A LEMKE-LIKE ALGORITHM FOR THE MULTICLASS NETWORK EQUILIBRIUM PROBLEM

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Abstract. We consider a nonatomic congestion game on a connected graph, with several classes of players. Each player wants to go from its origin vertex to its destination vertex at the minimum cost and all players of a given class share the same characteristics: cost functions on each arc, and origin-destination pair. Under some mild conditions, it is known that a Nash equilibrium exists, but the computation of an equilibrium in the multiclass case is an open problem for general functions. We consider the specific case where the cost functions are affine and propose an extension of Lemke's algorithm able to solve this problem. At the same time, it provides a constructive proof of the existence of an equilibrium in this case.

1. Introduction

Context. Being able to predict the impact of a new infrastructure on the traffic in a transportation network is an old but still important objective for transport planners. In 1952, Wardrop [28] noted that after some while the traffic arranges itself to form an equilibrium and formalized principles characterizing this equilibrium. With the terminology of game theory, the equilibrium is a Nash equilibrium for a congestion game with nonatomic players. In 1956, Beckmann [4] translated these principles as a mathematical program which turned out to be convex, opening the door to the tools from convex optimization. The currently most commonly used algorithm for such convex programs is probably the Frank-Wolfe algorithm [15], because of its simplicity and its efficiency, but many other algorithms with excellent behaviors have been proposed, designed, and experimented.

One of the main assumptions used by Beckmann to derive his program is the fact that all users are equally impacted by congestion. With the transportation terminology, it means that there is only one *class*. In order to improve the prediction of traffic patterns, researchers started in the 70s to study the *multiclass* situation where each class has its own way of being impacted by the congestion. Each class models a distinct mode of transportation, such as cars, trucks, or motorbikes. Dafermos [8, 9] and Smith [27] are probably the first who proposed a mathematical formulation of the equilibrium problem in the multiclass case. However, even if this problem has been the topic of many research works, an efficient algorithm for solving it remains to be designed, except in some special cases [13, 16, 20, 22]. In particular, there is no general algorithm in the literature for solving the problem when the cost of each arc is in an affine dependence with the flow on it.

Our main purpose is to propose such an algorithm.

Model. We are given a directed graph $D = (V, A)$ modeling the transportation network. A *route* is an s-t path of D and is called an s-t route. The set of all routes (resp. s-t routes) is denoted by $\mathcal{R}$ (resp. $\mathcal{R}_{(s,t)}$). The population of *players* is modeled as a bounded real interval I endowed with the Lebesgue measure λ, the *population measure*. The set I is partitioned into a finite number of measurable subsets $(I^k)_{k\in K}$ – the *classes* – modeling the players with same characteristics: they share a same collection of cost functions $(c_a^k : \mathbb{R}_+ \to \mathbb{R}_+)_{a\in A}$, a same origin s^k, and a same

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destination t^k. A player in I^k is said to be of *class* k. The set of vertices (resp. arcs) reachable from s^k is denoted V^k (resp. A^k).

A *strategy profile* is a measurable mapping $\sigma : I \to \mathcal{R}$ such that $\sigma(i) \in \mathcal{R}_{(s^k,t^k)}$ for all $k \in K$ and all $i \in I^k$. For each arc $a \in A$, the measure x_a^k of the set of all class k players i such that a is in $\sigma(i)$ is the *class k flow* on a in σ:

$$x_a^k = \lambda\{i \in I^k : \, a \in \sigma(i)\} \, .$$

The *total flow* on a is $x_a = \sum_{k \in K} x_a^k$. The cost of arc a for a class k player is then $c_a^k(x_a)$. For a class k player, the cost of a route r is defined as the sum of the costs of the arcs contained in r. Each player wants to select a minimum-cost route.

A strategy profile is a (pure) Nash equilibrium if each route is only chosen by players for whom it is a minimum-cost route. In other words, a strategy profile σ is a Nash equilibrium if for each class $k \in K$ and each player $i \in I^k$ we have

$$\sum_{a \in \sigma(i)} c_a^k(x_a) = \min_{r \in \mathcal{R}_{(s^k,t^k)}} \sum_{a \in r} c_a^k(x_a) \, . \tag{1}$$

This game enters in the category of *nonatomic congestion games with player-specific cost functions*, see Milchtaich [21]. The problem of finding a Nash equilibrium for such a game is called the *Multiclass Network Equilibrium Problem*.

Contribution. Our results concern the case when the cost functions are affine and stricly increasing: for all $k \in K$ and $a \in A^k$, there exist $\alpha_a^k > 0$ and $\beta_a^k \geq 0$ such that $c_a^k(x) = \alpha_a^k x + \beta_a^k$ for all $x \in \mathbb{R}_+$. In this case, the Multiclass Network Equilibrium Problem can be written as a linear complementarity problem. In 1965, Lemke [19] designed a pivoting algorithm for solving a linear complementarity problem under a quite general form. This algorithm has been adapted and extended several times – see for instance [1, 3, 5, 7, 12, 25] – to be able to deal with linear complementarity problems that do not directly fit in the required framework of the original Lemke algorithm.

We show that there exists a pivoting Lemke-like algorithm solving the Multiclass Network Equilibrium Problem when the costs are affine. To our knowledge, it is the first algorithm solving this problem. We prove its efficiency through computational experiments. Moreover, the algorithm provides the first constructive proof of the existence of an equilibrium for this problem. The initial proof of the existence from Schmeidler [26] uses a non-constructive approach with the help of a general fixed point theorem.

On our track, we extend slightly the notion of basis used in linear programming and linear complementarity programming to deal directly with unsigned variables. Even if it is natural, we are not aware of previous use of such an approach. An unsigned variable can be replaced by two variables – one for the nonnegative part and one for the nonpositive part. Such an operation considerably increases the size of the matrices, while, in our approach, we are able to deal directly with the unsigned variables.

Related works. We already gave some references of works related to ours with respect to the linear complementarity. The work by Schiro *et al.* [25] is one of them and deals actually with a problem more general than ours. They propose a pivotal algorithm to solve it. However, our problem is not covered by their termination results (the condition of their Proposition 5 is not satisfied by our problem). Another close work is the one by Eaves [12], which allows additional affine constraints on the variables, but the constraints we need – flow constraints – do not enter in this framework. Note also the work by De Schutter and De Moor [11], devoted to the "Extended Linear Complementarity Problem" which contains our problem. They propose a method that exhaustively enumerates all solutions and all extreme rays, without giving *a priori* guarantee for the existence of a solution.

Papers dealing with algorithms for solving the Multiclass Network Equilibrium Problems propose in general a Gauss-Seidel type diagonalization method, which consists in sequentially fixing the flows for all classes but one and solving the resulting single-class problem by methods of convex programming, see [13, 14, 16, 20] for instance. For this method, a condition ensuring the convergence to an equilibrium is not always stated, and, when there is one, it requires that "the interaction between the various users classes be relatively weak compared to the main effects (the latter translates a requirement that a complicated matrix norm be less than unity)" [20]. Such a condition does clearly not cover the case with affine cost functions. Another approach is proposed by Marcotte and Wynter [22]. For cost functions satisfying the "nested monotonicity" condition – a notion developed by Cohen and Chaplais [6] – they design a descent method for which they are able to prove the convergence to a solution of the problem. However, we were not able to find any paper with an algorithm solving the problem when the costs are polynomial functions, or even affine functions.

Structure of the paper. In Section 2, we explain how to write the Multiclass Network Equilibrium Problem as a linear complementarity problem. We get the formulation $(AMNEP(\boldsymbol{e}))$ on which the remaining of the paper focuses. Section 3 presents the notions that underly the Lemke-like algorithm. All these notions, likes basis, secondary ray, pivot, and so on, are classical in the context of the Lemke algorithm. They require however to be redefined in order to be able to deal with the features of $(AMNEP(\boldsymbol{e}))$. The algorithm is then described in Section 4. We also explain why it provides a constructive proof of the existence of an equilibrium. Section 5 is devoted to the experiments and shows the efficiency of the proposed approach.

All proofs are in the Appendix.

2. Formulation as a linear complementarity problem

In this section, we formulate the Multiclass Network Equilibrium Problem as a complementarity problem which turns out to be linear when the cost functions are affine.

From now on, we assume that the cost functions are increasing. In the single-class case, i.e. $|K| = 1$, the equilibrium flows are optimal solutions of a convex optimization problem, see Beckmann [4]. If the flows $\boldsymbol{x}^{k'}$ for $k' \neq k$ are fixed, finding the equilibrium flows for the class k is again a single-class problem which can be formulated as a convex optimization problem. With the help of the Karush-Kuhn-Tucker conditions, we get that the equilibrium flows (x_a^k) coincide with the solutions of a system of the following form, where $\boldsymbol{b} = (b_v^k)$ is a given vector with $\sum_{v \in V^k} b_v^k = 0$ for all k.

$$(MNEP_{gen}) \qquad \begin{array}{ll} \displaystyle\sum_{a \in \delta^+(v)} x_a^k = \sum_{a \in \delta^-(v)} x_a^k + b_v^k & k \in K, v \in V^k \\[2ex] c_{uv}^k(x_{uv}) + \pi_u^k - \pi_v^k - \mu_{uv}^k = 0 & k \in K, (u,v) \in A^k \\[2ex] x_a^k \mu_a^k = 0 & k \in K, a \in A^k \\[2ex] x_a^k \geq 0, \mu_a^k \geq 0, \pi_v^k \in \mathbb{R} & k \in K, a \in A^k, v \in V^k . \end{array}$$

Actually in our model, we should have moreover $b_v^k = 0$ for $v \notin \{s^k, t^k\}$, and the inequalities $b_{s^k}^k > 0$ and $b_{t^k}^k < 0$, but we relax this condition to deal with a slightly more general problem. Moreover, in this more general form, we can easily require the problem to be non-degenerate, see Section 3.2.

Finding solutions for such systems is a *complementarity problem*, the word "complementarity" coming from the condition $x_a^k \mu_a^k = 0$ for all (a,k) such that $a \in A^k$.

We have thus the following proposition.

Proposition 1. *$(\boldsymbol{x}^k)_{k\in K}$ is an equilibrium flow if and only if there exist $\boldsymbol{\mu}^k \in \mathbb{R}_+^{A^k}$ and $\boldsymbol{\pi}^k \in \mathbb{R}^{V^k}$ for all k such that $(\boldsymbol{x}^k, \boldsymbol{\mu}^k, \boldsymbol{\pi}^k)_{k\in K}$ is a solution of the complementarity problem $(MNEP_{gen})$.*

When the cost functions are affine $c_a^k(x) = \alpha_a^k x + \beta_a^k$, solving the Multiclass Network Equilibrium Problem amounts thus to solve the following linear complementarity problem

$$(MNEP)\qquad \begin{array}{ll} \displaystyle\sum_{a\in\delta^+(v)} x_a^k = \sum_{a\in\delta^-(v)} x_a^k + b_v^k & k\in K, v\in V^k \\ \displaystyle\alpha_{uv}^k \sum_{k'\in K} x_{uv}^{k'} + \pi_u^k - \pi_v^k - \mu_{uv}^k = -\beta_{uv}^k & k\in K, (u,v)\in A^k \\ x_a^k \mu_a^k = 0 & k\in K, a\in A^k \\ x_a^k \geq 0, \mu_a^k \geq 0, \pi_v^k \in \mathbb{R} & k\in K, a\in A^k, v\in V^k\ . \end{array}$$

Similarly as for the Lemke algorithm, we rewrite the problem as an optimization problem. It will be convenient for the exposure of the algorithm, see Section 3. This problem is called the *Augmented Multiclass Network Equilibrium Problem.* It uses a vector $\boldsymbol{e} = (e_a^k)$ defined for all $k \in K$ and $a \in A^k$. Problem $(AMNEP(\boldsymbol{e}))$ is

$$(AMNEP(\boldsymbol{e}))\qquad \begin{array}{lll} \min & \omega & \\ \text{s.t.} & \displaystyle\sum_{a\in\delta^+(v)} x_a^k = \sum_{a\in\delta^-(v)} x_a^k + b_v^k & k\in K, v\in V^k \\ & \displaystyle\alpha_{uv}^k \sum_{k'\in K} x_{uv}^{k'} + \pi_u^k - \pi_v^k - \mu_{uv}^k + e_{uv}^k\omega = -\beta_{uv}^k & k\in K, (u,v)\in A^k \\ & x_a^k \mu_a^k = 0 & k\in K, a\in A^k \\ & x_a^k \geq 0, \mu_a^k \geq 0, \omega\geq 0, \pi_v^k \in \mathbb{R} & k\in K, a\in A^k, v\in V^k\ . \end{array}$$

Some choices of $\boldsymbol{e}$ allow to find easily feasible solutions to problem $(AMNEP(\boldsymbol{e}))$. In Section 3, $\boldsymbol{e}$ will be chosen in such a way. A key remark is that solving $(MNEP)$ amounts to find an optimal solution for $(AMNEP(\boldsymbol{e}))$ with $\omega = 0$.

Without loss of generality, we impose that $\pi_{s^k}^k = 0$ for all $k \in K$ and it holds throughout the paper. It allows to rewrite problem $(AMNEP(\boldsymbol{e}))$ under the form

$$\begin{array}{ll} \min & \omega \\ \text{s.t.} & \overline{M}^{\boldsymbol{e}} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\mu} \\ \omega \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ M^T \end{pmatrix} \boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix} \\ & \boldsymbol{x}\cdot\boldsymbol{\mu} = 0 \\ & \boldsymbol{x} \geq \mathbf{0},\ \boldsymbol{\mu} \geq \mathbf{0},\ \omega \geq 0,\ \boldsymbol{\pi} \in \mathbb{R}^{\sum_k V^k \setminus \{s^k\}}, \end{array}$$

where $\overline{M}^{\boldsymbol{e}}$ and C are defined as follows. (The matrix $\overline{M}^{\boldsymbol{e}}$ is denoted with a superscript $\boldsymbol{e}$ in order to emphasize its dependency on $\boldsymbol{e}$).

We define $M = \mathrm{diag}((M^k)_{k\in K})$ where M^k is the incidence matrix of the directed graph (V^k, A^k) from which the s^k-row has been removed:

$$M^k_{v,a} = \begin{cases} 1 & \text{if } a \in \delta^+(v), \\ -1 & \text{if } a \in \delta^-(v), \\ 0 & \text{otherwise .} \end{cases}$$

We also define $C^k = \mathrm{diag}((\alpha^k_a)_{a\in A^k})$ for $k \in K$, and then C the real matrix $C = (\underbrace{(C^k, \cdots, C^k}_{|K| \text{ times}})_{k\in K})$.

Then let

$$\overline{M}^{\boldsymbol{e}} = \begin{pmatrix} M & \mathbf{0} & \mathbf{0} \\ C & -I & \boldsymbol{e} \end{pmatrix}.$$

For $k \in K$, the matrix M^k has $|V^k| - 1$ rows and $|A^k|$ columns, while C^k is a square matrix with $|A^k|$ rows and columns. Then the whole matrix $\overline{M}^{\boldsymbol{e}}$ has $\sum_{k\in K}(|A^k| + |V^k| - 1)$ rows and $2\left(\sum_{k\in K}|A^k|\right) + 1$ columns.

3. Bases, pivots, and rays

3.1. **Bases.** We define $\mathcal{X}$ and $\mathcal{M}$ to be two disjoint copies of $\{(a, k) : k \in K, a \in A^k\}$. We denote by $\phi^x(a, k)$ (resp. $\phi^\mu(a, k)$) the element of $\mathcal{X}$ (resp. $\mathcal{M}$) corresponding to (a, k). The set $\mathcal{X}$ models the set of all possible indices for the 'x' variables and $\mathcal{M}$ the set of all possible indices for the 'μ' variables for problem $(AMNEP(\boldsymbol{e}))$. We consider moreover a dummy element o as the index for the 'ω' variable.

We define a *basis* for problem $(AMNEP(\boldsymbol{e}))$ to be a subset B of the set $\mathcal{X} \cup \mathcal{M} \cup \{o\}$ such that the square matrix of size $\sum_{k\in K}\left(|A^k| + |V^k| - 1\right)$ defined by

$$\left(\begin{array}{c|c} \overline{M}^{\boldsymbol{e}}_B & \begin{array}{c} \mathbf{0} \\ M^T \end{array} \end{array} \right)$$

is nonsingular. Note that this definition is not standard. In general, a basis is defined in this way but without the submatrix $\begin{pmatrix} \mathbf{0} \\ M^T \end{pmatrix}$ corresponding to the 'π' columns. We use this definition in order to be able to deal directly with the unsigned variables 'π'. We will see that this approach is natural (and could be used for linear programming as well). However, we are not aware of a previous use of such an approach.

As a consequence of this definition, since M^T has $\sum_{k\in K}(|V^k| - 1)$ columns, a basis is always of cardinality $\sum_{k\in K}|A^k|$.

Remark 1. In particular, since the matrix is nonsingular and since M^T has $\sum_{k\in K}|A^k|$ rows, the first $\sum_{k\in K}(|V^k|-1)$ rows of $\overline{M}^{\boldsymbol{e}}_B$ have each a nonzero entry. This property is used below, especially in the proof of Lemma 4.

The following additional notation is useful: given a subset $Z \subseteq \mathcal{X} \cup \mathcal{M} \cup \{o\}$, we denote by Z^x the set $(\phi^x)^{-1}(Z \cap \mathcal{X})$ and by Z^μ the set $(\phi^\mu)^{-1}(Z \cap \mathcal{M})$. In other words, (a, k) is in Z^x if and only if $\phi^x(a, k)$ is in Z, and similarly for Z^μ.

3.2. **Basic solutions and non-degeneracy.** Let B a basis. If it contains o, the unique solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ of

$$
(2) \qquad \left\{
\begin{array}{ll}
\left(\begin{array}{c|c} \overline{M}_B^{\boldsymbol{e}} & \begin{array}{c} \mathbf{0} \\ M^T \end{array} \end{array} \right) \begin{pmatrix} \boldsymbol{x}_{B^x} \\ \boldsymbol{\mu}_{B^\mu} \\ \omega \\ \boldsymbol{\pi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix} & \\
x_a^k = 0 & \text{for all } (a,k) \notin B^x \\
\mu_a^k = 0 & \text{for all } (a,k) \notin B^\mu \;.
\end{array}
\right.
$$

is called the *basic solution* associated to B.

If B does not contain o, we define similarly its associated *basic solution*. It is the unique solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ of

$$
(3) \qquad \left\{
\begin{array}{ll}
\left(\begin{array}{c|c} \overline{M}_B^{\boldsymbol{e}} & \begin{array}{c} \mathbf{0} \\ M^T \end{array} \end{array} \right) \begin{pmatrix} \boldsymbol{x}_{B^x} \\ \boldsymbol{\mu}_{B^\mu} \\ \boldsymbol{\pi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix} & \\
x_a^k = 0 & \text{for all } (a,k) \notin B^x \\
\mu_a^k = 0 & \text{for all } (a,k) \notin B^\mu \\
\omega = 0 \;. &
\end{array}
\right.
$$

A basis is said to be *feasible* if the associated basic solution is such that $\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega} \geq 0$.

The problem $(AMNEP(\boldsymbol{e}))$ is said to *satisfy the non-degeneracy assumption* if, for any feasible basis B, the associated basic solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ is such that

$$
\Big((a,k) \in B^x \Rightarrow \bar{x}_a^k > 0 \Big) \text{ and } \Big((a,k) \in B^\mu \Rightarrow \bar{\mu}_a^k > 0 \Big) \;.
$$

Note that if we had defined the vector $\boldsymbol{b}$ to be 0 on all vertices $v \notin \{s^k, t^k\}$, the problem would not in general satisfy the non-degeneracy assumption. An example of a basis for which the condition fails to be satisfied is the basis B^{ini} defined in Section 3.5. Remark 3 in that section details the example.

3.3. **Pivots and polytope.** The following lemmas are key results that will eventually lead to the Lemke-like algorithm. They are classical for the usual definition of bases. Since we have extended the definition, we have to prove that they still hold.

Lemma 1. *Let B be a feasible basis for problem $(AMNEP(\boldsymbol{e}))$ and assume non-degeneracy. Let i be an index in $\mathcal{X} \cup \mathcal{M} \cup \{o\} \setminus B$. Then there is at most one feasible basis $B' \neq B$ in the set $B \cup \{i\}$.*

The operation consisting in computing B' given B and the *entering index* i is called the *pivot operation.*

If we are able to determine an index in $\mathcal{X} \cup \mathcal{M} \cup \{o\} \setminus B$ for any basis B, Lemma 1 leads to a "pivoting" algorithm. At each step, we have a current basis B^{curr}, we determine the entering index i, and we compute the new basis in $B^{curr} \cup \{i\}$, if it exists, which becomes the new current basis B^{curr}; and so on. Next lemma allows to characterize situations where there is no new basis, i.e. situations for which the algorithm gets stuck.

The feasible solutions of $(AMNEP(\boldsymbol{e}))$ belong to the polytope

$$\mathcal{P}(\boldsymbol{e}) = \left\{ (\boldsymbol{x}, \boldsymbol{\mu}, \omega, \boldsymbol{\pi}) : \overline{M}^{\boldsymbol{e}} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\mu} \\ \omega \end{pmatrix} + \begin{pmatrix} \boldsymbol{0} \\ M^T \end{pmatrix} \boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix}, \right.$$
$$\left. \boldsymbol{x} \geq \boldsymbol{0},\, \boldsymbol{\mu} \geq \boldsymbol{0},\, \boldsymbol{\pi} \geq \boldsymbol{0},\, \omega \in \mathbb{R}_+ \right\}.$$

Lemma 2. *Let B be a feasible basis for problem $(AMNEP(\boldsymbol{e}))$ and assume non-degeneracy. Let i be an index in $\mathcal{X} \cup \mathcal{M} \cup \{o\} \setminus B$. If there is no feasible basis $B' \neq B$ in the set $B \cup \{i\}$, then the polytope $\mathcal{P}(\boldsymbol{e})$ contains an infinite ray originating at the basic solution associated to B.*

3.4. **Complementarity and twin indices.** A basis B is said to be *complementary* if for every (a,k) with $a \in A^k$, we have $(a,k) \notin B^x$ or $(a,k) \notin B^\mu$: for each (a,k), one of the components x_a^k or μ_a^k is not activated in the basic solution. In case of non-degeneracy, it coincides with the condition $\boldsymbol{x} \cdot \boldsymbol{\mu} = 0$. An important point to be noted for a complementary basis B is that if $o \in B$, then there is (a_0, k_0) with $a_0 \in A^{k_0}$ such that

- $(a_0, k_0) \notin B^x$ and $(a_0, k_0) \notin B^\mu$, and
- for all $(a,k) \neq (a_0,k_0)$ with $a \in A^k$, exactly one of the relations $(a,k) \in B^x$ and $(a,k) \in B^\mu$ is satisfied.

This is a direct consequence of the fact that there are exactly $\sum_{k \in K} |A^k|$ elements in a basis and that each (a,k) is not present in at least one of B^x and B^μ. In case of non-degeneracy, this point amounts to say that $x_a^k = 0$ or $\mu_a^k = 0$ for all (a,k) with $a \in A^k$ and that there is exactly one such pair, denoted (a_0, k_0), such that both are equal to 0.

We say that $\phi^x(a_0,k_0)$ and $\phi^\mu(a_0,k_0)$ for such (a_0,k_0) are the *twin indices.*

3.5. **Initial feasible basis.** A good choice of $\boldsymbol{e}$ gives an easily computable initial feasible complementary basis to problem $(AMNEP(\boldsymbol{e}))$.

An *s-arborescence* in a directed graph is a spanning tree rooted at s that has a directed path from s to any vertex of the graph. We arbitrarily define a collection $\mathcal{T} = (T^k)_{k \in K}$ where $T^k \subseteq A^k$ is an s^k-arborescence of (V^k, A^k). Then the vector $\boldsymbol{e} = (e_a^k)_{k \in K}$ is chosen with the help of $\mathcal{T}$ by

$$e_a^k = \begin{cases} 1 & \text{if } a \notin T^k \\ 0 & \text{otherwise}\,. \end{cases} \tag{4}$$

Lemma 3. *Let the set of indices $Y \subseteq \mathcal{X} \cup \mathcal{M} \cup \{o\}$ be defined by*

$$Y = \{\phi^x(a,k) : a \in T^k, k \in K\} \cup \{\phi^\mu(a,k) : a \in A^k \setminus T^k, k \in K\} \cup \{o\}\,.$$

Then, one of the following situations occurs:

- *$Y \setminus \{o\}$ is a complementary feasible basis providing an optimal solution of problem $(AMNEP(\boldsymbol{e}))$ with $\omega = 0$.*
- *There exists (a_0,k_0) such that $B^{ini} = Y \setminus \{\phi^\mu(a_0,k_0)\}$ is a feasible complementary basis for problem $(AMNEP(\boldsymbol{e}))$.*

We emphasize that B^{ini} depends on the chosen collection $\mathcal{T}$ of arborescences. Note that the basis B^{ini} is polynomially computable.

Remark 2. A short examination of the proof makes clear that the following claim is true: *Assuming non-degeneracy, if B is a feasible basis such that $B^x = \{(a,k) : a \in T^k,\, k \in K\}$, then $B = B^{ini}$.* The fact that the T^k are arborescences fixes completely $\boldsymbol{x}$, and then $\boldsymbol{\pi}$. The fact that B is a feasible basis forces ω to be equal to the maximal value of $-\alpha_{uv}^k x_{uv} - \beta_{uv}^k - \pi_u^k + \pi_v^k$ (except of course if this value is nonpositive, in which case we have already solved our problem), which in turn fixes the values of the μ_{uv}^k.

Remark 3. As already announced in Section 3.2, if we had defined the vector $\boldsymbol{b}$ to be 0 on all vertices $v \notin \{s^k, t^k\}$, the problem would not satisfy the non-degeneracy assumption as soon as there is $k \in K$ such that T^k has a vertex of degree 3 (which happens when (V^k, A^k) has no Hamiltonian path). In this case, the basis B^{ini} shows that the problem is degenerate. Since the unique solution $\boldsymbol{x}^k_{T^k}$ of $M^k_{T^k}\boldsymbol{x}^k_{T^k} = \boldsymbol{b}^k$ consists in sending the whole demand on the unique route in T^k from s^k to t^k, we have for all arcs $a \in T^k$ not belonging to this route $x^k_a = 0$ while $(a,k) \in B^{ini,x}$.

3.6. **No secondary ray.** Let $(\bar{\boldsymbol{x}}^{ini}, \bar{\boldsymbol{\mu}}^{ini}, \bar{\omega}^{ini}, \bar{\boldsymbol{\pi}}^{ini})$ be the feasible basic solution associated to the initial basis B^{ini}, computed according to Lemma 3 and with $\boldsymbol{e}$ given by Equation (4). The following inifinite ray

$$\rho^{ini} = \left\{(\bar{\boldsymbol{x}}^{ini}, \bar{\boldsymbol{\mu}}^{ini}, \bar{\omega}^{ini}, \bar{\boldsymbol{\pi}}^{ini}) + t(\mathbf{0}, \boldsymbol{e}, 1, \mathbf{0}) : \, t \geq 0\right\},$$

has all its points in $\mathcal{P}(\boldsymbol{e})$. This ray with direction $(\mathbf{0}, \boldsymbol{e}, 1, \mathbf{0})$ is called the *primary ray*. In the terminology of the Lemke algorithm, another infinite ray originating at a solution associated to a feasible complementary basis is called a *secondary ray*. Recall that we defined $\pi^k_{s^k} = 0$ for all $k \in K$ in Section 2 (otherwise we would have a trivial secondary ray). System $(AMNEP(\boldsymbol{e}))$ has no secondary ray for the chosen $\boldsymbol{e}$.

Lemma 4. *Let $\boldsymbol{e}$ be defined by Equation* (4)*. Under the non-degeneracy assumption, there is no secondary ray in $\mathcal{P}(\boldsymbol{e})$.*

3.7. **A Lemke-like algorithm.** Assuming non-degeneracy, the combination of Lemma 1 and the point explicited in Section 3.4 give rise to a Lemke-like algorithm. Two feasible complementary bases B and B' are said to be *neighbors* if B' can be obtained from B by a pivot operation using one of the twin indices as an entering index, see Section 3.4. Note that is is a symmetrical notion: B can then also be obtained from B' by a similar pivot operation. The abstract graph whose vertices are the feasible complementary bases and whose edges connect neighbor bases is thus a collection of paths and cycles. According to Lemma 3, we can find in polynomial time an initial feasible complementary basis for $(AMNEP(\boldsymbol{e}))$ with the chosen vector $\boldsymbol{e}$. This initial basis has exactly one neighbor according to Lemma 2 since there is a primary ray and no secondary ray (Lemma 4).

Algorithm 1 explains how to follow the path starting at this initial feasible complementary basis. Function `EnteringIndex`(B, i') is defined for a feasible complementary basis B and an index $i' \notin B$ being a twin index of B and computes the other twin index $i \neq i'$. Function `LeavingIndex`(B, i) is defined for a feasible complementary basis B and an index $i \notin B$ and computes the unique index $j \neq i$ such that $B \cup \{i\} \setminus \{j\}$ is a feasible complementary basis (see Lemma 1).

Since there is no secondary ray (Lemma 4), a pivot operation is possible because of Lemma 2 as long as there are twin indices. By finiteness, a component in the abstract graph having an endpoint necessarily has another endpoint. It implies that the algorithm reaches at some moment a basis B without twin indices. Such a basis is such that $o \notin B$ (Section 3.4), which implies that we have a solution of problem $(AMNEP(\boldsymbol{e}))$ with $\omega = 0$, i.e. a solution of problem $(MNEP)$, and thus a solution of our initial problem.

4. Algorithm and main result

We are now in a position to describe the full algorithm under the non-degeneracy assumption.

(1) For each $k \in K$, compute a collection $\mathcal{T} = (T^k)$ where $T^k \subseteq A^k$ is an s^k-arborescence of (V^k, A^k).
(2) Define $\boldsymbol{e}$ as in Equation (4) (which depends on $\mathcal{T}$).
(3) Define $Y = \{\phi^x(a,k) : \, a \in T^k, k \in K\} \cup \{\phi^\mu(a,k) : \, a \in A^k \setminus T^k, k \in K\} \cup \{o\}$.
(4) If $Y \setminus \{o\}$ is a complementary feasible basis providing an optimal solution of problem $(AMNEP(\boldsymbol{e}))$ with $\omega = 0$, then we have a solution of problem $(MNEP)$, see Lemma 3.

input : The matrix $\overline{M}^{\boldsymbol{e}}$, the matrix M, the vectors $\boldsymbol{b}$ and $\boldsymbol{\beta}$, an initial feasible complementary basis B^{ini}
output: A feasible basis B^{end} with $o \notin B^{end}$.

$\phi^{\mu}(a_0, k_0) \leftarrow$ twin index in $\mathcal{M}$;
$i \leftarrow \texttt{EnteringIndex}(B^{ini}, \phi^{\mu}(a_0, k_0))$;
$j \leftarrow \texttt{LeavingIndex}(B^{ini}, i)$;
$B^{curr} \leftarrow B^{ini} \cup \{i\} \setminus \{j\}$;
while *There are twin indices* **do**
  $i \leftarrow \texttt{EnteringIndex}(B^{curr}, j)$;
  $j \leftarrow \texttt{LeavingIndex}(B^{curr}, i)$;
  $B^{curr} \leftarrow B^{curr} \cup \{i\} \setminus \{j\}$;
end
$B^{end} \leftarrow B^{curr}$;
return B^{end};

Algorithm 1: Lemke-like algorithm

(5) Otherwise, let B^{ini} be defined as in Lemma 3 and apply Algorithm 1, which returns a basis B^{end}.
(6) Compute the basic solution associated to B^{end}.

All the elements proved in Section 3 leads finally to the following result.

Theorem 1. *Under the non-degeneracy assumption, this algorithm solves problem $(MNEP)$, i.e. the Multiclass Network Equilibrium Problem with affine costs.*

This result provides actually a constructive proof of the existence of an equilibrium for the Multiclass Network Equilibrium Problem when the cost are affine and strictly increasing, even if the non-degeneracy assumption is not satisfied. If we compute $\boldsymbol{b} = (b_v^k)$ strictly according to the model, we have

$$b_v^k = \begin{cases} \lambda(I^k) & \text{if } v = s^k \\ -\lambda(I^k) & \text{if } v = t^k \\ 0 & \text{otherwise .} \end{cases} \tag{5}$$

In this case, the non-degeneracy assumption is not satisfied as it has been noted at the end of Section 3.5 (Remark 3). Anyway, we can slightly perturb $\boldsymbol{b}$ and $-\boldsymbol{\beta}$ in such a way that any feasible complementary basis of the perturbated problem is still a feasible complementary basis for the original problem. Such a perturbation exists by standard arguments, see [7]. Theorem 1 ensures then the termination of the algorithm on a feasible complementary basis B whose basic solution is such that $\omega = 0$. It provides thus a solution for the original problem.

It shows also that the problem of finding such an equilibrium belongs to the PPAD complexity class. The PPAD class – defined by Papadimitriou [24] in 1994 – is the complexity class of functional problems for which we know the existence of the object to be found because of a (oriented) path-following argument. There are PPAD-complete problems, i.e. PPAD problems as hard as any problem in the PPAD class, see [18] for examples of such problems. A natural question would be whether the Multiclass Network Equilibrium Problem with affine costs is PPAD-complete. We do not know the answer. Antother natural question is whether the problem belongs to other complexity classes often met in the context of congestion games, such as the PLS class [17] or the

Classes	Grid	Vertices	Arcs	Pivots	Algorithm 1 (seconds)	Inversion (seconds)
2	2 × 2	4	8	2	<0.01	<0.01
	4 × 4	16	48	21	0.01	0.03
	6 × 6	36	120	54	0.08	0.5
	8 × 8	64	224	129	0.9	4.0
3	2 × 2	4	8	4	<0.01	<0.01
	4 × 4	16	48	33	0.03	0.1
	6 × 6	36	120	97	0.4	1.9
	8 × 8	64	224	183	2.6	12
4	2 × 2	4	8	3	<0.01	<0.01
	4 × 4	16	48	41	0.06	0.3
	6 × 6	36	120	126	0.9	4.7
	8 × 8	64	224	249	5.4	25
10	2 × 2	4	8	11	<0.01	0.02
	4 × 4	16	48	107	0.7	4.1
	6 × 6	36	120	322	15	70
	8 × 8	64	224	638	87	385
50	2 × 2	4	8	56	0.3	2.6
	4 × 4	16	48	636	105	511

Table 1. Performances of the complete algorithm for various instance sizes

CLS class [10]. However, these latter classes require the existence of some potential functions which is not likely to be the case for our problem.

Another consequence of Theorem 1 is that if the demands $\lambda(I^k)$ and the cost parameters α_a^k, β_a^k are rational numbers, then there exists an equilibrium inducing rational flows on each arc and for each class k. It is reminiscent of a similar result for two players matrix games: if the matrices involve only rational entries, there is an equilibrium involving only rational numbers [23].

5. Computational experiments

5.1. **Instances.** The experiments are made on $n \times n$ grid graphs (Manhattan instances). For each pair of adjacent vertices u and v, both arcs (u, v) and (v, u) are present. We built several instances on these graphs with various sizes n, various numbers of classes, and various cost parameters α_a^k, β_a^k. The cost parameters were chosen uniformly at random such that for all a and all k

$$\alpha_a^k \in [1, 10] \quad \text{and} \quad \beta_a^k \in [0, 100] \, .$$

5.2. **Results.** The algorithm has been coded in C++ and tested on a PC Intel® Core™ i5-2520M clocked at 2.5 GHz, with 4 GB RAM. The experiments are currently in progress. However, some preliminary computational results are given in Table 1. Each row of the table contains average figures obtained on five instances on the same graph and with the same number classes, but with various origins, destinations, and costs parameters.

The columns "Classes", "Vertices", and "Arcs" contain respectively the number of classes, the number of vertices, and the number of arcs. The column "Pivots" contains the number of pivots performed by the algorithm. They are done during Step 5 in the description of the algorithm in Section 4 (application of Algorithm 1). The column "Algorithm 1" provides the time needed for the whole execution of this pivoting step. The preparation of this pivoting step requires a first matrix inversion, and the final computation of the solution requires such an inversion as well. The times needed to perform these inversions are given in the column "Inversion". The total time needed by the complete algorithm to solve the problem is the sum of the "Algorithm 1" time and twice the "Inversion" time, the other steps of the algorithm taking a negligible time.

It seems that the number of pivots remains always reasonable. Even if the time needed to solve large instances is sometimes important with respect to the size of the graph, the essential computation time is spent on the two matrix inversions. The program has not been optimized. Since there are several efficient techniques known for inverting matrices, the results can be considered as very positive.

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Appendix

Proof of Proposition 1. The proof is based on the following fact: $(\boldsymbol{x}^k)_{k\in K}$ is an equilibrium if and only if for each $k \in K$, the vector $\boldsymbol{x}^k$ is an equilibrium flow of the game where all $\boldsymbol{x}^{k'}$, $k' \neq k$ are fixed. Using the result of Beckmann *et al.* [4] for single-class problems, we get that $(\boldsymbol{x}^k)_{k\in K}$ is an equilibrium if and only if for each $k \in K$, the vector $\boldsymbol{x}^k$ is a solution of the following problem.

$$
(P^k) \qquad \begin{aligned}
&\min \sum_{a\in A^k} \int_0^{x_a^k} c_a^k(u + x_a^{-k})\, du \\
\text{s.t.} &\sum_{a\in\delta^+(v)} x_a^k = \sum_{a\in\delta^-(v)} x_a^k + b_v^k && v \in V^k, \\
&x_a^k \geq 0 && a \in A^k\ ,
\end{aligned}
$$

where $x_a^{-k} = \sum_{k'\neq k} x_a^{k'}$. According to the Karush-Kuhn-Tucker conditions, $\boldsymbol{x}^k$ solves problem (P^k) if and only if there exist $\boldsymbol{\mu}^k \in \mathbb{R}_+^{A^k}$ and $\boldsymbol{\pi}^k \in \mathbb{R}^{V^k}$ such that $(\boldsymbol{x}^k, \boldsymbol{\mu}^k, \boldsymbol{\pi}^k)_{k\in K}$ satisfies the constraints of problem $(MNEP_{gen})$. □

Proof of Lemma 1. Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ be the basic solution associated to B and let $Y = B \cup \{i\}$. The set of solutions

$$
\begin{cases}
\left(\begin{array}{c|c} \overline{M}_Y^{\boldsymbol{e}} & \begin{matrix} \mathbf{0} \\ M^T \end{matrix} \end{array} \right) \begin{pmatrix} \boldsymbol{x}_{Y^x} \\ \boldsymbol{\mu}_{Y^\mu} \\ \omega \\ \boldsymbol{\pi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix} \\
x_a^k = 0 \quad \text{for all } (a,k) \notin Y^x \\
\mu_a^k = 0 \quad \text{for all } (a,k) \notin Y^\mu
\end{cases}
$$

is a one-dimensional line in $\mathbb{R}^{1+\sum_{k\in K}(2|A^k|+|V^k|-1)}$ (the space of all variables) and passing through $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$. The bases in Y correspond to intersections of this line with the boundary of

$$
Q = \{(\boldsymbol{x}, \boldsymbol{\mu}, \omega, \boldsymbol{\pi}) :\ x_a^k \geq 0, \mu_a^k \geq 0, \omega \geq 0,\ \text{for all } k \in K \text{ and } a \in A^k\}\ .
$$

This latter set being convex (it is a polyhedron), the line intersect at most twice its boundary under the non-degeneracy assumption. □

Proof of Lemma 2. The proof is similar as the one of Lemma 1, of which we take the same notions and notations. If B is the only feasible basis, then the line intersects the boundary of Q exactly once. Because of the non-degeneracy assumption, it implies that there is an infinite ray originating at $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ and whose points are all feasible.

□

Proof of Lemma 3. The subset Y has cardinality $\sum_{k\in K} |A^k| + 1$. To show that Y contains a feasible complementary basis, we proceed by studying the solutions of the system

$$
(S^{\boldsymbol{e}}) \qquad \begin{cases} \left(\begin{array}{c|c} \overline{M}^{\boldsymbol{e}}_Y & \begin{array}{c} \mathbf{0} \\ M^T \end{array} \end{array} \right) \begin{pmatrix} \boldsymbol{x}_{Y^x} \\ \boldsymbol{\mu}_{Y^\mu} \\ \omega \\ \boldsymbol{\pi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix} \\ x_a^k = 0 \quad \text{for all } (a,k) \notin Y^x \\ \mu_a^k = 0 \quad \text{for all } (a,k) \notin Y^\mu \ . \end{cases}
$$

It is convenient to rewrite problem $(S^{\boldsymbol{e}})$ in the following form.

For all $k \in K$,

$$
(6) \qquad \begin{cases} M^k_{T^k} x^k_{T^k} = b^k \\ \alpha^k_{uv} \sum_{k' \in K} x^{k'}_{uv} + \pi^k_u - \pi^k_v - \mu^k_{uv} + e^k_{uv}\omega = -\beta^k_{uv} & \text{for all } (u,v) \in A^k \\ x^k_a = 0 & \text{for all } a \notin T^k \\ \mu^k_a = 0 & \text{for all } a \in T^k \ . \end{cases}
$$

The matrix $M^k_{T^k}$ is nonsingular (see for instance the book by Ahuja *et al.* [2]). It gives a unique solution $x^k_{T^k}$ of the first equation of (6), and since $x^k_a = 0$ for $a \notin T^k$, we get a unique solution $\boldsymbol{x}$ to system $(S^{\boldsymbol{e}})$.

We look now at the second equation of (6) for k and (u,v) such that $(u,v) \in T^k$. We get that any solution of system $(S^{\boldsymbol{e}})$ satisfies the equalities

$$
\alpha^k_{uv} \sum_{k' \in K} x^{k'}_{uv} + \pi^k_u - \pi^k_v = -\beta^k_{uv}, \qquad \text{for all } k \in K \text{ and } (u,v) \in T^k \ .
$$

Indeed, if $(u,v) \in T^k$, we have $e^k_{uv} = 0$ and $\mu^k_{uv} = 0$. Recall that we defined $\pi^k_{s^k} = 0$. Since T^k is a spanning tree of (V^k, A^k) for all k, these equations completely determine $\boldsymbol{\pi}$.

We look then at the second equation of (6), this time for k and (u,v) such that $(u,v) \notin T^k$. We get that any solution of system $(S^{\boldsymbol{e}})$ satisfies the equalities

$$
(7) \qquad \alpha^k_{uv} \sum_{k' \neq k} x^{k'}_{uv} - \mu^k_{uv} + \omega + \pi^k_u - \pi^k_v = -\beta^k_{uv}, \qquad \text{for all } k \in K \text{ and } (u,v) \notin T^k \ .
$$

Indeed, if $(u,v) \notin T^k$, we have $e^k_{uv} = 1$ and $x^k_{uv} = 0$.

If $\alpha^k_{uv} x_{uv} + \beta^k_{uv} + \pi^k_u - \pi^k_v \geq 0$ for all $k \in K$ and $(u,v) \notin T^k$, then we have an optimal solution of problem $(AMNEP(\boldsymbol{e}))$ with $\omega = 0$, and we get the first point of Lemma 3. We can thus assume that $\alpha^k_{uv} x_{uv} + \beta^k_{uv} + \pi^k_u - \pi^k_v < 0$ for at least one triple u, v, k. Let u_0, v_0, k_0 be such a triple minimizing $\alpha^k_{uv} x_{uv} + \beta^k_{uv} + \pi^k_u - \pi^k_v$ and let $a_0 = (u_0, v_0)$. Note that Equation (7) implies that

$$
(8) \qquad \mu^k_{uv} \geq \mu^{k_0}_{u_0 v_0}, \qquad \text{for all } k \in K \text{ and } (u,v) \notin T^k \ .
$$

We finish the proof by showing that B^{ini}, defined as $Y \setminus \{\phi^\mu(a_0, k_0)\}$, is a feasible complementary basis for problem $(AMNEP(\boldsymbol{e}))$. For B^{ini}, system (2) has a unique solution. Indeed, the first part of the proof devoted to the solving of $(S^{\boldsymbol{e}})$ has shown that $\boldsymbol{x}$ and $\boldsymbol{\pi}$ are uniquely determined, without having to compute the values of the μ^k_a's. By definition of (a_0, k_0), since $\phi^\mu(a_0, k_0)$ is not in B^{ini}, we have

$$
\mu^{k_0}_{u_0 v_0} = 0 \quad \text{and} \quad \omega = -\alpha^{k_0}_{u_0 v_0} x_{u_0 v_0} - \beta^{k_0}_{u_0 v_0} - \pi^{k_0}_{u_0} + \pi^{k_0}_{v_0} \ .
$$

Finally, Equation (7) determines the values of the μ^k_{uv} for $k \in K$ and $(u,v) \notin T^k$, and Equation (8) ensures that these values are nonnegative. Therefore, B^{ini} is a basis, and it is feasible because all x^k_a and μ^k_a in the solution are nonnegative. Furthermore, for each (a,k) with $a \in A^k$, at least one of $\phi^x(a,k)$ and $\phi^\mu(a,k)$ is not in B^{ini}.

Hence, the subset B^{ini} is a feasible complementary basis. $\square$

Proof of Lemma 4. Suppose that $\mathcal{P}(\boldsymbol{e})$ contains an infinite ray

$$\rho = \left\{ (\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}}) + t(\boldsymbol{x}^{dir}, \boldsymbol{\mu}^{dir}, \omega^{dir}, \boldsymbol{\pi}^{dir}) : \, t \geq 0 \right\},$$

where $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ is a feasible complementary basic solution associated to a basis B.

We first show that $\boldsymbol{x}^{dir} = 0$. For a contradiction, suppose that it is not the case and let k be such that $\boldsymbol{x}^{dir,k}$ is not zero. Since the points of ρ must satisfy the system $(AMNEP(\boldsymbol{e}))$ for all $t \geq 0$, we have that $(\boldsymbol{x}^{dir}, \boldsymbol{\mu}^{dir}, \omega^{dir}, \boldsymbol{\pi}^{dir})$ must satisfy for all $v \in V^k$

$$\sum_{a \in \delta^+(v)} x_a^{dir,k} = \sum_{a \in \delta^-(v)} x_a^{dir,k},$$

which shows that $\boldsymbol{x}^k$ is a circulation in the directed graph (V^k, A^k). Moreover, we must have for all $(u, v) \in A^k$

$$\alpha_{uv}^k \sum_{k' \in K} x_{uv}^{dir,k'} + \pi_u^{dir,k} - \pi_v^{dir,k} - \mu_{uv}^{dir,k} + e_{uv}^k \omega^{dir} = 0. \tag{9}$$

where we have $\pi_{s^k}^{dir,k} = 0$ since $\pi_{s^k}^k = 0$ for any feasible solution of $(AMNEP(\boldsymbol{e}))$, see Section 2. The following relations must also be satisfied:

$$\boldsymbol{x}^{dir} \cdot \boldsymbol{\mu}^{dir} = 0, \tag{10}$$

and

$$\boldsymbol{x}^{dir} \geq \boldsymbol{0}, \boldsymbol{\mu}^{dir} \geq \boldsymbol{0}, \omega^{dir} \geq 0 \, . \tag{11}$$

Take now any circuit C in $D = (V, A)$ in the support of $\boldsymbol{x}^{dir,k}$. Since we have supposed that $\boldsymbol{x}^{dir,k}$ is not zero and since it is a circulation, such a circuit necessarily exists. According to Equations (10) and (11), we have $\mu_a^{dir,k} = 0$ for each $a \in C$. The sum $\sum_{a \in C} e_a^k$ is nonzero since no tree T^k can contain all arcs in C. Summing Equation (9) for all arcs in C, we get

$$\omega^{dir} = -\frac{\sum_{a \in C} \alpha_a^k \sum_{k' \in K} x_a^{dir,k'}}{\sum_{a \in C} e_a^k} < 0 \, .$$

It is in contradiction with Equation (11). It implies that $x_a^{dir,k} = 0$ for all $k \in K$ and $a \in A^k$.

We show now that $\boldsymbol{\pi}^{dir} = 0$. We start by noting that Equation (9) becomes

$$\pi_u^{dir,k} - \pi_v^{dir,k} - \mu_{uv}^{dir,k} = 0, \quad \text{for all } k \in K \text{ and } (u, v) \in T^k \, .$$

Since T^k is an s^k-arborescence, we have $0 = \pi_{s^k}^{dir,k} \geq \pi_v^{dir,k}$ for all $v \in V^k$, according to Equation (11).

Define now F^k to be the set of arcs $a \in A^k$ such that $(a, k) \in B^x$. Using Remark 1 of Section 3.1, $\overline{M}_B^{\boldsymbol{e}}$ has a nonzero entry on each of its first $\sum_{k \in K}(|V^k| - 1)$ rows, which implies that the set F^k spans all vertices in $V^k \setminus \{s^k\}$.

According to the non-degeneracy assumption, $\bar{x}_a^k$ is non-zero on all arcs of F^k. The complementarity condition for all points of the ray give that $\bar{\boldsymbol{x}} \cdot \boldsymbol{\mu}^{dir} + \boldsymbol{x}^{dir} \cdot \bar{\boldsymbol{\mu}} = 0$, and since $\boldsymbol{x}^{dir} = \boldsymbol{0}$, we have $\bar{\boldsymbol{x}} \cdot \boldsymbol{\mu}^{dir} = 0$. Hence $\mu_{uv}^{dir,k} = 0$ for all $(u, v) \in F^k$, and Equation (9) becomes

$$\pi_u^{dir,k} - \pi_v^{dir,k} + e_{uv}^k \omega^{dir} = 0 \quad \text{for all } k \in K \text{ and } (u, v) \in F^k \, . \tag{12}$$

Thus, according to Equation (11), we have $0 = \pi_{s^k}^{dir,k} \leq \pi_v^{dir,k}$ for all $v \in V^k$. Since we have already shown the reverse inequality, we have $\pi_v^{dir,k} = 0$ for all $v \in V^k$.

Now, if $T^k \neq F^k$ for at least one k, we get the existence of an arc $(u, v) \in F^k$ for which $e^k_{uv} = 1$, while $\pi^{dir,k}_u = \pi^{dir,k}_v = 0$. Equation (12) implies then that $\omega^{dir} = 0$. Still using $\boldsymbol{x}^{dir} = \mathbf{0}$, we get then, again with the help of Equation (9), that $\boldsymbol{\mu}^{dir} = \mathbf{0}$, which contradicts the fact that ρ is an infinite ray.

Therefore, we have $T^k = F^k$ for all k. Using Remark 2 of Section 3.5, we are at the initial basic solution: $B = B^{ini}$. According to Equation (9), and since $\boldsymbol{x}^{dir} = \mathbf{0}$ and $\boldsymbol{\pi}^{dir} = \mathbf{0}$, we have $\mu^{dir,k}_{uv} = e^k_{uv}\omega^{dir}$ for all $k \in K$ and $(u, v) \in A^k$. Thus $(\boldsymbol{x}^{dir}, \boldsymbol{\mu}^{dir}, \omega^{dir}, \boldsymbol{\pi}^{dir}) = \omega^{dir}(\mathbf{0}, \boldsymbol{e}, 1, \mathbf{0})$ for $\omega^{dir} \geq 0$, and ρ is necessarily the primary ray ρ^{ini}.

Then there is no secondary ray, as required. $\square$

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