Hedetniemi's conjecture for Kneser hypergraphs

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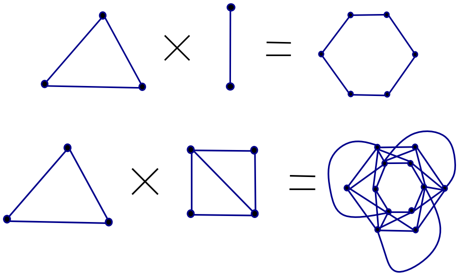
CERMICS, Optimisation et Systèmes

Product of graphs and Hedetniemi's conjecture

Let G, H be two graphs.

Categorical product $G \times H$:

$$V(G \times H) = V(G) \times V(H)$$
$$E(G \times H) = \{(u,v)(u',v') : \ uu' \in E(G), \ vv' \in E(H)\}$$



Hedetniemi's conjecture (1966)

$$\chi(G \times H) = \min(\chi(G), \chi(H))$$

Some known cases

Hedetniemi's conjecture has been proved in (not so many) special cases.

Theorem (El-Zahar-Sauer 1985)

If $\min(\chi(G), \chi(H)) \leq 4$, *Hedetniemi's conjecture is true.*

Theorem (Duffus-Sands-Woodrow 1985)

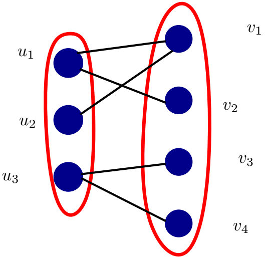
Let G *and* H *be connected graphs with* $\min(\chi(G), \chi(H)) > n$. *If both* G *and* H *contain a clique of size* n, *then* $\chi(G \times H) > n$.

In particular, it implies that Hedetniemi's conjecture is true for such graphs when $\min(\chi(G), \chi(H)) = n + 1$.

Product of hypergraphs and Zhu's conjecture

$\{u_1, u_2, u_3\} \in E(\mathcal{G})$ $\{v_1, v_2, v_3, v_4\} \in E(\mathcal{H})$

$\{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_3, v_3), (u_3, v_4)\} \in E(\mathcal{G} \times \mathcal{H})$



Let $\mathcal{G}, \mathcal{H}$ be two hypergraphs. Categorical product $\mathcal{G} \times \mathcal{H}$:

$$V(\mathcal{G} \times \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$$

$$E(\mathcal{G} \times \mathcal{H}) = \{e \subseteq V(\mathcal{G} \times \mathcal{H}) : \ \pi_{\mathcal{G}}(e) \in E(\mathcal{G}), \ \pi_{\mathcal{H}}(e) \in E(\mathcal{H})\}$$

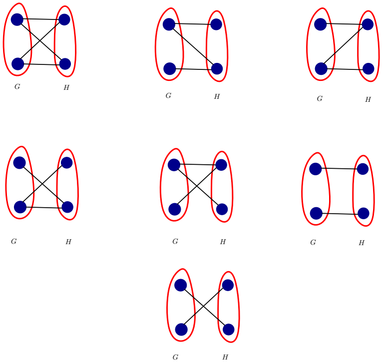
Proper coloring of a hypergraph: no monochromatic edges.

Zhu's conjecture (1992)

$$\chi(\mathcal{G} \times \mathcal{H}) = \min(\chi(\mathcal{G}), \chi(\mathcal{H}))$$

Product of graphs vs of hypergraphs, and coloring

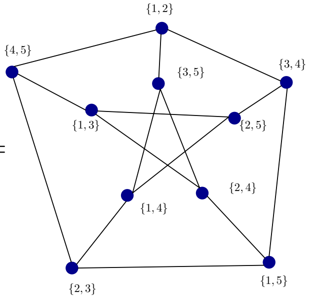
Product of two graphs seen as hypergraphs: will have edges of cardinality 3 and 4.



But the chromatic number does not depend of the definition of the product.

Zhu's conjecture is a true generalization of Hedetniemi's conjecture.

Kneser graphs

$\mathrm{KG}(5,2) =$ 

n, k two integers s.t. $n \geq 2k$.

Kneser graph $\mathrm{KG}(n,k)$:

$$V(\mathrm{KG}(n,k)) = \binom{[n]}{k}$$

$$E(\mathrm{KG}(n,k))) = \left\{ AB : A, B \in \binom{[n]}{k},\ A \cap B = \emptyset \right\}$$

Kneser hypergraphs and main result

n, k, r three integers s.t. $n \geq rk$.

Kneser hypergraph $\mathrm{KG}^r(n,k)$:

$$V(\mathrm{KG}^r(n,k)) = \binom{[n]}{k}$$

$$E(\mathrm{KG}^r(n,k))) = \left\{ \{A_1, \ldots, A_r\} : A_i \in \binom{[n]}{k},\ A_i \cap A_j = \emptyset \text{ for } i \neq j \right\}$$

Theorem (Hajiabolhassan-M.)

$$\chi\left(\mathrm{KG}^r(n,k) \times \mathrm{KG}^r(n',k')\right) = \min\left(\chi\left(\mathrm{KG}^r(n,k)\right), \chi\left(\mathrm{KG}^r(n',k')\right)\right)$$

Known to be true for $r = 2$ (Hell 1980).

Hedetniemi's conjecture for Kneser graphs

Proof of the general case explained for:

Theorem (Hell, 1980)

$\chi\left(\mathrm{KG}(n,k)\times\mathrm{KG}(n',k')\right)=\min\left(\chi\left(\mathrm{KG}(n,k)\right),\chi\left(\mathrm{KG}(n',k')\right)\right).$

Proof uses a combinatorial approach proposed by Matoušek (2003) for proving

$$\chi(\mathrm{KG}(n,k))=n-2k+2$$

(Lovász' theorem 1979).

Approach extended by Ziegler for computing $\chi(\mathrm{KG}^r(n,k))$.

Tucker's lemma

Theorem

Suppose there exists a map

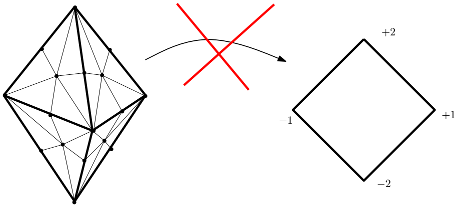
$$\lambda : \{+,-,0\}^m \setminus \{0,\dots,0\} \longrightarrow \{-1,+1,-2,+2,\dots,-\ell,+\ell\}$$

s.t.

- $\lambda(-\boldsymbol{y}) = -\lambda(\boldsymbol{y})$ *for all* $\boldsymbol{y}$
- $\lambda(\boldsymbol{y}) + \lambda(\boldsymbol{z}) \neq 0$ *if* $\boldsymbol{y} \preceq \boldsymbol{z}$.

Then $\ell \geq m$.

"$\boldsymbol{y} \preceq \boldsymbol{z}$" means "$y_i \neq 0 \Rightarrow y_i = z_i$".



Proof of Hedetniemi's conjecture for Kneser graphs

Assume

- $\mathrm{KG}(n,k) \times \mathrm{KG}(n',k')$ colored with t colors.
- $n - 2k \leq n' - 2k'$

With the help of coloring, build a map

$$\begin{array}{rccc} \lambda: & \{+,-,0\}^{n+n'} \setminus \{0,\ldots,0\} & \longrightarrow & \{-1,+1,\ldots,-(t+n'+2k-2), t+n'+2k-2\} \\ & (\boldsymbol{x},\boldsymbol{x}') & \longmapsto & \underbrace{s(\boldsymbol{x},\boldsymbol{x}')}_{\text{sign}} \ \underbrace{v(\boldsymbol{x},\boldsymbol{x}')}_{\text{absolute value}} \end{array}$$

satisfying condition of Tucker's lemma

- $\lambda(-\boldsymbol{x},-\boldsymbol{x}') = -\lambda(\boldsymbol{x},\boldsymbol{x}')$
- $\lambda(\boldsymbol{x},\boldsymbol{x}') + \lambda(\boldsymbol{y},\boldsymbol{y}') \neq 0$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{x}' \preceq \boldsymbol{y}'$.

Second point satisfied because of coloring condition: no two adjacent vertices get the same color.

Thus, $t + n' + 2k - 2 \geq n + n'$, i.e.

$$\boxed{t \geq n - 2k + 2}$$

$$(\overbrace{0,+,+,-,0,\ldots,0,-,0}^{\boldsymbol{x}},\overbrace{-,+,0,0,+,\ldots,0,+,+}^{\boldsymbol{x}'})$$

$$\begin{array}{ll} \boldsymbol{x}^+ = \{i:\ x_i = +\} & \boldsymbol{x}'^+ = \{i:\ x'_i = +\} \\ \boldsymbol{x}^- = \{i:\ x_i = -\} & \boldsymbol{x}'^- = \{i:\ x'_i = -\} \end{array}$$

⋆ If $|\boldsymbol{x}^+| \geq k$, $|\boldsymbol{x}^-| \geq k$, $|\boldsymbol{x}'^+| \geq k'$, and $|\boldsymbol{x}'^-| \geq k'$: existence of

- (A^+, A'^+) s.t. $A^+ \subseteq \boldsymbol{x}^+$ and $A'^+ \subseteq \boldsymbol{x}'^+$.
- (A^-, A'^-) s.t. $A^- \subseteq \boldsymbol{x}^-$ and $A'^- \subseteq \boldsymbol{x}'^-$.

(A^+, A'^+) and (A^-, A'^-) are both vertices of $\mathrm{KG}(n,k) \times \mathrm{KG}(n',k')$ and are colored.

$s(\boldsymbol{x},\boldsymbol{x}') =$ corresponding sign $\qquad v(\boldsymbol{x},\boldsymbol{x}') =$ minimal such color.

⋆ If not: most difficult part

Back to Kneser hypergraphs

n, k, r three integers s.t. $n \geq rk$.

Kneser hypergraph $\mathrm{KG}^r(n,k)$:

$$V(\mathrm{KG}^r(n,k)) = \binom{[n]}{k}$$

$$E(\mathrm{KG}^r(n,k)) = \left\{ \{A_1, \ldots, A_r\} : A_i \in \binom{[n]}{k},\ A_i \cap A_j = \emptyset \text{ for } i \neq j \right\}$$

Categorical product $\mathcal{G} \times \mathcal{H}$:

$$V(\mathcal{G} \times \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$$

$$E(\mathcal{G} \times \mathcal{H}) = \{ e \subseteq V(\mathcal{G} \times \mathcal{H}) :\ \pi_{\mathcal{G}}(e) \in E(\mathcal{G}),\ \pi_{\mathcal{H}}(e) \in E(\mathcal{H}) \}$$

Proof

Theorem

$\chi\left(\mathrm{KG}^r(n,k)\times\mathrm{KG}^r(n',k')\right)=\min(\chi(\mathrm{KG}^r(n,k)),\chi(\mathrm{KG}^r(n',k'))).$

Proof via two steps:

1. **Proposition**

   Theorem is true for r prime.

   Proved via Ziegler's extension of Matoušek approach, used to compute

   $$\chi(\mathrm{KG}^r(n,k))=\left\lceil\frac{n-r(k-1)}{r-1}\right\rceil \quad \text{(Alon-Frankl-Lovász theorem)}$$

2. **Proposition**

   If theorem true for r_1 and r_2, then it is true for r_1r_2.

Case r prime

Assume

- r prime
- $\mathrm{KG}^r(n,k) \times \mathrm{KG}^r(n',k')$ colored with t colors.
- $n - rk \leq n' - rk'$

$Z_r =$ rth roots of unity

With the help of coloring, build a map

$$\begin{array}{rccc} \lambda: & (Z_r \cup \{0\})^{n+n'} \setminus \{0,\ldots,0\} & \longrightarrow & Z_r \times [t+n'+rk-2] \\ & (\boldsymbol{x},\boldsymbol{x}') & \longmapsto & (\underbrace{s(\boldsymbol{x},\boldsymbol{x}')}_{\text{sign}}, \underbrace{v(\boldsymbol{x},\boldsymbol{x}')}_{\text{absolute value}}) \end{array}$$

satisfying condition of a "Z_r-Tucker" lemma

- $\lambda(\omega\boldsymbol{x},\omega\boldsymbol{x}') = \omega\lambda(\boldsymbol{x},\boldsymbol{x}')$ for all $\omega \in Z_r$
- condition on $\{\lambda(\boldsymbol{x}^1,\boldsymbol{x}'^1),\ldots,\lambda(\boldsymbol{x}^\ell,\boldsymbol{x}'^\ell)\}$ when $(\boldsymbol{x}^1,\boldsymbol{x}'^1) \preceq \cdots \preceq (\boldsymbol{x}^\ell,\boldsymbol{x}'^\ell)$

Second point satisfied by coloring condition: no r adjacent vertices get the same color.

Thus, $(r-1)(t-1) + n' + rk - 1 \geq n + n'$, i.e.

$$\boxed{t \geq \frac{n - r(k-1)}{r-1}}$$

Case r nonprime

Assume

- $r = r_1 r_2$
- $\mathrm{KG}^r(n,k) \times \mathrm{KG}^r(n',k')$ colored with t colors.

Theorem is true for $\mathrm{KG}^{r_1}(A,k) \times \mathrm{KG}^{r_1}(A',k')$, where $A \subseteq [n]$ and $A' \subseteq [n']$ are of carefully chosen size.

Theorem is true for $\mathrm{KG}^{r_2}(n,|A|) \times \mathrm{KG}^{r_2}(n',|A'|)$: if t too small, contradiction.

A general lower bound

General Kneser hypergraph $\mathrm{KG}^r(\mathcal{H})$ defined for a hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ and integer $r \geq 2$ by

$$\begin{array}{rcl} V(\mathrm{KG}^r(\mathcal{H})) & = & E(\mathcal{H}) \\ E(\mathrm{KG}^r(\mathcal{H})) & = & \{\{e_1,\ldots,e_r\} : e_1,\ldots,e_r \in E(\mathcal{H}),\ e_i \cap e_j = \emptyset \text{ for all } i,j \text{ with } i \neq j\}. \end{array}$$

$\mathrm{cd}_r(\mathcal{H})$ = minimum number of vertices to be removed s.t. induced subhypergraph r-colorable

Theorem (Hajiabolhassan-M.)

Let $\mathcal{H}_1,\ldots,\mathcal{H}_t$ be hypergraphs. Then

$$\chi\left(\mathrm{KG}^r(\mathcal{H}_1)\times\cdots\times\mathrm{KG}^r(\mathcal{H}_t)\right) \geq \frac{1}{r-1}\min_{\ell=1,\ldots,t}\mathrm{cd}_r(\mathcal{H}_\ell).$$

New graphs for Hedetniemi's conjecture

$U_1, \ldots, U_s$ pairwise disjoint, $\ell_1, \ldots, \ell_s$ integers

$$\mathcal{B} = \left\{ A \in \binom{[n]}{k} : |A \cap U_i| \leq \ell_i \right\}$$

bases of the truncation of a partition matroid

Proposition (Hajiabolhassan-M.)

Graphs $\mathrm{KG}^2(\mathcal{B})$ *satisfy Hedetniemi's conjecture.*

Holds also if $\mathcal{B}$ is the set of spanning trees of some "dense" graph.

Open question

Categorical product of two hypergraphs $\mathcal{G} \times \mathcal{H}$:

$$V(\mathcal{G} \times \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$$

$$E(\mathcal{G} \times \mathcal{H}) = \{e \subseteq V(\mathcal{G} \times \mathcal{H}) : \ \pi_{\mathcal{G}}(e) \in E(G), \ \pi_{\mathcal{H}}(e) \in E(H)\}$$

Let $r \neq r'$. Do we have

$$\chi\left(\mathrm{KG}^{r}(n,k) \times \mathrm{KG}^{r'}(n',k')\right) = \min\left(\chi\left(\mathrm{KG}^{r}(n,k)\right), \chi(\mathrm{KG}^{r'}(n',k'))\right)$$

?

Thank you

Proof, cont'd

Building of $\lambda(\boldsymbol{x},\boldsymbol{x}') = s(\boldsymbol{x},\boldsymbol{x}')v(\boldsymbol{x},\boldsymbol{x}')$:

⋆ If $|\boldsymbol{x}^+| \geq k$, $|\boldsymbol{x}^-| \geq k$, $|\boldsymbol{x}'^+| \geq k'$, and $|\boldsymbol{x}'^+| \geq k'$: existence of

- (A^+, A'^+) s.t. $A^+ \subseteq \boldsymbol{x}^+$ and $A'^+ \subseteq \boldsymbol{x}'^+$.
- (A^-, A'^-) s.t. $A^- \subseteq \boldsymbol{x}^-$ and $A'^- \subseteq \boldsymbol{x}'^-$.

(A^+, A'^+) and (A^-, A'^-) are both vertices of $\mathrm{KG}(n,k) \times \mathrm{KG}(n',k')$ and are colored.

$$s(\boldsymbol{x},\boldsymbol{x}') = \text{ corresponding sign} \qquad v(\boldsymbol{x},\boldsymbol{x}') = \text{ minimal such color.}$$

⋆ If not:

$$g(\boldsymbol{x}) = \begin{cases} 2k-1 & \text{if } |\boldsymbol{x}^+| \geq k \text{ and } |\boldsymbol{x}^-| \leq k-1 \\ 2k-1 & \text{if } |\boldsymbol{x}^+| \leq k-1 \text{ and } |\boldsymbol{x}^-| \geq k \\ |\boldsymbol{x}^+| + |\boldsymbol{x}^-| & \text{otherwise.} \end{cases}$$

$v(\boldsymbol{x},\boldsymbol{x}') = g(\boldsymbol{x}) + g(\boldsymbol{x}')$

$s(\boldsymbol{x},\boldsymbol{x}') =$ (most difficult part) □

Case r nonprime

Assume

- $r = r_1 r_2$
- $\mathrm{KG}^r(n,k) \times \mathrm{KG}^r(n',k')$ colored with t colors.
- $n - rk \leq n' - rk'$

Suppose for a contradiction that $t < \frac{n-r(k-1)}{r-1}$.

$A \subseteq [n]$ with $|A| = t(r_1 - 1) + r_1(k-1) + 1$
$A' \subseteq [n']$ with $|A'| = t(r_1 - 1) + r_1(k'-1) + 1$

$$\Downarrow$$

Monochromatic edge $\{(X^{(1)}, X'^{(1)}), \ldots, (X^{(r_1)}, X'^{(r_1)})\}$ in $\mathrm{KG}^{r_1}(A,k) \times \mathrm{KG}^{r_1}(A',k')$.
$\Rightarrow$ Coloring of (A, A').

$$\Downarrow$$

Monochromatic edge $\{(A^{(1)}, A'^{(1)}), \ldots, (A^{(r_2)}, A'^{(r_2)})\}$ in $\mathrm{KG}^{r_2}(n,|A|) \times \mathrm{KG}^{r_2}(n',|A'|)$.

$$\Downarrow$$

Contradiction