Kneser hypergraphs

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Kneser hypergraphs

m, ℓ, r three integers s.t. $m \geq r\ell$.

Kneser hypergraph $\mathrm{KG}^r(m,\ell)$:

$$V(\mathrm{KG}^r(m,\ell)) = \binom{[m]}{\ell}$$

$$E(\mathrm{KG}^r(m,\ell)) = \left\{\{A_1,\ldots,A_r\} : A_i \in \binom{[m]}{\ell},\ A_i \cap A_j = \emptyset \text{ for } i \neq j\right\}$$

Chromatic number

Theorem (Alon-Frankl-Lovász theorem)

$$\chi(\mathrm{KG}^r(m,\ell)) = \left\lceil \frac{m - r(\ell - 1)}{r-1} \right\rceil$$

All proofs:

- if true for r_1 and r_2, then true for $r_1 r_2$.
- true when r is prime.

Original proof for the case r prime: as for Lovász-Kneser conjecture, uses box complexes but Dold's theorem instead BU theorem.

A combinatorial proof

Ziegler (2003) proposed a combinatorial proof via a Z_p-Tucker's lemma.

Assume p prime and $\mathrm{KG}^p(m,\ell)$ properly colored with t colors.

$Z_p = p$th roots of unity

With the help of coloring, build a map

$$\begin{array}{rccc} \lambda: & (Z_p \cup \{0\})^m \setminus \{\mathbf{0}\} & \longrightarrow & Z_p \times [t + p\ell - 2] \\ & \boldsymbol{x} & \longmapsto & (\underbrace{s(\boldsymbol{x})}_{\text{sign}}, \underbrace{v(\boldsymbol{x})}_{\text{absolute value}}) \end{array}$$

satisfying condition of a "Z_p-Tucker" lemma

- $\lambda(\omega\boldsymbol{x}) = \omega\lambda(\boldsymbol{x})$ for $\omega \in Z_p$
- condition on $\{\lambda(\boldsymbol{x}^1), \ldots, \lambda(\boldsymbol{x}^p)\}$ when $\boldsymbol{x}^1 \preceq \cdots \preceq \boldsymbol{x}^p$.

Second point satisfied by coloring condition: no p adjacent vertices get the same color.

Thus, $(p-1)(t-1) + p\ell - 1 \geq m$, i.e.

$$\boxed{t \geq \frac{m - p(\ell - 1)}{p - 1}}$$

Z_p-Tucker lemma

Lemma.

Let $\alpha \in \{0, 1, \ldots, k\}$. If there exists

$$\begin{array}{rccc} \lambda : & (Z_p \cup \{0\})^m \setminus \{\mathbf{0}\} & \longrightarrow & Z_p \times [k] \\ & \boldsymbol{x} & \longmapsto & (s(\boldsymbol{x}), v(\boldsymbol{x})) \end{array}$$

such that

- $\lambda(\omega \boldsymbol{x}) = \omega \lambda(\boldsymbol{x})$ for $\omega \in Z_p$
- $v(\boldsymbol{x}) = v(\boldsymbol{y}) \leq \alpha$ and $\boldsymbol{x} \prec \boldsymbol{y} \quad \Longrightarrow \quad s(\boldsymbol{x}) = s(\boldsymbol{y})$
- $v(\boldsymbol{x}^1) = \cdots = v(\boldsymbol{x}^p) \geq \alpha + 1$ and $\boldsymbol{x}^1 \prec \cdots \prec \boldsymbol{x}^p \quad \Longrightarrow$ $s(\boldsymbol{x}^i)$'s not pairwise distinct,

then $m \leq \alpha + (k - \alpha)(p - 1)$.

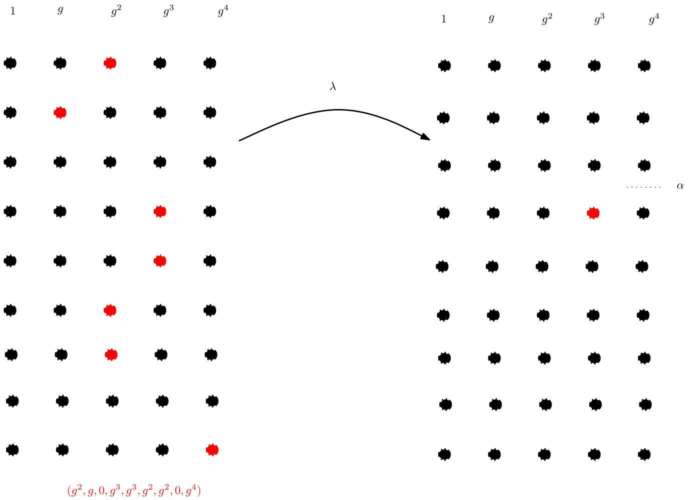
Octahedral Tucker's lemma.

Let $\lambda : \{+, -, 0\}^m \setminus \{\mathbf{0}\} \to \{\pm 1, \ldots, \pm k\}$ s.t.

- $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$
- $\boldsymbol{x} \preceq \boldsymbol{y} \Longrightarrow \lambda(\boldsymbol{x}) + \lambda(\boldsymbol{y}) \neq 0$

Then $m \leq k$.

$$Z_p = \{1, g, g^2, \ldots, g^{p-1}\}.$$



Z_p-Tucker: a proof

$$\mathsf{K} = \underbrace{Z_p * \cdots * Z_p}_{m \text{ times}}$$

$$\mathsf{L} = \underbrace{Z_p * \cdots * Z_p}_{\alpha \text{ times}} * \underbrace{\partial\triangle^{p-1} * \cdots * \partial\triangle^{p-1}}_{k-\alpha \text{ times}}$$

Lemma (Z_p-Tucker lemma – reformulation)

If there exists $\lambda : \mathrm{sd}(\mathsf{K}) \longrightarrow_{Z_p} \mathsf{L}$, *then* $m \leq \alpha + (k-\alpha)(p-1)$.

Proof.

Dold's theorem: connectivity(sd(K)) $<$ dim(L).

connectivity(sd(K)) $= m - 2$ and dim(L) $= \alpha + (k-\alpha)(p-1) - 1$.

$\square$

Ziegler's proof

- ⋆ $c : \binom{[m]}{\ell} \to [t]$ proper coloring of $\mathrm{KG}^p(m,\ell)$ with t colors.
- ⋆ Extension for any $U \subseteq [m]$: $c(U) = \max\{c(A) :\ A \subseteq U,\ |A| = \ell\}$.
- ⋆ $\boldsymbol{x}^\omega = \{i :\ x_i = \omega\}$
- ⋆

$$\lambda(\boldsymbol{x}) = \begin{cases} (x_a, |\boldsymbol{x}|) & \text{if } |\boldsymbol{x}| \leq p(\ell-1) \text{ and } a = \min\{i :\ x_i \neq 0\} \\ (\omega, c(\boldsymbol{x}^\omega) + p(\ell-1)) & \text{if } |\boldsymbol{x}| \geq p(\ell-1)+1 \text{ and } \omega = \arg\max\{c(\boldsymbol{x}^{\omega'}) :\ \omega' \in Z_p\} \end{cases}$$

- ⋆ $\alpha = p(\ell-1)$.

A Zig-zag theorem for Kneser hypergraphs

Theorem (M. 2014)

Let p be a prime number. Any proper coloring c of $\mathrm{KG}^p(m,\ell)$ with t colors contains a complete p-uniform p-partite hypergraph with parts $U_1,\ldots,U_p$ satisfying the following properties.

- *It has $m-p(\ell-1)$ vertices.*
- *The values of $|U_j|$ differ by at most one.*
- *The vertices of U_j get distinct colors.*

Generalizes Alon-Frankl-Lovász theorem and almost generalizes the Zig-zag theorem for Kneser graphs.

Proof uses a "Z_p-Fan" lemma due to Hanke, Sanyal, Schultz, Ziegler (2009), M. (2006)

Z_p-Fan lemma

Theorem

Let

$$\begin{array}{rccc} \lambda: & (Z_p \cup \{0\})^m \setminus \{\mathbf{0}\} & \longrightarrow & Z_p \times [k] \\ & \boldsymbol{x} & \longmapsto & (s(\boldsymbol{x}), v(\boldsymbol{x})) \end{array}$$

be such that

- $\lambda(\omega \boldsymbol{x}) = \omega \lambda(\boldsymbol{x})$ *for* $\omega \in Z_p$
- $v(\boldsymbol{x}) = v(\boldsymbol{y})$ *and* $\boldsymbol{x} \prec \boldsymbol{y} \implies s(\boldsymbol{x}) = s(\boldsymbol{y})$.

Then there exists an m-chain $\boldsymbol{x}^1 \prec \cdots \prec \boldsymbol{x}^m$ *such that*

$$\lambda(\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m\}) = \{(s_1, v_1), \ldots, (s_m, v_m)\}$$

with $v_1 < \cdots < v_m$ *and* $s_i \neq s_{i+1}$ *for* $i \in [m-1]$.

Local chromatic number of Kneser hypergraphs

Let $\mathcal{H} = (V, E)$ be a uniform hypergraph. For $X \subseteq V$,

$$\mathcal{N}(X) = \{v : \exists e \in E \text{ s.t. } e \setminus X = \{v\}\}.$$

$\mathcal{N}[X] := X \cup \mathcal{N}(X)$.

$$\psi(\mathcal{H}) = \min_{c} \max_{e \in E,\, v \in e} |c(\mathcal{N}[e \setminus \{v\}])|,$$

where minimum taken over all proper colorings c.

Consequence of the Zig-zag theorem for Kneser hypergraphs:

Theorem

$$\psi(\mathrm{KG}^p(m, \ell)) \geq \min\left(\left\lceil \frac{m - p(\ell - 1)}{p} \right\rceil + 1, \left\lceil \frac{m - p(\ell - 1)}{p - 1} \right\rceil\right)$$

for any prime number p.

Hedetniemi's conjecture for Kneser hypergraphs

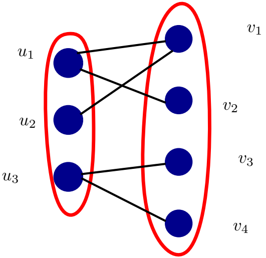
Theorem (Hajiabolhassan-M. 2014)

$$\chi\left(\mathrm{KG}^r(m,\ell)\times\mathrm{KG}^r(m',\ell')\right)=\min\left(\chi\left(\mathrm{KG}^r(m,\ell)\right),\chi\left(\mathrm{KG}^r(m',\ell')\right)\right)$$

Product of hypergraphs and Zhu's conjecture

$\{u_1, u_2, u_3\} \in E(\mathcal{G})$ $\{v_1, v_2, v_3, v_4\} \in E(\mathcal{H})$

$\{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_3, v_3), (u_3, v_4)\} \in E(\mathcal{G} \times \mathcal{H})$



Let $\mathcal{G}, \mathcal{H}$ be two hypergraphs. Categorical product $\mathcal{G} \times \mathcal{H}$:

$$V(\mathcal{G} \times \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$$

$$E(\mathcal{G} \times \mathcal{H}) = \{e \subseteq V(\mathcal{G} \times \mathcal{H}) : \; \pi_{\mathcal{G}}(e) \in E(\mathcal{G}), \; \pi_{\mathcal{H}}(e) \in E(\mathcal{H})\}$$

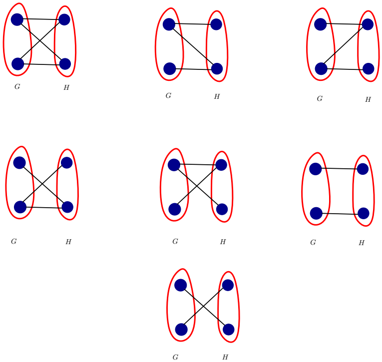
Proper coloring of a hypergraph: no monochromatic edges.

Zhu's conjecture (1992)

$$\chi(\mathcal{G} \times \mathcal{H}) = \min(\chi(\mathcal{G}), \chi(\mathcal{H}))$$

Product of graphs vs of hypergraphs, and coloring

Product of two graphs seen as hypergraphs: will have edges of cardinality 3 and 4.



But the chromatic number does not depend of the definition of the product.

Zhu's conjecture is a true generalization of Hedetniemi's conjecture.

New graphs for Hedetniemi's conjecture

$U_1, \ldots, U_s$ pairwise disjoint, $\ell_1, \ldots, \ell_s$ integers

$$\mathcal{B} = \left\{ A \in \binom{[n]}{k} : \; |A \cap U_i| \leq \ell_i \right\}$$

bases of the truncation of a partition matroid

Proposition (Hajiabolhassan-M.)

Graphs $\mathrm{KG}^2(\mathcal{B})$ *satisfy Hedetniemi's conjecture.*

Holds also if $\mathcal{B}$ is the set of spanning trees of some "dense" graph.

Thank you