$ suppliers, $c = 6$ customers, at most $\sigma = 2$ suppliers per table, at most $\gamma = 3$ customers per table

Optimal solution:

dinner 1:	table 1:	suppliers: customers:	1,2 1,2,3	table 2:	suppliers: customers:	3,4 4,5,6
dinner 2:	table 1:	suppliers: customers:	3 1,2,3	table 2:	suppliers: customers:	2,5 4,5,6
dinner 3:	table 1:	suppliers: customers:	4,5 1,2,3	table 2:	suppliers: customers:	1 4,5,6

Is there always an optimal solution in which the customers are split into groups, the members of each group staying together for all dinners?

Open question: $t = \gamma = 1$

What is the optimal solution when there is only one table and exactly one customer per table?

Thank you.