Bounded lexicographical polytopes

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LIPN

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Part I
Definitions and Top-lex polytopes
Definitions and goal

**Lexicographical order** $\preceq$

$x \preceq y \iff x = y$, or the first nonzero coordinate of $(y - x)$ is positive.
Definitions and goal

Lexicographical order \( \preccurlyeq \)

\[ x \preccurlyeq y \iff x = y, \text{ or the first nonzero coordinate of } (y - x) \text{ is positive.} \]

Top-lex set: \( \ell, u, s \in \mathbb{Z}^n, \ell \leq s \leq u \)

\[ X_{\ell,u}^{s} := \{ x \in \mathbb{Z}^n : \ell \leq x \leq u, x \preccurlyeq s \} \]
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Example

- $p^{(n-1)} := (s_1, \ldots, s_{n-1} - 1, u_n)$
Definitions and Top-lex polytopes

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Example

- \( p^{(n-1)} := (s_1, \ldots, s_{n-1} - 1, u_n) \)
- \( p^{(n-2)} := (s_1, \ldots, s_{n-2} - 1, u_{n-1}, u_n) \)
Definitions and Top-lex polytopes

Lexicographical order ≼

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Example

- \( p^{(n-1)} := (s_1, \ldots, s_{n-1} - 1, u_n) \)
- \( p^{(n-2)} := (s_1, \ldots, s_{n-2} - 1, u_{n-1}, u_n) \)
- \( p^{(i)} := (s_1, \ldots, s_i - 1, u_{i+1}, \ldots, u_n) \)
Lexicographical order $\preceq$

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\[ X_{\ell,u}^{\preceq s} := \{ x \in \mathbb{Z}^n : \ell \leq x \leq u, x \preceq s \} \]

Goal

Describe \( \text{conv}(X_{\ell,u}^{\preceq s}) \)
Definitions and Top-lex polytopes

Lexicographical order \( \preceq \)

\[ x \preceq y \iff x = y, \text{ or the first nonzero coordinate of } (y - x) \text{ is positive.} \]

Top-lex set: \( \ell, u, s \in \mathbb{Z}^n, \ell \leq s \leq u \)

\[ X_{\ell, u}^{\preceq s} := \{ x \in \mathbb{Z}^n : \ell \leq x \leq u, x \preceq s \} \]
Valid inequalities

We construct $cx \leq d$ valid for $\text{conv}(X_{\ell,u}^{s})$ with $cs = d$, $c \geq 0$ and $c_n = 1$. 
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- $c p^{(n-1)} = d - (c_{n-1} - (u_n - s_n))$
- $c_{n-1} = u_n - s_n$
Valid inequalities

We construct $cx \leq d$ valid for $\text{conv}(X_{\ell,u}^s)$ with $cs = d$, $c \geq 0$ and $c_n = 1$.

- $cp^{(n-1)} = d - (c_{n-1} - (u_n - s_n))$
- $cp^{(n-2)} = d - (c_{n-2} - c_{n-1}(u_{n-1} - s_{n-1} + 1))$

\[ c_{n-1} = u_n - s_n \]
\[ c_{n-2} = c_{n-1}(u_{n-1} - s_{n-1} + 1) \]
Valid inequalities

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- $cp^{(i)} = d - (c_i - c_{i+1}(u_{i+1} - s_{i+1} + 1))$

$c_{n-1} = u_n - s_n$

$c_{n-2} = c_{n-1}(u_{n-1} - s_{n-1} + 1)$

$c_i = c_{i+1}(u_{i+1} - s_{i+1} + 1)$
Valid inequalities

We construct \( cx \leq d \) valid for \( \text{conv}(X_{\ell,u}^{s}) \) with \( cs = d \), \( c \geq 0 \) and \( c_n = 1 \).

- \( cp^{(n-1)} = d - (c_{n-1} - (u_n - s_n)) \)
- \( cp^{(n-2)} = d - (c_{n-2} - c_{n-1}(u_{n-1} - s_{n-1} + 1)) \)
- \( cp^{(i)} = d - (c_i - c_{i+1}(u_{i+1} - s_{i+1} + 1)) \)

\[
A_n(x) := (x_n - s_n) + (u_n - s_n) \sum_{i=1}^{n-1} (x_i - s_i) \prod_{j=i+1}^{n-1} (u_j - s_j + 1) \leq 0
\]
Valid inequalities

Inequalities

For $k = 1, \ldots, n$, 

$$A_k(x) := (x_k - s_k) + (u_k - s_k) \sum_{i=1}^{k-1} (x_i - s_i) \prod_{j=i+1}^{k-1} (u_j - s_j + 1) \le 0$$

is valid for $\text{conv}(X_{\leq s}^\ell,u)$. 
Definitions and Top-lex polytopes

Maximal points

Maximal points

\[ M^s \leq s := \{ s, p^{(1)}, \ldots, p^{(n-1)}, p^{(n)} \} \]
Maximal points

\[ M \lessdot^s := \{ s, p^{(1)}, \ldots, p^{(n-1)}, p^{(n)} \} \]
Maximal points

Maximal points

\[ M^{\leq s} := \{ s, p^{(1)}, \ldots, p^{(n-1)}, p^{(n)} \} \]

Two nice properties

- \( X_{\ell,u}^{\leq s} \subseteq M^{\leq s} + \mathbb{R}^n \)
- It is “easy” to obtain \( \text{conv}(M^{\leq s}) \)
Outline of the proof

Description

\[ \text{conv}(X_{\ell,u}^s) = (\text{conv}(M^s) + \mathbb{R}^n) \cap \{x \geq \ell\} \]
Outline of the proof

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\[ \text{conv}(X_{\ell,u}^{\leq s}) = (\text{conv}(\mathcal{M}^{\leq s}) + \mathbb{R}^n) \cap \{x \geq \ell\} \]
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Outline of the proof

Description

\[
\text{conv}(X_{\ell,u}^s) = (\text{conv}(M^s) + \mathbb{R}_-) \cap \{ x \geq \ell \}
\]
Flow model of $\text{conv}(M \preceq \leq s)$
Flow model of $\text{conv}(M\preceq s)$

Example

$p^{(i)} = (s_1, \ldots, s_i - 1, u_{i+1}, \ldots, u_n)$
Flow model of $\text{conv}(M \leq s)$

Extended formulation

\begin{align*}
x_i &= u_i y_i + (s_i - 1) t_i + s_i z_i \\
y_i &= y_{i-1} + t_{i-1} \\
z_i &= z_{i+1} + t_{i+1} \\
y_n + t_n + z_n &= 1 \\
y_i, t_i, z_i &\geq 0
\end{align*}
Flow model of $\text{conv}(M \bowtie s)$

$$\text{conv}(M \bowtie s) = \text{proj}_x \{(x, y, t, z) \text{ satisfying } (1) - (5)\}$$
Flow model of $\text{conv}(M^{\preceq_s})$

$\text{conv}(M^{\preceq_s}) = \text{proj}_x \{ (x, y, t, z) \text{ satisfying } (1) - (5) \}$

$$\sum_{i=1}^{n} A_i(x) \geq -1$$

$$A_k(x) \leq 0 \text{ for } k = 1, \ldots, n$$
Definitions and Top-lex polytopes

Geometrical Intuition

\[ \sum_{i=1}^{n} A_i(x) \geq -1 \]

\[ A_k(x) \leq 0 \text{ for } k = 1, \ldots, n \]
Geometrical Intuition

\[ \sum_{i=1}^{n} A_i(x) \geq -1 \]

\[ A_k(x) \leq 0 \text{ for } k = 1, \ldots, n \]

\[ x \leq u \]
Geometrical Intuition

\[ A_k(x) \leq 0 \text{ for } k = 1, \ldots, n \]
\[ x \leq u \]

Lemma

\[ \text{conv}(M^\leq s) + \mathbb{R}_-^n = \{ x : x \leq u \text{ and } A_k(x) \leq 0 \} \]
Geometrical Intuition

\[ A_k(x) \leq 0 \text{ for } k = 1, \ldots, n \]
\[ x \leq u \]

Lemma

\[ \text{conv}(M^{<s}) + \mathbb{R}_-^n = \{ x : x \leq u \text{ and } A_k(x) \leq 0 \} \]

Proposition

\[ \text{conv}(X^{<s}_{\ell, u}) = \{ x : \ell \leq x \leq u \text{ and } A_k(x) \leq 0 \} \]
Bottom-lex set: \( \ell, u, r \in \mathbb{Z}^n, \ell \leq r \leq u \)

\( X_{\ell,u}^r := \{ x \in \mathbb{Z}^n : \ell \leq x \leq u, r \preceq x \} \)
**Bottom-lex polytope**

Bottom-lex set: \( \ell, u, r \in \mathbb{Z}^n, \ell \leq r \leq u \)

\[ X_{\ell,u}^{r^\preceq} := \{ x \in \mathbb{Z}^n : \ell \leq x \leq u, r \preceq x \} \]

**Proposition**

\[ \text{conv}(X_{\ell,u}^{r^\preceq}) = \{ x : \ell \leq x \leq u \text{ and } B_k(x) \leq 0 \} \]

Here \( B_k(x) = (r_k - x_k) + (r_k - \ell_k) \sum_{i=1}^{k-1} B_i(x) \) for \( k = 1, \ldots, n \).
Part II

Intersection of lex polytopes
Intersection of lexicographical polytopes

Proposition

The intersection of two lexicographical polytopes is an integer polytope.
Remember that $\preceq$ is a total order. Take $r \preceq s$. Then

Intersection “$\preceq \cap \preceq$”

$$\text{conv}(X_{\ell,u}^{\preceq r}) \cap \text{conv}(X_{\ell,u}^{\preceq s}) = \text{conv}(X_{\ell,u}^{\preceq r})$$
Intersection of top- and bottom-lex polytopes

Intersection "≼ ∩ ≽"

\[ \text{conv}(X_{\ell,u}^{s}) \cap \text{conv}(X_{\ell,u}^{r}) \supseteq \text{conv}(X_{\ell,u}^{s} \cap X_{\ell,u}^{r}) \]
Intersection of top- and bottom-lex polytopes

Intersection “$\preceq \cap \succeq$”

$$\text{conv}(X_{\ell,u}^{\preceq \succeq s}) \cap \text{conv}(X_{\ell,u}^{\preceq \succeq r}) \supseteq \text{conv}(X_{\ell,u}^{\preceq \succeq s} \cap X_{\ell,u}^{\preceq \succeq r})$$

Cases

- $r_1 + 1 \leq \pi \leq s_1 - 1$ for some $\pi \in \mathbb{Z}$
- $r_1 = s_1 - 1$
Intersection of top- and bottom-lex polytopes

Intersection “≪ ∩ ≻”

\[ \text{conv}(X_{s}^{\leq}) \cap \text{conv}(X_{r}^{\leq}) \supseteq \text{conv}(X_{s}^{\leq} \cap X_{r}^{\leq}) \]

Cases

- \( r_1 + 1 \leq \pi \leq s_1 - 1 \) for some \( \pi \in \mathbb{Z} \)
- \( r_1 = s_1 - 1 \)
- \( r_1 = s_1 \) reduces to \((n - 1)\)-dimensional case (induction)
Case $r_1 + 1 \leq \pi \leq s_1 - 1$

Let $\ell' = (\pi, \ell_2, \ldots, \ell_n)$. Then:

**Observation**

\[
\text{conv}(X_{r,\ell,u}^{\leq}) \cap \text{conv}(X_{s,\ell,u}^{\leq}) \cap \{x_1 \geq \pi\} = \text{conv}(X_{\ell',u}^{\leq s})
\]
Case $r_1 + 1 \leq \pi \leq s_1 - 1$

Let $\ell' = (\pi, \ell_2, \ldots, \ell_n)$. Then:

**Observation**

- $\text{conv}(X_{\ell, u}^{\leq r}) \cap \text{conv}(X_{\ell, u}^{\leq s}) \cap \{x_1 \geq \pi\} = \text{conv}(X_{\ell', u}^{\leq s})$
Case $r_1 + 1 \leq \pi \leq s_1 - 1$

Let $u' = (\pi, u_2, \ldots, u_n)$. Then:

Observation

\[ \text{conv}(X_{r, u}^{\leq}) \cap \text{conv}(X_{s, u}^{\leq}) \cap \{x_1 \leq \pi\} = \text{conv}(X_{r, u'}^{\leq}) \]
Case $r_1 + 1 \leq \pi \leq s_1 - 1$

Intersection "$\preceq \cap \succeq$"

- $\text{conv}(X_{l,u}^{\preceq s}) \cap \text{conv}(X_{l,u}^{r \preceq}) = \text{conv}(X_{l,u}^{\preceq s} \cap X_{l,u}^{r \preceq})$
Case $r_1 + 1 \leq \pi \leq s_1 - 1$

Intersection “$\preceq \cap \succeq$”

- $\text{conv}(X_l^{s}) \cap \text{conv}(X_r^{l}) = \text{conv}(X_l^{s} \cap X_r^{l})$

What if $r_1 = s_1 - 1$?
Case $r_1 + 1 \leq \pi \leq s_1 - 1$

Intersection "$\preceq \cap \succeq$"

- $\text{conv}(X_{l,u}^{\preceq s}) \cap \text{conv}(X_{l,u}^{r \preceq}) = \text{conv}(X_{l,u}^{\preceq s} \cap X_{l,u}^{r \preceq})$

What if $r_1 = s_1 - 1$?

Sufficient condition

$x_1 = r_1$ or $x_1 = s_1$ for all $x \in \text{vert}(\text{conv}(X_{l,u}^{\preceq s}) \cap \text{conv}(X_{l,u}^{r \preceq}))$
Case $r_1 = s_1 - 1$
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Part III

Link with knapsacks
Lex polytopes vs. Superincreasing knapsacks

Superincreasing knapsack

\[ K_u^{a,b} = \text{conv}\{x \in \mathbb{Z}^n : 0 \leq x \leq u, ax \leq b\} \text{ with } a_k \geq \sum_{i=k+1}^n a_i u_i \geq 0 \]
Lex polytopes vs. Superincreasing knapsacks

Superincreasing knapsack

\[ K_{u}^{a,b} = \text{conv}\{x \in \mathbb{Z}^n : 0 \leq x \leq u, ax \leq b\} \text{ with } a_k \geq \sum_{i=k+1}^{n} a_i u_i \geq 0 \]

Equivalence with Superincreasing knapsack

Superincreasing knapsacks are top-lexicographical polytopes, and conversely.
Lex polytopes vs. Superincreasing knapsacks

Proposition

Superincreasing knapsacks are Top-lexicographical polytopes.
Lex polytopes vs. Superincreasing knapsacks

Proposition

Superincreasing knapsacks are Top-lexicographical polytopes.

Proof

Let $s$ be the lexicographically greatest point in $K_{a,b}^u$. Then

$$K_{a,b}^u \subseteq \text{conv}(X_{0,u}^{\leq s})$$
Lex polytopes vs. Superincreasing knapsacks

Proposition
Superincreasing knapsacks are Top-lexicographical polytopes.

Proof
Let $s$ be the lexicographically greatest point in $K_{u}^{a,b}$. Then

$$K_{u}^{a,b} \subseteq \text{conv}(X_{0,u}^{s})$$

If $x \prec y$ then $ax \leq ay$. Indeed:
- there exists $k$ s.t. $x_k + 1 \leq y_k$ and $x_i = y_i$ for $i < k$
Lex polytopes vs. Superincreasing knapsacks

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Superincreasing knapsacks are Top-lexicographical polytopes.

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Let $s$ be the lexicographically greatest point in $K_{u}^{a,b}$. Then

$$K_{u}^{a,b} \subseteq \text{conv}(X_{0,u}^{\leq s})$$

If $x \prec y$ then $ax \leq ay$. Indeed:

- there exists $k$ s.t. $x_k + 1 \leq y_k$ and $x_i = y_i$ for $i < k$
- $ay - ax \geq \sum_{i > k} a_i(y_i - x_i) + a_k \geq \sum_{i > k} a_i(y_i - x_i + u_i) \geq 0$
Lex polytopes vs. Superincreasing knapsacks

Proposition
Superincreasing knapsacks are Top-lexicographical polytopes.

Proof
Let $s$ be the lexicographically greatest point in $K_{u,b}$. Then

$$K_{u,b} \subseteq \text{conv}(X_0^{\preceq s})$$

If $x \prec y$ then $ax \leq ay$. Indeed:

- there exists $k$ s.t. $x_k + 1 \leq y_k$ and $x_i = y_i$ for $i < k$
- $ay - ax \geq \sum_{i > k} a_i(y_i - x_i) + a_k \geq \sum_{i > k} a_i(y_i - x_i + u_i) \geq 0$

Then $\text{conv}(X_0^{\preceq s}) \subseteq K_{u,b}$
Lex polytopes vs. Superincreasing knapsacks

Proposition

Superincreasing knapsacks are Top-lexicographical polytopes, and conversely.

- It seems that our results were known in the knapsack setting (Gupte (2013))
- Lex setting gives simpler proofs and more geometrical insight
Bibliography
