A Strongly Polynomial Simplex Method for Totally Unimodular LP

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Outline

Problem:
Algorithm:
Analysis:
Result:
Outline

Problem: Standard form linear programming problem

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Algorithm: Dual simplex method with Dantzig’s rule + Tardos + Initial BFS

Analysis:

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Problem: Standard form linear programming problem
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Analysis: Kitahara-Mizuno
Result:
Outline

**Problem:** Standard form linear programming problem

**Algorithm:** Dual simplex method with Dantzig’s rule + Tardos + Initial BFS

**Analysis:** Kitahara-Mizuno

**Result:** Number of distinct solutions generated is bounded by a polynomial of $m$, $n$, and $\Delta$, where $\Delta$ is the maximum determinant of submatrices of a coefficient matrix.
If the coefficient matrix is totally unimodular ($\Delta = 1$) and all artificial problems solved are nondegenerate, then the algorithm is strongly polynomial.
Problem

- A standard form LP problem

\[
\begin{align*}
\text{min} & \quad c^T x, \\
\text{subject to} & \quad A x = b, \quad x \geq 0, \\
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m, \ c \in \mathbb{R}^n, \) and \( x \in \mathbb{R}^n. \)

- Assume that each element of \( A \) is an integer and \( \text{rank} \ A = m. \)

- Define

\[
\Delta = \max \{ |\text{det} \ D| \mid \text{D is a square submatrix of A} \}. 
\]
Our algorithm is based on the basic algorithm proposed by Tardos, in which we use the dual simplex method with Dantzig’s rule instead of the ellipsoid method (Khachiyan 1979) or the interior-point method (Karmarkar 1984) for solving artificial LP problems.

We also propose a procedure for finding an initial basic feasible solution by the simplex method.
Analysis

- K-M shows that the number of distinct solutions generated by the simplex method with Dantzig’s rule is bounded by

\[ mn^{\frac{\gamma}{\delta}} \log\left( m^{\frac{\gamma}{\delta}} \right), \quad (2) \]

where \( \delta \) and \( \gamma \) are the minimum and the maximum values of all the positive elements of basic feasible solutions.

- They also shows that the number by the dual simplex with Dantzig’s rule is bounded by

\[ m^2 \frac{\gamma_D}{\delta_D} \log\left( m^{\frac{\gamma_D}{\delta_D}} \right). \]
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When $b$ is integral, the number of distinct solutions generated by the primal simplex method with Dantzig’s rule is bounded by

$$m^2 n \Delta^2 \|b\|_\infty \log(m^2 \Delta^2 \|b\|_\infty).$$

When $c$ is integral, the number of distinct solutions generated by the dual simplex method with Dantzig’s rule is bounded by

$$m^2 n \Delta^2 \|c\|_\infty \log(mn \Delta^2 \|c\|_\infty).$$
Let $N = \{1, 2, \cdots, n\}$. For any $B \subset N$ and $\bar{B} = N - B$, we split

$$A = (A_B, A_{\bar{B}}), \quad c = \begin{pmatrix} c_B \\ c_{\bar{B}} \end{pmatrix}, \quad x = \begin{pmatrix} x_B \\ x_{\bar{B}} \end{pmatrix}.$$ 

The primal problem is represented as

$$\min \quad c_B^T x_B + c_{\bar{B}}^T x_{\bar{B}},$$

subject to $A_B x_B + A_{\bar{B}} x_{\bar{B}} = b, \; x \geq 0$. 

(3)
Kitahara-Mizuno (Primal)

- K-M shows that the number of distinct solutions generated by the simplex method with Dantzig’s rule is bounded by

\[
mn^{\gamma - \delta} \log(m^{\gamma - \delta}),
\]

where \(\delta\) and \(\gamma\) are the minimum and the maximum values of all the positive elements of basic feasible solutions.

- The bound is independent of \(c\).
Basic Feasible Solution (BFS)

- When $B$ is a index set of a BFS, the BFS is

$$x_B = (A_B)^{-1}b, \ x_\overline{B} = 0.$$  

- If $b$ is integral, each element is $\frac{p}{q}$ where

$$|p| \leq m\Delta\|b\|_{\infty}, \ q \leq \Delta.$$  

Hence $\gamma \leq m\Delta\|b\|_{\infty}$ and $\delta \geq 1/\Delta$.

- The number of distinct solutions is bounded by

$$m^2n\Delta^2\|b\|_{\infty} \log(m^2\Delta^2\|b\|_{\infty}).$$
Dual Problem

• The dual of the standard form LP is

$$\text{max } b^T y,$$

subject to $A^T y + z = c$, $z \geq 0$.

• The problem is expressed as

$$\text{max } b^T y,$$

subject to $A_B^T y + z_B = c_B$,

$A_B^T y + z_{\bar{B}} = c_{\bar{B}}$,

$z_B \geq 0$, $z_N \geq 0$. 
Kitahara-Mizuno (Dual)

- K-M shows that the number of distinct solutions generated by the dual simplex with Dantzig’s rule is bounded by

\[ m^2 \frac{\gamma_D}{\delta_D} \log(m \frac{\gamma_D}{\delta_D}). \]

where \( \delta_D \) and \( \gamma_D \) are the minimum and the maximum values of all the positive elements of \( z \) in dual basic feasible solutions.

- The bound is independent of \( b \).
Dual BFS

- When $B$ is a index set of a BFS, the dual BFS is
  \[ y = (A_B^T)^{-1}c_B, \quad z_B = 0, \quad z_B = c_B - A_B^T(A_B^T)^{-1}c_B \]

- If $c$ is integral, each element in $z$ is $\frac{p}{q}$ where
  \[ |p| \leq n\Delta \|c\|_\infty, \quad q \leq \Delta. \]

  Hence $\gamma_D \leq n\Delta \|c\|_\infty$ and $\delta_D \geq 1/\Delta$.

- The number of distinct solutions is bounded by
  \[ m^2 n\Delta^2 \|c\|_\infty \log(mn\Delta^2 \|c\|_\infty). \]
Introduction

Results of Kitahara-Mizuno

Results of Tardos

Our Algorithm

Feasibility and an initial dual BFS

Conclusion
Tardos’s Basic Algorithm

- Let $c'$ be the projection of $c$ onto the subspace \{\(x | Ax = 0\)\}.
  If \(c' = 0\) then any feasible solution is optimal.
- Let \(d = (n^2\Delta / \|c'\|_{\infty})c'\), \(\bar{d}_i = \lceil d_i \rceil\) for each \(i\).
- Solve the artificial dual problem

\[
\begin{align*}
\text{max} & \quad b^T y, \\
\text{subject to} & \quad A^T y + z = \bar{d}, \quad z \geq 0.
\end{align*}
\]

- If \(d_i - a_i^T \bar{y} \geq n\Delta\) (which is held by at least one \(i\)) at dual opt. sol. \((\bar{y}, \bar{z})\), then \(x_i^* = 0\) for any primal opt. sol. \(x^*\).
By repeating at most $n$ times of Tardos’s basic algorithm, we can find an optimal face of the primal LP problem.
How to solve the artificial problems

- We use the dual simplex method instead of the ellipsoid method (Khachiyan 1979) or the interior-point method (Karmarkar 1984) for solving artificial LP problems appeared in Tardos’s basic algorithm.
Algorithm

Step 0: If each element of $c$ is an integer and $\|c\|_\infty \leq n^2 \Delta$, then solve the LP problem (1) by the dual simplex method and stop. Otherwise set $K = N$ and go to Step 1.

Step 1: Let $c'_K$ be the projection of $c_K$ onto the subspace $\{x_K | A_K x_K = 0\}$. If $c'_K = 0$ then stop. Otherwise go to Step 2.
Algorithm (cont.)

Step 2: Let $d_K = (n^2 \Delta/\|c'_K\|_{\infty}) c'_K$, $\bar{d}_i = \lceil d_i \rceil$ for each $i \in K$. Consider an LP problem

$$\min \quad \bar{d}_K^T x_K,$$
subject to $A_K x_K = b$, $x_K \geq 0$

Check whether the dual feasible region $F = \{(y, z_K) | A_K^T y + z_K = \bar{d}_K, z_K \geq 0\}$ is empty. If it is empty then stop. Otherwise compute an initial basic feasible solution $(y^0, z_K^0) \in F$ and solve the LP problem by the dual simplex method with Dantzig’s rule.
Step 3: If the dual problem is unbounded, then stop. Otherwise compute the dual optimal basic feasible solution \((\bar{y}, \bar{z}_K)\) and an index set \(B \subset K\) of the optimal basis. Set \(J = \{i | d_i - a_i^T\bar{y} \geq n\Delta, i \in K\}\), where \(a_i\) is the \(i\)-th column of \(A_K\). Remove \(J\) from \(K\). Go to Step 1.
Our Algorithm

Tardos’s results

- If $d_i - a_i^T \bar{y} \geq n\Delta$ at the dual opt. sol. $(\bar{y}, \bar{z})$, then for any primal opt. sol. $x^*$, $x_i^* = 0$.
- $J = \{i | d_i - a_i^T \bar{y} \geq n\Delta, i \in K\}$ is not empty.

At each iteration of the algorithm, we can find at least one index $i$, for which $x_i = 0$ at any primal optimal solution. So we can remove the variable $x_i$ in the next iteration. By repeating the algorithm at most $n$, we will find the optimal face of the primal problem.
Problems in the Algorithm

- How to check the feasibility of the dual problem.
- How to compute an initial dual basic feasible solution.

We will discuss these problems in the next section.
Cases of Stopping

Case 1: Each element of $c$ is an integer and $\|c\|_\infty \leq n^2 \Delta$.

Case 2: $c'_K = 0$, where $c'_K$ is the projection of $c_K$.

Case 3: The dual feasible region $F$ is empty.

Case 4: The dual problem is unbounded.
Case 1: Each element of \( c \) is an integer and \( \|c\|_\infty \leq n^2 \Delta \).
\[ \Rightarrow \text{We can solve the original problem in polynomial time by the simplex method.} \]

Case 2: \( c'_K = 0 \), where \( c'_K \) is the projection of \( c_K \).

Case 3: The dual feasible region \( F \) is empty.

Case 4: The dual problem is unbounded.
Results in each Case of Stopping

Case 1: Each element of $c$ is an integer and $\|c\|_{\infty} \leq n^2 \Delta$.
⇒ We can solve the original problem in polynomial time by the simplex method.

Case 2: $c'_K = 0$, where $c'_K$ is the projection of $c_K$.
⇒ Optimal solution computed in the previous iteration is optimal for original.

Case 3: The dual feasible region $F$ is empty.

Case 4: The dual problem is unbounded.
Results in each Case of Stopping

Case 1: Each element of $\mathbf{c}$ is an integer and $\|\mathbf{c}\|_\infty \leq n^2 \Delta$.
\[ \Rightarrow \] We can solve the original problem in polynomial time by the simplex method.

Case 2: $\mathbf{c'}_K = \mathbf{0}$, where $\mathbf{c'}_K$ is the projection of $\mathbf{c}_K$.
\[ \Rightarrow \] Optimal solution computed in the previous iteration is optimal for original.

Case 3: The dual feasible region $\mathcal{F}$ is empty.
\[ \Rightarrow \] The original dual is infeasible.

Case 4: The dual problem is unbounded.
\[ \Rightarrow \] The original primal is infeasible.
How to check the feasibility of the dual feasible region $F = \{(y, z_K) | A^T_K y + z_K = \bar{d}_K, z_K \geq 0\}$.

How to compute an initial dual basic feasible solution of $F$. 

Problems
A Procedure

Let \( B \subset K \) be an index set of any basis and \( \bar{B} = K - B \). Then a basic solution of \( F \) is

\[
y = (A_B^T)^{-1}\bar{d}_B, \quad z_B = 0, \quad z_{\bar{B}} = \bar{d}_{\bar{B}} - A_{\bar{B}}^T(A_B^T)^{-1}\bar{d}_B.
\]

If \( z_{\bar{B}} \geq 0 \), then this is a basic feasible solution of \( F \). Otherwise let \( I = \{i | z_i < 0, i \in K\} \).

Define an artificial dual problem

\[
\max \sum_{i \in I} z_i,
\]

s.t. \( A_K^T y + z_K = \bar{d}_K \),

\[
z_i \geq 0 \quad (i \in K - I), \quad z_i \leq 0 \quad (i \in I).
\]

Obviously the solution above is a basic feasible solution of this problem.
A Procedure (cont.)

- Solve the problem (5) by the simplex method. When the problem is nondegenerate, the simplex method generates a sequence of BFSs.
- If we get an optimal solution where \( z_i < 0 \) for each \( i \in I \), then \( F \) is empty. Otherwise, a variable \( z_i (i \in I) \) must become zero at some basic feasible solution \((\hat{y}, \hat{z}_K)\).
- Define a new \( \hat{I} = \{i| \hat{z}_i < 0, i \in K\} \). Obviously \( |\hat{I}| < |I| \). If \( \hat{I} \) is empty, the solution \((\hat{y}, \hat{z}_K)\) is a basic feasible solution of \( F \). Otherwise it is a basic feasible solution of (5) for the new \( I = \hat{I} \).
- Repeat the above procedure until \( I = \emptyset \).
Main Result

The proposed algorithm solves at most \((1 + m)n\) artificial dual LP problems by the simplex method. If all the problems are nondegenerate, then the total number of basic solutions generated by the algorithm is bounded by

\[(m + 1)m^2n^4\Delta^3 \log(mn^3\Delta^3).
\]

The algorithm detects whether the problem (1) has an optimal solution. When (1) has an optimal solution, the algorithm finds an optimal basis. If the matrix \(A\) is totally unimodular, then the algorithm is strongly polynomial.
Problem: Standard form linear programming problem

Algorithm: Dual simplex method with Dantzig’s rule + Tardos + Initial BFS

Analysis: Kitahara-Mizuno

Result 1: Number of distinct solutions generated is bounded by a polynomial of $m$, $n$, and $\Delta$.

Result 2: If the coefficient matrix is totally unimodular and all artificial problems solved are nondegenerate, then the algorithm is strongly polynomial.
Announcement

ICCOPT V 2016 TOKYO
(The 5th International Conference on Continuous Optimization of the Mathematical Optimization Society)

Place: Roppongi, Tokyo, JAPAN

Dates: Aug. 6 (Sat) - 11 (Thu), 2016

Venue: National Graduate Institute for Policy Studies (GRIPS)