The splitting necklace problem

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May 26th, 2017

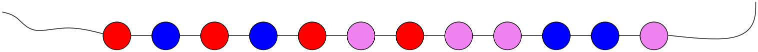
École des Ponts

Two thieves and a necklace

n beads, t types of beads, a_j (even) beads of each type.

Two thieves: Alice and Bob.

Beads fixed on the string.

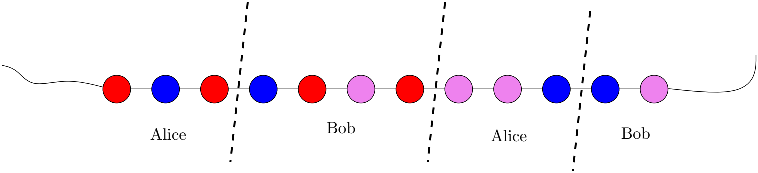


Fair splitting = each thief gets $a_j/2$ beads of type i

The splitting necklace theorem

Theorem (Alon, Goldberg, West, 1985-1986)

There is a fair splitting of the necklace with at most t cuts.



t is tight

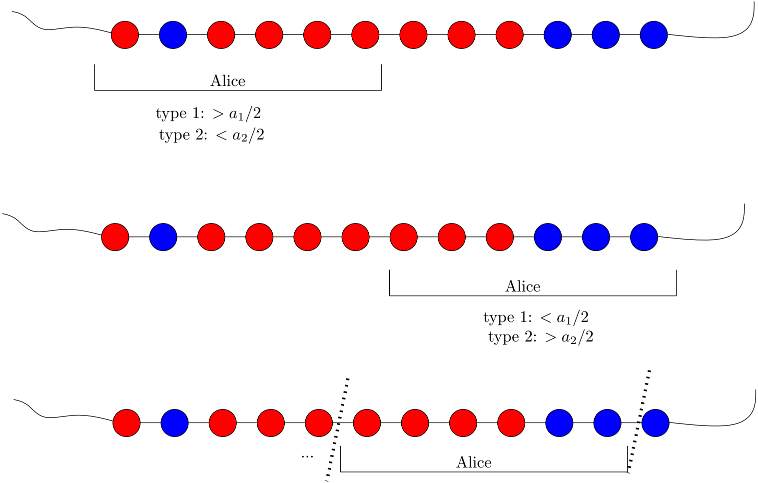
t cuts are sometimes necessary:

Plan

1. Proofs and algorithms
2. Generalizations
3. Open questions

Proofs and algorithms

Easy proof when there are two types of beads



Main tool: the Borsuk-Ulam theorem

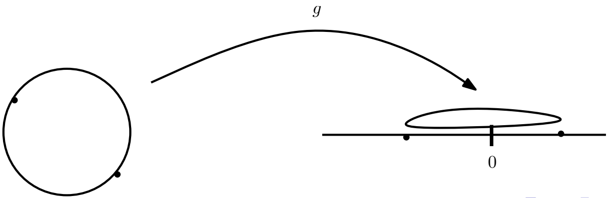
$\mathcal{S}^t$ = t-dimensional sphere in $\mathbb{R}^{t+1}$ (set of points in $\mathbb{R}^{t+1}$ at distance 1 from $\mathbf{0}$)

Theorem

Let g be a continuous map $\mathcal{S}^t \to \mathbb{R}^t$. If g is

- *continuous and*
- *antipodal (i.e., $g(-\boldsymbol{x}) = -g(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{S}^t$),*

then there exists $\boldsymbol{x}_0 \in \mathcal{S}^t$ s.t. $g(\boldsymbol{x}_0) = \mathbf{0}$.



Temperature and pression on earth

Classical application of the Borsuk-Ulam theorem:

On earth, there are always two antipodal points with same temperature and same pressure.

Explanation. For a point $\boldsymbol{x}$ on earth surface, define $t(\boldsymbol{x})$ and $p(\boldsymbol{x})$ to be respectively its current temperature and pressure (continuous).

$g \colon \mathcal{S}^2 \to \mathbb{R}^2_+$ defined by $g(\boldsymbol{x}) = \begin{pmatrix} t(\boldsymbol{x}) - t(-\boldsymbol{x}) \\ p(\boldsymbol{x}) - p(-\boldsymbol{x}) \end{pmatrix}$ is continous and antipodal.

Borsuk-Ulam: there exists a point $\boldsymbol{x}_0$ on earth surface such that $g(\boldsymbol{x}_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Continuous necklace theorem

Proof for any t via...

Theorem

Let $f_1, \ldots, f_t$ be piecewise continuous maps $[0,1] \to \mathbb{R}_+$. Then $[0,1]$ can be partitioned into $t+1$ intervals $I_1, \ldots, I_{t+1}$ and $\{1, \ldots, t+1\}$ can be partitioned into two sets A_1 and A_2 such that for every $j \in [t]$

$$\sum_{i \in A_1} \int_{I_i} f_j(u)du = \sum_{i \in A_2} \int_{I_i} f_j(u)du.$$

$\int_U f_j(u)du =$ amount of type j beads in $U \subseteq [0,1]$ when

$$f_j(u) = \begin{cases} 1 & \text{if type } j \text{ present at position } u \\ 0 & \text{otherwise} \end{cases}$$

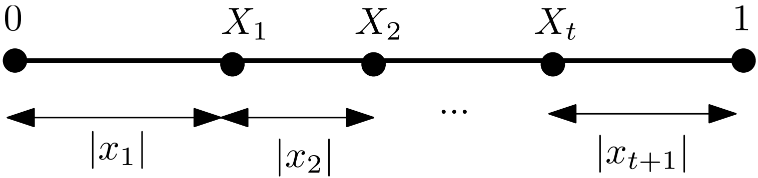
Encoding of a splitting

Encoding of a splitting into $t+1$ parts:

$(x_1, \ldots, x_{t+1}) \in \mathbb{R}^{t+1}$ such that $\displaystyle\sum_{i=1}^{t+1} |x_i| = 1$.

$|x_i|$ = length of ith part
sign of x_i = thief who get ith part

$X_i = \sum_{\ell=1}^{i} |x_\ell|$



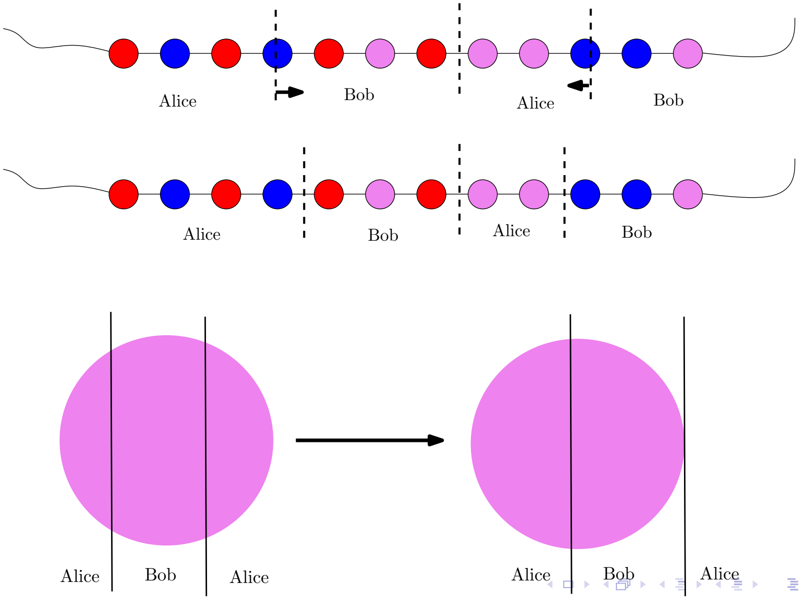
Repartition of the beads

$$\mathcal{S}^t \simeq \left\{ (x_1, \ldots, x_{t+1}) \in \mathbb{R}^{t+1} : \sum_{i=1}^{t+1} |x_i| = 1 \right\}$$

$$\begin{array}{cccc} f: & \mathcal{S}^t & \longrightarrow & \mathbb{R}^t \\ & \begin{pmatrix} x_1 \\ \vdots \\ x_{t+1} \end{pmatrix} & \longmapsto & \begin{pmatrix} \sum_{i=1}^{t+1} \operatorname{sgn}(x_i) \int_{X_{i-1}}^{X_i} f_1(u)du) \\ \vdots \\ \sum_{i=1}^{t+1} \operatorname{sgn}(x_i) \int_{X_{i-1}}^{X_i} f_t(u)du \end{pmatrix} \end{array}$$

Borsuk-Ulam: there exists a splitting $\boldsymbol{x}_0$ such that $f(\boldsymbol{x}_0) = \mathbf{0}$

From continuous to discrete



Elementary proofs and algorithms?

All known proofs rely on the Borsuk-Ulam theorem

Is there a direct/elementary proof?

Open question (Papadimitriou 1994)

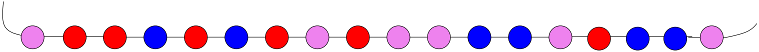
Is there a polynomial algorithm computing a fair splitting of the necklace with at most t cuts?

- Naive $O(n^t)$; less naive $O(n^{t-1})$ (proof using Ham-sandwhich and moment curve)
- Minimizing the number of cuts: NP-hard.

Generalizations

q thieves and a necklace

n beads, t types of beads, a_j (multiple of q) beads of each type.
q thieves: Alice, Bob, Charlie,...

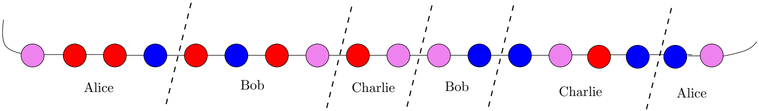


A generalization

Fair splitting = each thief gets a_j/q beads of type j

Theorem (Alon 1987)

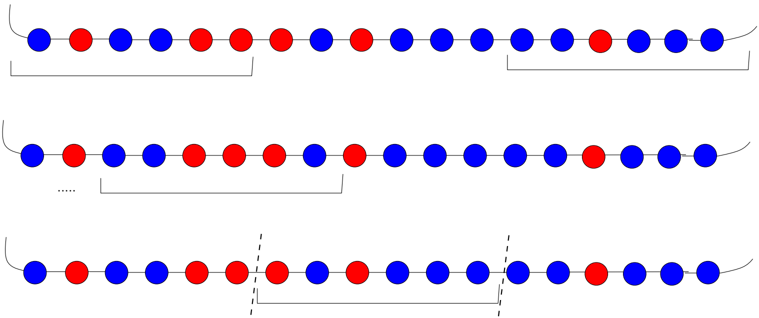
There is a fair splitting of the necklace with at most $(q-1)t$ cuts.



$(q-1)t$ is tight

$(q-1)t$ cuts are sometimes necessary:

An easy case: two types of beads (again!)



... and induction on the number of thieves.

Outline of the proof

Main tool: Dold's theorem

Continuous necklace theorem for q thieves

- Prove for q prime
- Prove that, if true for q' and q'', then true for $q = q'q''$

Rounding technique

Continuous necklace theorem for q thieves

Proof for any t via...

Theorem

Let $f_1, \ldots, f_t$ be piecewise continuous maps $[0,1] \to \mathbb{R}_+$. Then $[0,1]$ can be partitioned into $(q-1)t+1$ intervals $I_1, \ldots, I_{(q-1)t+1}$ and $[(q-1)t+1]$ can be partitioned into $A_1, \ldots, A_q$ such that for every $j \in \{1, \ldots, t\}$

$$\sum_{i \in A_1} \int_{I_i} f_j(u)du = \cdots = \sum_{i \in A_q} \int_{I_i} f_j(u)du.$$

$\int_U f_j(u)du$ = amount of type j beads in $U \subseteq [0,1]$ when

$$f_j(u) = \begin{cases} 1 & \text{if type } j \text{ present at position } u \\ 0 & \text{otherwise} \end{cases}$$

Encoding of a splitting

Z_q = qth roots of unity.

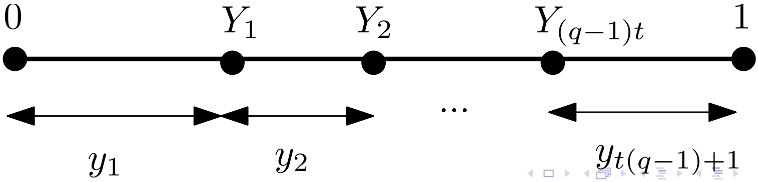
$$\Sigma^{(q-1)t} = \left\{ (\omega_i, y_i)_{i \in [(q-1)t+1]} \in (Z_q \times \mathbb{R}_+)^{(q-1)t+1} : \sum_{i=1}^{(q-1)t+1} y_i = 1 \right\}$$

where $(\omega, 0) = (\omega', 0)$.

Encoding of a splitting into $(q-1)t+1$ parts: $\boldsymbol{x} \in \Sigma^{(q-1)t}$

$y_i =$ length of ith part and $\omega_i =$ thief who get ith part

$Y_i = \sum_{\ell=1}^{i} y_\ell$



Repartition of the beads

$$\Sigma^{(q-1)t} = \left\{ (\omega_i, y_i)_{i \in [(q-1)t+1]} \in (Z_q \times \mathbb{R}_+)^{(q-1)t+1} : \sum_{i=1}^{(q-1)t+1} y_i = 1 \right\}$$

$$\begin{array}{cccc} f: & \Sigma^{(q-1)t} & \longrightarrow & \prod_{\omega \in Z_q} \mathbb{R}^t \\ & \boldsymbol{x} & \longmapsto & \left(\sum_{i:\ \omega_i = \omega} \int_{Y_{i-1}}^{Y_i} f_j(u) du \right)_{j,\omega} \end{array}$$

$$\omega \cdot (\omega', y) = (\omega\omega', y) \quad \text{and} \quad \omega \cdot (\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_q) = (\boldsymbol{z}_2, \ldots, \boldsymbol{z}_q, \boldsymbol{z}_1)$$

- f is equivariant $f(\omega \boldsymbol{x}) = \omega \cdot f(\boldsymbol{x})$
- f is continuous
- q prime
- Dold theorem: $\exists \boldsymbol{x}_0, \forall \omega \in Z_q,\ f(\boldsymbol{x}_0) = \omega \cdot f(\boldsymbol{x}_0)$

Continuous necklace theorem: nonprime case

Proposition

If the continuous necklace theorem is true for q' and q'' (whatever are the other parameters), then it is true for $q = q'q''$.

Proof. A super-thief = q'' thieves.

Make a first splitting among q' super-thieves: $(q'-1)t$ cuts.

For each super-thief: $(q''-1)t$ cuts.

In total: $q'(q''-1)t + (q'-1)t = (q'q''-1)t$ cuts. □

Results

- Complexity of finding a fair splitting of at most $(q-1)t$ cuts: unknown.
- Optimization version: NP-hard.
- No known algorithmic proof.

Yet another generalization for q thieves

n beads, t types of beads, a_j beads of each type, q thieves.

Fair splitting = each thief gets $\lfloor a_j/q \rfloor$ or $\lceil a_j/q \rceil$ beads of type j, for all j.

Theorem (Alon, Moshkovitz, Safra 2006)

There is a fair splitting of the necklace with at most $(q-1)t$ cuts.

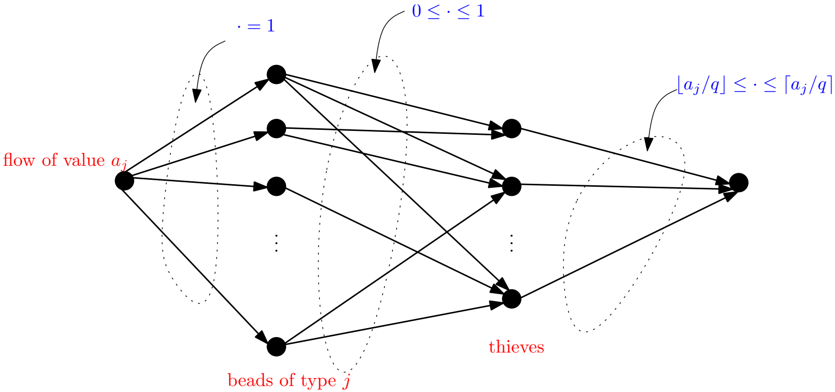
Direct proofs

- for $t = 2$
- $1 \leq a_j \leq q$ for all j.

Rounding via flow

Continuous necklace theorem: fair splitting with at most $(q-1)t$ cuts, but may be non-integral

For each type j, build a directed graph D_j



Existence of a flow $\Rightarrow$ existence of an integral flow.

Yet another generalization for q thieves?

Conjecture (M. 2008, Pálvölgyi 2009)

There is a fair splitting of the necklace with at most $(q-1)t$ cuts such that for each type j, we can decide which thieves receive $\lfloor a_j/q \rfloor$ and which receive $\lceil a_j/q \rceil$.

Proved for the following cases

- remainder of a_j/q is 0, 1, or $q-1$ for all j
- $t = 2$
- $1 \leq a_j \leq q$ for all i

Multidimensional continuous necklaces

Theorem (de Longueville, Živaljević 2008)

Let $\mu_1, \ldots, \mu_t$ *be continuous probability measures on* $[0,1]^d$*. Let* $m_1, \ldots, m_d$ *be positive integers such that* $m_1 + \cdots + m_d = (q-1)t$*. Then there exists a fair division of* $[0,1]^d$ *determined by* m_i *hyperplanes parallel to the* i*th coordinate hyperplane.*

The discrete version is not true (Lasoń 2015).

Open questions (summary)

Open questions

- Complexity of computing a fair splitting with at most t cuts when there are two thieves.
- Complexity of computing a fair splitting with at most $(q-1)t$ cuts when there are q thieves.
- Existence of a fair splitting with choice of the advantaged thieves.
- Elementary proof of the splitting necklace theorem (any version).

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Thank you