Sperner labellings : a combinatorial approach

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January 30, 2006

Abstract

In 2002, De Loera, Peterson and Su proved the following conjecture of Atanassov: let T be a triangulation of a d-dimensional polytope P with n vertices $v_1, v_2, \ldots, v_n$; label the vertices of T by 1, 2, ..., n in such a way that a vertex of T belonging to the interior of a face F of P can only be labelled by j if v_j is on F; then there are at least $n-d$ simplices labelled with $d+1$ different labels. We prove a generalization of this theorem which refines this lower bound and which is valid for a larger class of objects.

Key Words: chain map; fully-labelled simplex; labelling; polytopal body; polytope; Sperner's lemma; triangulation.

1 Introduction

1.1 Topic and goal

Sperner's lemma is a well-known combinatorial reformulation of Brouwer's fixed-point theorem. In 1996, Atanassov [1] conjectured a generalization of Sperner's lemma for a triangulation of a convex polytope P. By *triangulation*, we mean a geometric simplicial complex homeomorphic to P (the vertex set is not necessarily restricted to the vertex set of P). The conjecture is the following: if each vertex of a d-dimensional convex polytope P gets an unique label of $\{1, \ldots, n\}$, where n is the number of vertices of P, and if each other vertex of the triangulation gets a label of one of the vertices of the minimal face of P it belongs to (providing a *Sperner labelling* of the triangulation), then there are at least $n-d$ fully-labelled d-simplices. By fully-labelled simplex, we mean a simplex whose labels are all distinct. Sperner's lemma is a special case of Atanassov's conjecture with $n = d+1$.

In 2002, De Loera, Peterson and Su [2] gave two proofs of this conjecture: one with a geometric approach giving a lower bound of the smallest cover of a polytope; the other constructive using a path-following argument.

The purpose of our paper is to give a purely combinatorial proof of Atanassov's conjecture, which refines the lower bound, and which is valid for a larger class of objects. We call the elements of this class *polytopal bodies*. Polytopal bodies do not seem to have been defined before the present paper. Roughly speaking, a d-dimensional polytopal body is a pure d-dimensional polytopal complex P embedded in $\mathbb{R}^d$ such that (i) the boundary complex $\mathsf{B}(\mathsf{P})$ of P (see the definition later) is strongly connected and (ii) each polytope of $\mathsf{B}(\mathsf{P})$ having dimension $d-2$ belongs to exactly two $(d-1)$-polytopes of $\mathsf{B}(\mathsf{P})$. No assumption about convexity or simple connectivity of the boundary is needed. According to this definition, if P is a polytope, the set of all its faces $\mathsf{L}(P)$ (including P) is a polytopal body.

Let P be a polytopal body such that $\mathsf{B}(\mathsf{P})$ has n vertices. Let T be a triangulation of the underlying space $||\mathsf{P}||$ inducing triangulations of the polytopes of $\mathsf{B}(\mathsf{P})$. Notice that we

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have in particular that the vertex set of $\mathsf{B(P)}$ is contained in the vertex set of T. If we label the vertices of T in such a way that each vertex of $\mathsf{B(P)}$ gets a unique label in $\{1, 2, \ldots, n\}$, each vertex of T in the interior of $||\mathsf{P}||$ gets any label in $\{1, 2, \ldots, n\}$ and each other vertex of T gets a label of one of the minimal polytope (ordered by inclusion) of $\mathsf{B(P)}$ it belongs to, then we get a *Sperner labelling* of T. When the polytopal body is the set of all faces of a polytope, the definition coincides with the one given above.

We state here the main theorem of the present paper:

Theorem 1 *Let P be a d-dimensional polytopal body whose boundary complex $\mathsf{B(P)}$ has n vertices $v_1, \ldots, v_n$. Let T be a triangulation of the underlying space $||\mathsf{P}||$ of P, inducing a triangulation of each of the polytopes of $\mathsf{B(P)}$. Any Sperner labelling of T contains at least $n + \left\lceil \frac{\min_i \deg_{\mathsf{B(P)}}(v_i)}{d} \right\rceil - d - 1$ fully-labelled d-simplices such that any pair of these fully-labelled simplices receives two different labellings,*

where the *degree* of a vertex v of a polytopal complex C, denoted by $\deg_{\mathsf{C}}(v)$, is the number of edges of C which v belongs to.

As $\min_i \deg_{\mathsf{B(P)}}(v_i) \geq d$, we find the lower bound found in [2] for polytopes. Our bound is a real improvement: for the cyclic polytope $C(n)$ with n vertices in dimension 4 (see for instance p.15 of the book [5]), which is such that $G = (V(C(n)), E(C(n))) = K_n$ (complete graph of n vertices), the lower bound given by Theorem 1 is $n + \left\lceil \frac{n-1}{4} \right\rceil - 4 - 1 \sim \frac{5}{4}n$ $(n \to \infty)$. A lower bound that is asymptotic to $\sim n$ is obtained with the polytopal generalization of Sperner's lemma of [2].

1.2 Plan

The paper is divided into four parts:

In the first one (Section 2), we fix the basic notations and tools we use in the rest of the paper. We define in particular the notions of simplices, simplicial complexes, chains, chain maps. As we use GF(2) coefficients, a k-chain is seen as a formal sum (or, simply, a set) of k-simplices of a simplicial complex. We define also the notion of *strong connectivity* for a chain, and prove a lower bound for the number of simplices in a strongly connected chain: Proposition 1.

In the second one (Section 3), we define precisely what are a *polytopal body* P, a *triangulation* of a polytopal body and the *Sperner labelling* of such a triangulation.

In the third one (Section 4), we define *spread chains* and *quasi-triangulations*. A spread chain is a family of d-dimensional simplices whose vertices are vertices of the boundary complex of the d-dimensional polytopal body. A quasi-triangulation is a spread chain for which the parity of the number of simplices containing any generic point is odd if and only if the point is in the underlying space $||\mathsf{P}||$. Thus, a quasi-triangulation is a kind of binary cover in the sense defined in a paper of R. T. Firla and G. M. Ziegler ([3]). Conversely, a binary cover is not necessarily a quasi-triangulation, as shown in Section 4. We prove then properties of quasi-triangulations.

In the last part (Section 5), we prove Theorem 1 for the triangulations of d-dimensional polytopal body using properties of quasi-triangulations proved in Section 4 and using Proposition 1.

1.3 Outline of the proof

Let P be a d-dimensional polytopal body and let T be a triangulation of its underlying space $||\mathsf{P}||$ inducing triangulations of the polytopes of the boundary $\mathsf{B(P)}$. A Sperner labelling of T induces a simplicial map going from T (which is a simplicial complex) to the (abstract) simplicial complex whose simplices are k-subsets of the vertex set of $\mathsf{B(P)}$, $k \leq d+1$. This map induces itself a chain map. Here is the link with the fully-labelled simplices: a k-simplex in the image of this chain map corresponds to a fully-labelled simplex in T.

By induction, we prove that the image of the formal sum of all d-simplices of T by the chain map is a quasi-triangulation of P. We apply then Proposition 1 to one of the strongly connected components of the quasi-triangulation and obtain a lower bound for the number of simplices in the component, and thus for the number of simplices of the quasi-triangulation, and finally for the number of fully-labelled simplices of T.

2 Tools and Notation

We denote by $|A|$ the cardinality of A. $\lceil x \rceil$ is the smallest integer bigger than or equal to a real x. For a finite set Λ, $\binom{\Lambda}{k}$ is the set of k-subsets of Λ and $\binom{\Lambda}{\leq k}$ the set of subsets of Λ whose cardinality is less or equal to k. Given a sequence $a_0, \ldots, a_i, \ldots, a_k$, the sequence $a_0, \ldots, \hat{a}_j, \ldots, a_k$ is the same sequence with the a_j missing.

The set of proper faces ($\emptyset$ is included) of a polytope P is denoted by $\mathsf{F}(P)$. The set of all faces of P is denoted by $\mathsf{L}(P)$. We have then $\mathsf{L}(P) = \{P\} \cup \mathsf{F}(P)$.

For a set of points U, we denote by $\mathrm{conv}(U)$ the convex hull of the points of U.

For a compact set C of a topological space X, ∂C denotes the boundary of C, which is the intersection of the closure of C and the closure of the complement of C in X.

2.1 Simplices, complexes and chains

We give here a short introduction to the notions of simplices, complexes and chains. For a more complete study of this subject, see the book of Munkres [4]. We work with chains with coefficients in GF(2), thus we will not introduce the notion of an oriented simplex.

2.1.1 Simplices and simplicial complexes

An *(abstract) simplicial complex* is a collection K of subsets of a finite ground set with the property that $\sigma' \subseteq \sigma \in \mathsf{K}$ implies $\sigma' \in \mathsf{K}$. We define the *dimension* of K: $\dim(\mathsf{K}) = \max\{|\sigma| - 1 : \sigma \in \mathsf{K}\}$. The sets in K are called *(abstract) simplices* and the dimension of a simplex σ is $\dim(\sigma) = |\sigma| - 1$. If $\dim(\sigma) = d$, we say that σ is a d-simplex. $\emptyset$ has dimension -1.

The elements of K (resp. the subsets of a simplex σ) are called *faces*. A p-face of K is a face of K of dimension p. 0-faces are called *vertices*, and 1-faces *edges*. The set of the formers is denoted by $V(\mathsf{K})$, and the set of the latters by $E(\mathsf{K})$. $\emptyset$ is the only -1-face of K. For a p-simplex σ, the *facets* are the simplices $\sigma' \subseteq \sigma$ of dimension $p-1$.

For a finite set Λ, $\binom{\Lambda}{\leq k+1}$ is an example of k-dimensional simplicial complex.

Let K and L be two abstract simplicial complexes. A *simplicial map* of K into L is a mapping $f : V(\mathsf{K}) \to V(\mathsf{L})$ that maps simplices to simplices, i.e., such that $f(\sigma) \in \mathsf{L}$ whenever $\sigma \in \mathsf{K}$.

2.1.2 Geometric simplicial complexes, polytopal complexes and triangulations

A *polytopal complex* C is a collection of polytopes such that (i) $\emptyset$ is in C, (ii) for any $P \in \mathsf{C}$, all faces of P are in C, (iii) the intersection of any two polytopes in C is a face of both.

For instance, if P is a polytope, $\mathsf{L}(P)$ is a polytopal complex.

The polytopes in C are also called the *faces* of C. The maximal faces are called the *facets*. The *vertices* (resp. the *edges*) of C, denoted by $V(\mathsf{C})$ (resp. $E(\mathsf{C})$), are the 0-dimensional (resp. 1-dimensional) faces of C. The *degree* of a vertex v of C, denoted by $\deg_{\mathsf{C}}(v)$, is the number of edges of C it belongs to.

The *dimension* of C, denoted by $\dim \mathsf{C}$, is the largest dimension of a polytope in C. By $||\mathsf{C}||$, we mean the union of all polytopes of C, which we call the *underlying space* of C.

For instance, if P is a polytope, $||\mathsf{L}(P)|| = P$.

If all the maximal polytopes (by inclusion) of C have the same dimension, C is said to be *pure*.

If each $(d-1)$-dimensional polytope of a d-dimensional polytopal complex C is contained in one or two d-dimensional polytopes of C, we define the $(d-1)$-dimensional polytopal complex $\mathsf{B}(\mathsf{C})$, called the *boundary complex*, or simply the *boundary*, of C as follows: $P \in \mathsf{B}(\mathsf{C})$ if and only if P is a face of a $(d-1)$-dimensional polytope contained in exactly one d-dimensional polytope of C (P can be the $(d-1)$-dimensional polytope itself).

Clearly, a boundary complex is always pure.

We have also the following useful observation:

Observation 1 *If P is a polytope, we have $\mathsf{B}(\mathsf{L}(P)) = \mathsf{F}(P)$ (the boundary complex of the set of faces of a polytope P is the set of its proper faces).*

Finally, denoting by $V(P)$ the set of vertices of a polytope P, we make the following observation:

Observation 2 *Let F and G be two polytopes of a polytopal complex C.*
We have then $V(F \cap G) = V(F) \cap V(G)$.

If all polytopes of a polytopal complex are geometric simplices (a geometric simplex is the convex hull of $d+1$ affinely independent points), we call the polytopal complex a *geometric simplicial complex*. If C is a geometric simplicial complex, the collection of the vertex sets of the simplices $\{V(\sigma) : \sigma \in \mathsf{C}\}$ forms an abstract simplicial complex. Thus, in the sequel, a geometric simplicial complex will simultaneously be understood as an abstract simplicial complex.

A *triangulation* of a topological space X is a geometric simplicial complex T such that $||\mathsf{T}||$ is homeomorphic to X.

A d-dimensional polytopal complex is said to be *strongly connected* if for every pair $P, P' \in \mathsf{C}$, each of them of dimension d, there is a sequence of d-dimensional polytopes of C, $P = P_0, P_1, \ldots, P_r = P'$, such that either $r = 0$, or $r \geq 1$ and $P_i \cap P_{i+1}$ is a $(d-1)$-face of both for all $i = 0, \ldots, r-1$.

2.1.3 Chains

Let K be an abstract simplicial complex. The *chain complex* $\mathcal{C}(\mathsf{K})$ is :

$$\ldots \to C_3(\mathsf{K}) \xrightarrow{\partial} C_2(\mathsf{K}) \xrightarrow{\partial} C_1(\mathsf{K}) \xrightarrow{\partial} C_0(\mathsf{K}) \to \ldots,$$

where $C_p(\mathsf{K})$ is the free abelian group of all formal linear combinations of p-faces of K with coefficients in GF(2). Any element c of $C_p(\mathsf{K})$ is called a *p-chain*.

For $c \in C_p(\mathsf{K})$, $\mu_c(\sigma)$ is the coefficient of σ in c, and the *support* of c, denoted by supp c, is the set $\{\sigma \in \mathsf{K} : \mu_c(\sigma) = 1\}$. We say that a c' is a *subchain* of c, where $c, c' \in C_p(\mathsf{K})$, if and only if $\mu_c(\sigma) = 0 \Rightarrow \mu_{c'}(\sigma) = 0$. This inclusion will be denoted by $c' \subseteq c$. Let $c \in C_p(\mathsf{K})$: by $\sigma \in c$, we mean $\sigma \in \mathsf{K}$ such that $\mu_c(\sigma) = 1$ (this is an abuse of notation: we should write $\sigma \in \text{supp } c$).

By $V(c)$, we mean the set of vertices of all simplices $\sigma \in c$: $V(c) = \cup_{\sigma \in c} V(\sigma)$. By $E(c)$, we mean the set of edges of all simplices $\sigma \in c$: $E(c) = \cup_{\sigma \in c} E(\sigma)$.

For $v \in V(c)$, we define $\deg_c(v) := |\{e \in E(c) : v \in e\}|$.

We define the *boundary operator* ∂ for a simplicial complex K as follows: ∂ is a homomorphism of free groups: $C_p(\mathsf{K}) \to C_{p-1}(\mathsf{K})$ and if σ is a p-simplex $\{v_0, \ldots, v_p\}$, $p \geq 1$, $\partial\sigma := \sum_{i=0}^{p} \{v_0, \ldots, \hat{v}_i, \ldots, v_p\}$.

The boundary operator satisfies:

$$\partial\partial = \partial^2 = 0 \tag{1}$$

because it is obviously true for simplices ($[\ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots]$ arises twice).

A *chain map* ν is a collection of homomorphisms $\nu_p : C_p(\mathsf{K}) \to C_p(\mathsf{L})$ such that

$$\partial \nu_p = \nu_{p-1} \partial \tag{2}$$

for all p. If f is a simplicial map of K to L, we define a collection $f_\#$ of homomorphisms $f_{\#p} : C_p(\mathsf{K}) \to C_p(\mathsf{L})$ by defining it on simplices as follows: for σ a p-simplex, we have

$$f_{\#p}(\sigma) = \begin{cases} f(\sigma) \text{ if } \dim f(\sigma) = p \\ 0 \text{ otherwise.} \end{cases}$$

$f_\#$ is then a chain map (it can be easily checked, again, first for simplices).

A d-chain is said to be *strongly connected* is for every pair $\sigma, \sigma' \in c$, there is a sequence of d-simplices of c, $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_r = \sigma'$, such that either $r = 0$, or $r \geq 1$ and $\sigma_i \cap \sigma_{i+1}$ is a $(d-1)$-face of both for all $i = 0, \ldots, r-1$.

2.2 A bound for the number of simplices in the support of a strongly connected chain

We give now a proposition which will allow us to give a lower bound of the number of simplices in the image of the labellings seen as a chain map:

Proposition 1 *Let K be a simplicial complex of dimension d. Let c a strongly connected chain of $C_d(\mathsf{K})$. Then*

$$|\text{supp } c| \geq |V(c)| + \left\lceil \frac{\min_{v \in V(c)} \deg_c(v)}{d} \right\rceil - d - 1.$$

Proof: We proceed by induction on $k := |\text{supp } c|$. If $k = 1$, the proposition is trivial. Let us suppose that $k > 1$. It is well known that in a connected graph with at least 2 vertices, there is always a vertex whose removal does not disconnect the graph. Considering the graph whose vertices are the d-simplices, and such that two simplices are connected by an edge if they have a facet in common, we see that there is a $\sigma \in c$ such that $c' := c - \sigma$ is still a strongly connected chain. For $v \in V(c')$, $\deg_{c'}(v) \geq \deg_c(v) - d$.

As σ has a $(d-1)$-face in common with another d-simplex of K, at least d vertices among the $d+1$ of σ remains in $V(c')$, thus either $|V(c')| = |V(c)|$ or $|V(c')| = |V(c)| - 1$. Moreover, if $V(c') = V(c) - 1$, $\min_{v \in V(c)} \deg_c(v) = d$, and then $\min_{v \in V(c')} \deg_{c'}(v) \geq \min_{v \in V(c)} \deg_c(v)$. Thus:

$$\begin{aligned} |\text{supp } c| - 1 = |\text{supp } c'| &\geq |V(c')| + \left\lceil \frac{\min_{v \in V(c')} \deg_{c'}(v)}{d} \right\rceil - d - 1 \\ &\geq |V(c)| + \left\lceil \frac{\min_{v \in V(c)} \deg_c(v)}{d} \right\rceil - 1 - d - 1, \end{aligned}$$

which implies

$$|\text{supp } c| \geq |V(c)| + \left\lceil \frac{\min_{v \in V(c)} \deg_c(v)}{d} \right\rceil - d - 1.$$

∎

3 Bodies and Sperner labellings

3.1 Bodies: the definition

Definition 1 *A d-*dimensional polytopal body P *is a d-dimensional polytopal complex embedded in $\mathbb{R}^d$ such that*

1. *if $d = 0$, $\mathsf{P} = \{v, \emptyset\}$, where v is a point.*
2. *if $d = 1$, $\mathsf{P} = \{[v, w], v, w, \emptyset\}$, where v, w, are two different points in $\mathbb{R}$, and $[v, w]$ is the segment linking these two points.*
3. *if $d > 1$,*
   (a) *$\mathsf{B}(\mathsf{P})$, the boundary of P, is strongly connected,*
   (b) *each $(d-2)$-dimensional face of $\mathsf{B}(\mathsf{P})$ belongs to exactly two $(d-1)$-dimensional faces of $\mathsf{B}(\mathsf{P})$.*

Note that this implies that P is strongly connected as well.

If P is a polytope, $\mathsf{L}(P)$ is a polytopal body. The converse is not true: the Figure 1 shows a 3-dimensional polytopal body (whose boundary has 16 vertices and 16 facets), whose underlying space is not a polytope.

3.2 Some properties of polytopal bodies

We state now some properties of polytopal bodies.

Observation 3 *Let P be a polytopal body, and $\tau \subseteq V\big(\mathsf{B}(\mathsf{P})\big)$. Either there is no element G of $\mathsf{B}(\mathsf{P})$ such that $\tau \subseteq V(G)$, or there is a unique minimal (by inclusion) element F of $\mathsf{B}(\mathsf{P})$ such that $\tau \subseteq V(F)$.*

Indeed, if there is at least one G in $\mathsf{B}(\mathsf{P})$ such that $\tau \subseteq V(G)$, take all the faces F_i such that $\tau \subseteq V(F_i)$. Observation 2 allows us to write $V(\cap_i F_i) = \cap_i V(F_i)$. Thus $\cap_i F_i$ is then the minimal element of $\mathsf{B}(\mathsf{P})$ whose vertex set contains τ.

For a polytope P (whose faces are ordered by inclusion) and a chain c (of abstract simplices):

$$c|_P := \sum_{\tau \in c : P \text{ is the minimal face of } \mathsf{L}(P) \text{ s.t. } \tau \subseteq V(P)} \tau.$$

According to this definition, if $V(P) \cap V(c) = \emptyset$, then $c|_P = 0$.

For instance, let P be a 4-dimensional hypercube. $\mathsf{L}(P)$ is a 4-dimensional polytopal body. Let F be a 3-dimensional face of P, let G be a face (a square) of F with $V(G) = \{1, 2, 3, 4\}$, and let c be the 3-simplex $[1, 2, 3, 4]$. Then $c|_F = 0$ (because F is not minimal) and $c|_G = [1, 2, 3, 4]$. As another example, take the 3-dimensional polytopal body of Figure 1 and F the face whose vertices are v_1, v_{12}, v_{11}, v_4. Let $c := [v_1, v_{12}, v_{11}, v_4] + [v_1, v_{12}, v_7, v_8]$. Then $c|_F = [v_1, v_{12}, v_{11}, v_4]$. For $c' := [v_1, v_{11}] + [v_{11}, v_{12}]$, we have $c'|_F = [v_1, v_{11}]$.

Observation 4 *For two chains c and c', $(c + c')|_F = c|_F + c'|_F$.*

Observation 5 *Let P be a polytope and c be a chain such that $V(c) \subseteq V(P)$. We have the following equality:*

$$c = \sum_{F \in \mathsf{L}(P)} c|_F.$$

This last observation can be seen as a consequence of Observation 3 for the polytopal body $\mathsf{L}(P)$.

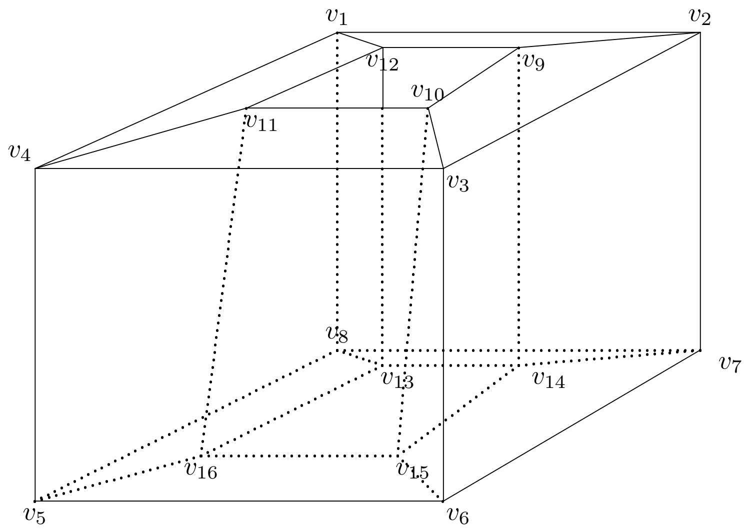

Figure 1: A 3-dimensional polytopal body whose boundary has 16 facets

3.3 Triangulations of polytopal bodies

We define a triangulation of a polytopal body as follows:

Definition 2 *A* triangulation *of a polytopal body is a triangulation of its underlying space inducing triangulations of the faces of its boundary.*

If the boundary $\mathsf{B}(\mathsf{P})$ of a polytopal body P has two neighboring facets having the same supporting hyperplane, we could have triangulations of $||\mathsf{P}||$ which do not induce triangulations of the facets of $\mathsf{B}(\mathsf{P})$. A triangulation of a polytope P is a triangulation of $\mathsf{L}(P)$. Thus we cover the case studied in [2].

We state an easy property about such triangulations of polytopal bodies, which will be useful for the induction in the proof of Theorem 2. Let T be a triangulation of a polytopal body P (we consider T as an abstract simplicial complex) satisfying the assumption above. By $(\partial\mathsf{T})|_F$, we mean the triangulation of the face $F \in \mathsf{B}(\mathsf{P})$ induced by the triangulation T:

Observation 6 *Let F be one of the facets of $\mathsf{B}(\mathsf{P})$. Let T be the formal sum of all d-simplices of T. Then $(\partial T)|_F$ is the formal sum of all $(d-1)$-simplices of $(\partial\mathsf{T})|_F$.*

Indeed, let τ be a simplex of $(\partial T)|_F$. Then τ is a facet of a unique d-simplex σ of T, and $\tau \subseteq V(F)$. As F is a polytope, this implies that τ is a simplex of $(\partial\mathsf{T})|_F$.

Conversely, let τ be a (abstract) $(d-1)$-dimensional simplex of $(\partial\mathsf{T})|_F$. Then $\tau \subseteq V(F)$ and, as the vertices of τ are affinely independent, they are not in a proper face of F. Moreover, τ is a simplex of ∂T. Hence, $\tau \in (\partial T)|_F$.

3.4 Sperner labellings

Given a simplicial complex K, a *labelling* is a surjective function λ mapping the vertex set to a set, called the *set of labels*. If we denote by Λ this set of labels, λ induces a simplicial map from K into the abstract simplicial complex $\binom{\Lambda}{\leq \dim(\mathsf{K})+1}$, which induces itself a chain map. This last chain map will be denoted $\lambda_{\#}$.

A labelling λ of a triangulation T of a d-dimensional polytopal body P is a *Sperner labelling* if
(i) the set of labels is the vertex set of $\mathsf{B}(\mathsf{P})$,
(ii) each vertex of $\mathsf{B}(\mathsf{P})$ gets itself as a label and
(iii) each vertex of T belonging to a face of $\mathsf{B}(\mathsf{P})$ gets a label of one of the vertices of the minimal face of $\mathsf{B}(\mathsf{P})$ it belongs to (minimal with respect to inclusion).

Formally written λ is a Sperner labelling if: for $v \in V(\mathsf{T})$, $\lambda(v) \in V\big(\mathsf{B}(\mathsf{P})\big)$ and for $F \in \mathsf{B}(\mathsf{P})$, $v \in F \Rightarrow \lambda(v) \in V(F)$.

There is no condition for the vertices in the interior of $||\mathsf{P}||$: they can get any vertex of $\mathsf{B}(\mathsf{P})$ as a label. This definition is equivalent to the one given at the beginning of the present paper for polytopes (and thus equivalent to the one given in the paper [2]).

For instance, for the polytopal body of the Figure 1, vertices of T inside the face F defined by the vertices v_1, v_{12}, v_{11}, v_4 can only get v_1, v_4, v_{11} or v_{12}. Vertices of T inside the edge $[v_1, v_4]$ can get v_1 or v_4. Nevertheless, $\lambda(v_1) = v_1, \lambda(v_4) = v_4, \ldots$

4 Spread chains and quasi-triangulations

We define now the main tool of our result: the *spread* chain with respect to a polytopal body P. A spread chain has, for the moment, nothing to do with labellings. This notion concerns only chains whose vertex set is a subset of the vertex set of the boundary of the polytopal body. The motivation for introducing the notion of spread chains is that the Sperner labelling of a triangulation of a polytopal body induces a chain map whose image is a spread chain.

We consider the abstract simplicial complex $\binom{V(\mathsf{B}(\mathsf{P}))}{\leq d+1}$ whose simplices are the subsets of $V\big(\mathsf{B}(\mathsf{P})\big)$ of at most $d+1$ elements.

Definition 3 *Let P be a d-dimensional polytopal body. A d-chain $c \in C_d(\binom{V(\mathsf{B}(\mathsf{P}))}{\leq d+1})$ is* spread *on P, if for every simplex $\sigma \in \partial c$, there exists a face F of $\mathsf{B}(\mathsf{P})$ such that $\sigma \subseteq V(F)$.*

According to this definition 0 is always a spread chain. If we replace in the definition above the existence of a face F by the existence of a facet F such that $\sigma \subseteq V(F)$, we get an equivalent definition.

For instance, if P is a square with four vertices 1,2,3 and 4, in this order. The chain $[1, 2, 3]$ is not spread with respect to $\mathsf{L}(P)$ because $[1, 3]$ is not in a facet of P. But $[1, 2, 3] + [1, 3, 4]$ is spread.

The following consequence of Observation 3 is useful:

Observation 7 *c is spread if and only if $\partial c = \sum_{F \in \mathsf{B}(\mathsf{P})} (\partial c)|_F$.*

We will make an intensive use of the following lemma:

Lemma 1 *If c is spread on P and F is a facet of $\mathsf{B}(\mathsf{P})$, then $(\partial c)|_F$ is spread on $\mathsf{L}(F)$.*

Proof: Indeed, let $c' := (\partial c)|_F$ and let $\tau \in \partial c'$. To show that c' is spread, we only have to show that $\tau \subseteq V(H)$ for some face H of $\mathsf{B}(\mathsf{L}(F))$. Because of Observation 1, we have to show that $\tau \subseteq V(H)$ for some proper face H of F. As $\partial^2 c = 0$, $\tau \in \partial(\partial c - c')$. Let $\sigma \in \partial c - c'$ such that $\tau \in \partial\sigma$. c is spread and, by definition of σ, the minimal face whose vertex set contains σ is not F. Thus, there is a face F' of $\mathsf{B}(\mathsf{P})$ such that $F' \neq F$ and $\tau \subseteq V(F')$. Hence $\tau \subseteq V(F) \cap V(F')$, which means, because of Observation 2, $\tau \subseteq V(H)$, where $H := F \cap F'$ is a face of F. It remains to show that H is actually a proper face of

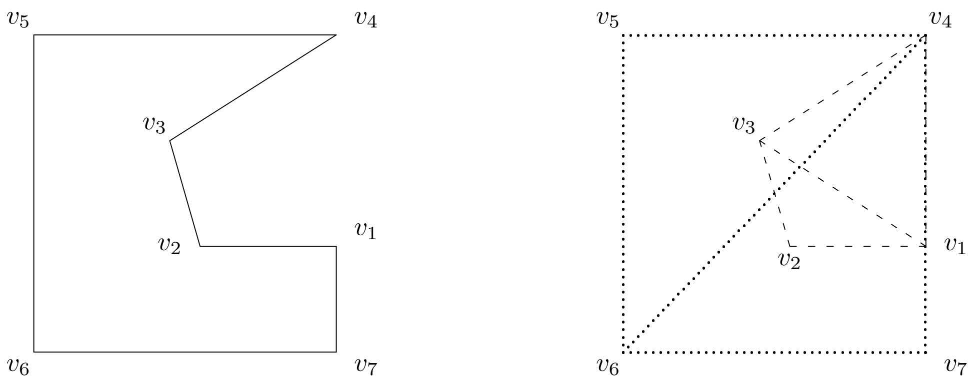

Figure 2: A binary cover is not necessarily a quasi-triangulation (v_7, v_1 and v_4 are aligned)

F, but this is straightforward: if $F = F \cap F'$, we have simultaneously $F \subseteq F'$, $F \neq F'$, F is facet of $\mathsf{B}(\mathsf{P})$, and F' a face of $\mathsf{B}(\mathsf{P})$. Contradiction. ∎

We define a *quasi-triangulation* of a polytopal body recursively:

Definition 4 *For a d-dimensional polytopal body P, c is a* quasi-triangulation *if and only if*

(i) c is a spread chain, and

(ii) either $\dim \mathsf{P} = 0$ *and $c \neq 0$, or, for every facet F of* $\mathsf{B}(\mathsf{P})$, $(\partial c)|_F$ *is a quasi-triangulation of* $\mathsf{L}(F)$.

For instance, if $||\mathsf{P}||$ is a segment $[v, w]$, the only quasi-triangulation of $\mathsf{L}(P)$ is $[v, w]$ (this explains why the following theorems are easy to prove for $d = 0$ or $d = 1$).

Of course, the formal sum of all (abstract) d-simplices of a triangulation T of a d-dimensional polytopal body P such that $V(\mathsf{T}) = V(\mathsf{B}(\mathsf{P}))$ is a quasi-triangulation.

In order to have an intuitive idea of what a quasi-triangulation is, one can see it as a family of (abstract) d-simplices which has the following property: every generic point g in $||\mathsf{P}||$ (resp. not in $||\mathsf{P}||$) is such that there is an odd (resp. even) number of d-simplices σ of this family whose convex hull $\mathrm{conv}(\sigma)$ contains g. So it is a binary cover in the sense of Firla and Ziegler ([3]).

We formalize this property with the following proposition (this property is not used for proving the other results of the present paper):

Proposition 2 *Let c be a quasi-triangulation of a d-dimensional polytopal body* P. *Then* $\cup_{\sigma \in c} \mathrm{conv}(\sigma)$ *is a binary cover of* $||\mathsf{P}||$.

Proof: We prove it by induction on d. For $d = 0$, it is straightforward.

Let c be a quasi-triangulation of P. Take g a generic point in $\mathbb{R}^d$ (this means that g and any d vertices of $\mathsf{B}(\mathsf{P})$ are affinely independent).

Consider any generic half-line l in $\mathbb{R}^d$ emanating from g. Generic means whenever we take $(d-1)$ vertices of $\mathsf{B}(\mathsf{P})$, l does not intersect the convex hull of those vertices.

Let σ be a simplex of c. If $\mathrm{conv}(\sigma)$ contains g, since g is generic, g is in the interior of $\mathrm{conv}(\sigma)$ and the vertices of σ are affinely independent. Hence l intersects $\partial\mathrm{conv}(\sigma) = \cup_{\tau \in \partial\sigma} \mathrm{conv}(\tau)$ once. If $\mathrm{conv}(\sigma)$ does not contain g, l intersects $\cup_{\tau \in \partial\sigma} \mathrm{conv}(\tau)$ 0 or 2 times.

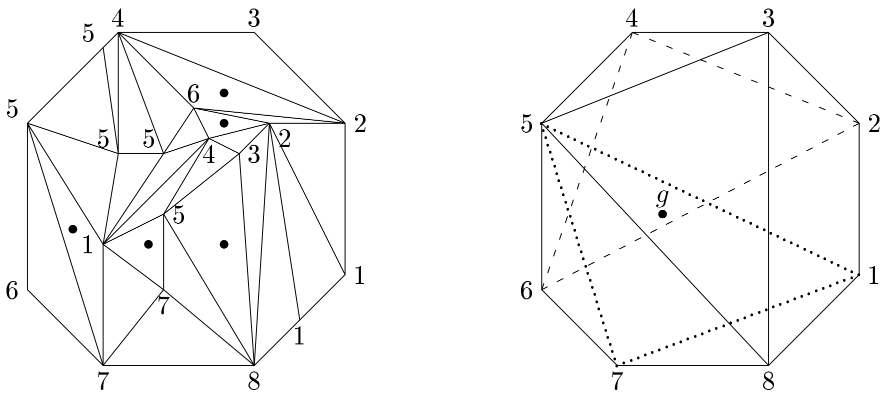

Figure 3: An illustration of Theorem 2

Thus, modulo 2, the number of simplices σ of c such that $\text{conv}(\sigma)$ contains g is equal to the number of simplices τ of $\sum_{\sigma\in c}\partial\sigma$ such that l intersects $\text{conv}(\tau)$.

But this last sum is precisely ∂c. As c is spread, $\partial c = \sum_{F\in\mathsf{B}(\mathsf{P})}(\partial c)|_F$. Let s be the number of intersections of l and $||\mathsf{B}(\mathsf{P})||$. If $s \neq 0$, let $p_1, p_2, \ldots, p_s$ be those intersections, and $F_1, F_2, \ldots, F_s$ be the facets where the intersections take place. As $(\partial c)|_{F_i}$ is a quasi-triangulation of $\mathsf{L}(F_i)$, by induction, there is an odd number of simplices τ of $(\partial c)|_{F_i}$ such that $\text{conv}(\tau)$ contains p_i. Hence, modulo 2, there are s simplices τ in ∂c such that l intersects $\text{conv}(\tau)$. This means that there are s modulo 2 simplices σ of c such that $\text{conv}(\sigma)$ contains g. As $||\mathsf{P}||$ is bounded, s is odd if and only if g is in the interior of $||\mathsf{P}||$. ∎

We use in this proof the fact the P is embedded in $\mathbb{R}^d$: when l leaves or enters $||\mathsf{P}||$, it intersects $||\mathsf{B}(\mathsf{P})||$. The same holds for any d-dimensional simplex of $\mathbb{R}^d$.

There are binary covers which are not quasi-triangulations: for instance, take the 2-dimensional polytopal body of Figure 2 (on left). We define $V_1 := \{v_1, v_4, v_5, v_6, v_7\}$ and $V_2 := \{v_1, v_2, v_3, v_4\}$. Let P_1 be the convex hull of V_1 and P_2 the convex hull of V_2. Let T_1 (resp. T_2) be a triangulation of P_1 (resp. P_2) such that $V(\mathsf{T}_i) = V_i$, $i = 1, 2$. The sum T of those two triangulations is a binary cover, but may not be a quasi-triangulation: for instance if the simplex $[v_6, v_7, v_4]$ is in T_1, the formal sum of all 2-simplices of T cannot be spread, because there is no facet containing the vertices v_7 and v_4 simultaneously.

To see the relevance of this notion, we state here the following theorem, announced in the Introduction. It will be proved at the end of this section:

Theorem 2 *If λ is a Sperner labelling of a triangulation T of a d-dimensional polytopal body P, then $\lambda_{\#}T$ is a quasi-triangulation of P, where T is the formal sum of all d-simplices of T.*

We illustrate this theorem with Figure 3. Almost every point of the right octagon is in an odd number of triangles, which are image of fully-labelled simplices of the left octagon. The point g is covered 5 times. The five corresponding fully-labelled simplices are marked with a thick dot.

We can also check Theorem 1 on this figure: we can find at least 6 fully-labelled simplices such that any pair of them receives two different labelling: $[1, 5, 7]$, $[2, 4, 6]$, $[3, 5, 8]$, $[2, 3, 4]$, $[4, 5, 6]$, $[5, 7, 8]$.

Quasi-triangulations have important properties, as stated in the following theorem:

Theorem 3 *Let $d \geq 0$. (A) If c_1 and c_2 are spread chains of a d-dimensional polytopal body P, and $c = c_1 + c_2$ then c is also a spread chain. Moreover c is a quasi-triangulation if and only if exactly one of c_1 and c_2 is a quasi-triangulation. (B) For Q a $(d+1)$-dimensional polytopal body, c' is a quasi-triangulation of Q if and only if c' is spread and there is a facet F of $\mathsf{B}(\mathsf{Q})$ such that $(\partial c')|_F$ is a quasi-triangulation of $\mathsf{L}(F)$.*

From (A), we can deduce that a spread chain that is not a quasi-triangulation is a kind of "even binary cover": every generic point is contained in an even number of d-simplices of that spread chain. (B) shows that in fact for c to be a quasi-triangulation, it is sufficient to check for an arbitrary facet F that $(\partial c)|_F$ is a quasi-triangulation, and thus that (ii) of the definition of a quasi-triangulation is too strong.

Proof: We proceed by induction on d.

For $d = 0$, the proof is easy. Let us suppose $d \geq 1$.

Proof of (A): The fact that c is spread is straightforward. It remains to show that c is a quasi-triangulation if and only if exactly one of c_1 and c_2 is a quasi-triangulation.

$(\partial c)|_F = (\partial c_1)|_F + (\partial c_2)|_F$ for every facet F of $\mathsf{B}(\mathsf{P})$. By Lemma 1 and by (A) of Theorem 3 for $d-1$ (which is already proved by induction), $(\partial c)|_F$ is a quasi-triangulation of $\mathsf{L}(F)$ if and only if exactly one of $(\partial c_1)|_F$ and $(\partial c_2)|_F$ is a quasi-triangulation of $\mathsf{L}(F)$. We need only that either $(\partial c_1)|_F$ is a quasi-triangulation of $\mathsf{L}(F)$ for every facet F, or $(\partial c_2)|_F$ is so. However, we have it by (B) of Theorem 3 for $d-1$. (A) is proved.

Proof of (B): We call facets F_1, F_2 of $\mathsf{B}(\mathsf{Q})$ *neighboring*, if $F_1 \cap F_2$ is a facet of both. Let c' be spread on the $(d+1)$-dimensional polytopal body Q and let F_1, F_2 be two neighboring facets of $\mathsf{B}(\mathsf{Q})$. If we show that $(\partial c')|_{F_1}$ is a quasi-triangulation of $\mathsf{L}(F_1)$ if and only if $(\partial c')|_{F_2}$ is a quasi-triangulation of $\mathsf{L}(F_2)$, using the strong connectivity of $\mathsf{B}(\mathsf{Q})$ (Point 3a. in Definition 1), we obtain the complete statement (B).

So, let us show that $(\partial c')|_{F_1}$ is a quasi-triangulation of $\mathsf{L}(F_1)$ if and only if $(\partial c')|_{F_2}$ is a quasi-triangulation of $\mathsf{L}(F_2)$. Let $F_{12} := F_1 \cap F_2$ (a facet of both).

Since c' is spread, we have, using Observation 7:

$$\partial c' = \sum_{J \in \mathsf{B}(\mathsf{Q})} (\partial c')|_J.$$

Applying ∂ again:

$$0 = \sum_{J \in \mathsf{B}(\mathsf{Q})} \partial[(\partial c')|_J].$$

In particular:

$$\sum_{J \in \mathsf{B}(\mathsf{Q})} \big(\partial[(\partial c')|_J]\big)\big|_{F_{12}} = 0. \tag{3}$$

A simplex σ of $\big(\partial[(\partial c')|_J]\big)\big|_{F_{12}}$ is such that $\sigma \subseteq V(J) \cap V(F_{12}) = V(J \cap F_{12})$ (Observation 2) and such that there is no proper face H of F_{12} such that $\sigma \subseteq V(H)$. Thus $\big(\partial[(\partial c')|_J]\big)\big|_{F_{12}} \neq 0$ implies that $J \cap F_{12}$ is not a proper face of F_{12}. Using Point 3b. in Definition 1, we get that $J \in \{F_1, F_2, F_{12}\}$.

Hence, the equality (3) reduces to:

$$(\partial e_1)|_{F_{12}} + (\partial e_2)|_{F_{12}} + (\partial f)|_{F_{12}} = 0, \tag{4}$$

where $e_1 := (\partial c')|_{F_1}$, $e_2 := (\partial c')|_{F_2}$ and $f := (\partial c')|_{F_{12}}$.

f is spread on $\mathsf{L}(F_1)$. Let $G \neq F_{12}$ be another facet of F_1 (which exists because $d \geq 1$). A simplex τ of $(\partial f)|_G$ is such that $\tau \subseteq V(G) \cap V(F_{12}) = V(G \cap F_{12})$ (Observation 2), that is there is a smaller face than G containing τ. Hence $(\partial f)|_G = 0$, and f is not a quasi-triangulation of $\mathsf{L}(F_1)$.

Using induction, we apply (B) for $d-1$: $(\partial f)|_{F_{12}}$ is not a quasi-triangulation of $\mathsf{L}(F_{12})$. Still using induction, we can apply (A) for $d-1$ on the equation (4) above and get that $(\partial e_1)|_{F_{12}}$ is a quasi-triangulation of $\mathsf{L}(F_{12})$ if and only if $(\partial e_2)|_{F_{12}}$ is so. Thus, by (B) for $d-1$, $e_1 = (\partial c')|_{F_1}$ is a quasi-triangulation of $\mathsf{L}(F_1)$ if and only if $e_2 = (\partial c')|_{F_2}$ is a quasi-triangulation of $\mathsf{L}(F_2)$. ∎

Proof of Theorem 2: For $d = 0$, this is trivial. We proceed by induction. Suppose that $d \geq 1$. We have to check point (i) and (ii) of the definition of a quasi-triangulation

Checking (i): if c is a strongly connected component of $\lambda_{\#}T$, then c is spread on P: indeed, let $\tau \in \partial c$. Suppose $\tau \notin \partial(\lambda_{\#}T)$. Then there exists another strongly connected component c' of $\lambda_{\#}T$ such that $\tau \in \partial c'$. But then c is not maximal. Hence $\tau \in \partial(\lambda_{\#}T)$. As $\lambda_{\#}\partial = \partial\lambda_{\#}$, there is $\epsilon \in \partial T$ such that $\lambda_{\#}\epsilon = \tau$. The labels of ϵ are vertices of a face F of $\mathsf{B}(\mathsf{P})$. And thus $\tau \subseteq V(F)$. As $\lambda_{\#}T$ is the sum of its strongly connected components, $\lambda_{\#}T$ is spread.

Checking (ii): We have to prove that $[\partial(\lambda_{\#}T)]|_F$ is a quasi-triangulation of $\mathsf{L}(F)$ for any facet F of $\mathsf{B}(\mathsf{P})$. Using $\partial\lambda_{\#} = \lambda_{\#}\partial$, we have $[\partial(\lambda_{\#}T)]|_F = [\lambda_{\#}(\partial T)]|_F$. So, we have to prove that $[\lambda_{\#}(\partial T)]|_F$ is a quasi-triangulation of $\mathsf{L}(F)$.

Let σ a $(d-1)$-simplex of ∂T such that $\lambda(\sigma) \subseteq V(F)$ and such that there is no smaller face whose vertex set contains $\lambda(\sigma)$ (in other words, $\lambda(\sigma)$ contributes to $[\lambda_{\#}(\partial T)]|_F$). By definition of a Sperner labelling, there is no face $H \neq F$, such that $\sigma \subseteq V(H)$. Hence $[\lambda_{\#}(\partial T)]|_F = (\lambda_{\#}[(\partial T)|_F])\,|_F$. With the notation $e := \lambda_{\#}[(\partial T)|_F]$, our objective becomes to prove that $e|_F$ is a quasi-triangulation of $\mathsf{L}(F)$.

$(\partial \mathsf{T})|_F$ is a triangulation of F, and λ is a Sperner labelling of its vertices. Observation 6 and induction imply that e is a quasi-triangulation of $\mathsf{L}(F)$.

For a proper face G of F, there is a facet G' of F such that $G \neq G'$ (such a facet exists because $d \geq 1$). $[\partial(e|_G)]|_{G'} = 0$ because a nonzero term would be a simplex whose vertices are in $V(G)$, and G' would not be the minimal face containing those vertices. Hence, $e|_G$ is not a quasi-triangulation of $\mathsf{L}(F)$ for any proper face G of F, and thus, using Theorem 3 (A) and the equality $e = e|_F + \sum_{G \in \mathsf{F}(F)} e|_G$ (Observation 5), we see that $e|_F$ is a quasi-triangulation. ∎

Theorem 4 *If λ is a Sperner labelling of a triangulation T of d-dimensional polytopal body P, then at least one (and actually an odd number) of the strongly connected components of $\lambda_{\#}T$ is a quasi-triangulation of P.*

Proof: According to Theorem 2 $c := \lambda_{\#}T$ is a quasi-triangulation. Let c_1 be a strongly connected component of c, and $c_2 := c - c_1$. Clearly, c_1 and c_2 are spread, since ∂c_1, $\partial c_2 \subseteq \partial c$.

We are now done by Theorem 3 by induction on the size of the support of c: indeed, either c_1 is quasi-triangulation, and then there is nothing to prove, or c_2 is a quasi-triangulation and then we are done by the induction hypothesis. ∎

Corollary 1 *If λ is a Sperner labelling of a triangulation T of P, then there exists a strongly connected component c of $\lambda_{\#}T$ which is a quasi-triangulation of P, and in particular every simplicial face of $\mathsf{B}(\mathsf{P})$ has a vertex set which is the face of some simplex of c.*

Proof: Indeed, the first part is just a restatement of Theorem 4, and the second part also follows by definition of a quasi-triangulation since the only quasi-triangulation of $\mathsf{L}(\sigma)$, where σ is a geometric simplex, is the abstract simplicial complex $V(\sigma)$ seen as a chain (the checking of this affirmation is straightforward). ∎

5 Generalized Sperner lemma

We turn now to the proof of our main result.

Proof of Theorem 1: Let T be the formal sum of the d-simplices of T, and let c as in Corollary 1. As vertices and edges of $\mathsf{B}(\mathsf{P})$ are simplices, we know that $V\big(\mathsf{B}(\mathsf{P})\big) = V(c)$ (equality comes from the fact that the labels are the vertices of $\mathsf{B}(\mathsf{P})$) and $E\big(\mathsf{B}(\mathsf{P})\big) \subseteq E(c)$. The theorem is then a direct consequence of Proposition 1:

$$|\text{supp } c| \geq |V(c)| + \left\lceil \frac{\min_{v \in V(c)} \deg_c(v)}{d} \right\rceil - d - 1 \geq |V\big(\mathsf{B}(\mathsf{P})\big)| + \left\lceil \frac{\min_{v \in V(\mathsf{B}(\mathsf{P}))} \deg_{\mathsf{B}(\mathsf{P})}(v)}{d} \right\rceil - d - 1.$$

Different simplices in c correspond to simplices in T getting different labellings. ∎

6 Conclusion

The main point we wanted to communicate is that, contrary to previous work, convexity plays no particular role in the existence of such lower bounds. The crucial points are the strong connectivity of the boundary complex and the fact that any $(d-2)$-dimensional face of the boundary complex is contained in two facets, where d is the dimension of the polytopal body. At the same time, we obtain an improved bound.

How to deal with triangulable compact sets whose boundary is not strongly connected is an open question. It seems that the bound of Theorem 1 has to be decreased as a function of the number of strongly connected components of the boundary.

Aknowledgement The author thanks the referees for their help and for all the precious remarks

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