THE CHROMATIC NUMBER OF ALMOST STABLE KNESER HYPERGRAPHS

FRÉDÉRIC MEUNIER

Abstract. Let $V(n,k,s)$ be the set of k-subsets S of $[n]$ such that for all $i,j\in S$, we have $|i-j|\geq s$ We define almost s-stable Kneser hypergraph $KG^r\binom{[n]}{k}^{\sim}_{s\text{-stab}}$ to be the r-uniform hypergraph whose vertex set is $V(n,k,s)$ and whose edges are the r-uples of disjoint elements of $V(n,k,s)$.

With the help of a Z_p-Tucker lemma, we prove that, for p prime and for any $n\geq kp$, the chromatic number of almost 2-stable Kneser hypergraphs $KG^p\binom{[n]}{k}^{\sim}_{2\text{-stab}}$ is equal to the chromatic number of the usual Kneser hypergraphs $KG^p\binom{[n]}{k}$, namely that it is equal to $\left\lceil\frac{n-(k-1)p}{p-1}\right\rceil$.

Related results are also proved, in particular, a short combinatorial proof of Schrijver's theorem (about the chromatic number of stable Kneser graphs) and some evidences are given for a new conjecture concerning the chromatic number of usual s-stable r-uniform Kneser hypergraphs.

1. Introduction and main results

1.1. **Introduction.** Let $[a]$ denote the set $\{1,\ldots,a\}$. The Kneser graph $KG^2\binom{[n]}{k}$ for integers $n\geq 2k$ is defined as follows: its vertex set is the set of k-subsets of $[n]$ and two vertices are connected by an edge if they have an empty intersection.

Kneser conjectured [Kne55] in 1955 that its chromatic number $\chi\left(KG^2\binom{[n]}{k}\right)$ is equal to $n-2k+2$. It was proved to be true by Lovász in 1978 in a famous paper [Lov78], which is the first and one of the most spectacular applications of algebraic topology in combinatorics.

Soon after this result, Schrijver [Sch78] proved that the chromatic number remains the same when we consider the subgraph $KG^2\binom{[n]}{k}_{2\text{-stab}}$ of $KG^2\binom{[n]}{k}$ obtained by restricting the vertex set to the k-subsets that are 2-*stable*, that is, that do not contain two consecutive elements of $[n]$ (where 1 and n are considered to be also consecutive).

Let us recall that a *hypergraph* $\mathcal{H}$ is a set family $\mathcal{H}\subseteq 2^V$, with *vertex set* V. An hypergraph is said to be r-*uniform* if all its *edges* $S\in\mathcal{H}$ have the same cardinality r. A *proper coloring with* t *colors* of $\mathcal{H}$ is a map $c:V\to[t]$ such that there is no monochromatic edge, that is, such that in each edge there are two vertices i and j with $c(i)\neq c(j)$. The smallest number t such that there exists such a proper coloring is called *the chromatic number* of $\mathcal{H}$ and denoted by $\chi(\mathcal{H})$.

In 1986, solving a conjecture of Erdős [Erd76], Alon, Frankl and Lovász [AFL86] found the chromatic number of *Kneser hypergraphs.* The Kneser hypergraph $KG^r\binom{[n]}{k}$ is an r-uniform hypergraph which has the k-subsets of $[n]$ as vertex set and whose edges are formed by the r-tuple of disjoint k-subsets of $[n]$. If n,k,r,t are positive integers such that $n\geq(t-1)(r-1)+rk$, then $\chi\left(KG^r\binom{[n]}{k}\right)>t$. Combined with a lemma by Erdős giving an explicit proper coloring, it implies that $\chi\left(KG^r\binom{[n]}{k}\right)=\left\lceil\frac{n-(k-1)r}{r-1}\right\rceil$. The proof found by Alon, Frankl and Lovász used tools from algebraic topology.

In 2001, Ziegler gave a combinatorial proof of this theorem [Zie02], which makes no use of topological tools. He was inspired by a combinatorial proof of the Lovász theorem found by Matoušek [Mat04]. A subset $S\subseteq[n]$ is s-*stable* if any two of its elements are at least "at distance s apart"

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on the n-cycle, that is, if $s \leq |i-j| \leq n-s$ for distinct $i, j \in S$. Define then $KG^r\binom{[n]}{k}_{s\text{-stab}}$ as the hypergraph obtained by restricting the vertex set of $KG^r\binom{[n]}{k}$ to the s-stable k-subsets. At the end of his paper, Ziegler made the supposition that the chromatic number of $KG^r\binom{[n]}{k}_{r\text{-stab}}$ is equal to the chromatic number of $KG^r\binom{[n]}{k}$ for any $n \geq kr$. This supposition generalizes both Schrijver's theorem and the Alon-Frankl-Lovász theorem. Alon, Drewnowski and Łuczak make this supposition an explicit conjecture in [ADŁ09].

Conjecture 1. *Let n, k, r be non-negative integers such that $n \geq rk$. Then*

$$\chi\left(KG^r\binom{[n]}{k}_{r\text{-stab}}\right) = \left\lceil \frac{n-(k-1)r}{r-1} \right\rceil.$$

1.2. **Main results.** We prove a weaker form of Conjecture 1 – Theorem 1 below – but which strengthes the Alon-Frankl-Lovász theorem. Let $V(n,k,s)$ be the set of k-subsets S of $[n]$ such that for all $i, j \in S$, we have $|i-j| \geq s$. We define the almost s-stable Kneser hypergraphs $KG^r\binom{[n]}{k}^{\sim}_{s\text{-stab}}$ to be the r-uniform hypergraph whose vertex set is $V(n,k,s)$ and whose edges are the r-tuples of disjoint elements of $V(n,k,s)$. Note that this kind of edges has already been considered and named *quasistable* in a paper by Björner and de Longueville [BdL03].

Theorem 1. *Let p be a prime number and n, k be non negative integers such that $n \geq pk$. We have*

$$\chi\left(KG^p\binom{[n]}{k}^{\sim}_{2\text{-stab}}\right) \geq \left\lceil \frac{n-(k-1)p}{p-1} \right\rceil.$$

Combined with the lemma by Erdős, we get that

$$\chi\left(KG^p\binom{[n]}{k}^{\sim}_{2\text{-stab}}\right) = \left\lceil \frac{n-(k-1)p}{p-1} \right\rceil.$$

Moreover, we will see that it is then possible to derive the following corollary. Denote by $\mu(r)$ the number of prime divisors of r counted with multiplicities. For instance, $\mu(6) = 2$ and $\mu(12) = 3$. We have

Corollary 1. *Let n, k, r be non-negative integers such that $n \geq rk$. We have*

$$KG^r\binom{[n]}{k}^{\sim}_{2^{\mu(r)}\text{-stab}} = \left\lceil \frac{n-(k-1)r}{r-1} \right\rceil.$$

For stable Kneser hypergraphs, what happens when $s \geq r$? This question does not seem to have attracted attention yet. As a first step, we prove the following proposition, which deals with Kneser graphs. It generalizes the fact that odd-length cycles have their chromatic number equaling 3.

Proposition 1. *Let k and s be two positive integers such that $s \geq 2$. We have*

$$\chi\left(KG^2\binom{[ks+1]}{k}_{s\text{-stab}}\right) = s+1.$$

1.3. **Plan.** The first section (Section 2) gives the main notations and tools used in the paper. Section 3 proves Theorem 1 and Corollary 1. Using a similar method, we are able to write a very short combinatorial proof of Schrijver's theorem in Section 4. Section 5 introduces preliminary results for the study of s-stable r-uniform Kneser hypergraphs when $s \geq r$ – in particular Proposition 1 – and proposes a conjecture (Conjecture 2) regarding their chromatic number. Section 6 is a collection of concluding remarks.

2. Notations and tools

$Z_p = \{\omega, \omega^2, \ldots, \omega^p\}$ is the cyclic group of order p, with generator ω.

We write σ^{n-1} for the $(n-1)$-dimensional simplex with vertex set $[n]$ and by σ^{n-1}_{k-1} the $(k-1)$-skeleton of this simplex, that is the set of faces of σ^{n-1} having k or less vertices.

If A and B are two sets, we write $A \uplus B$ for the set $(A \times \{1\}) \cup (B \times \{2\})$. For two simplicial complexes, K and L, with vertex sets $V(\mathsf{K})$ and $V(\mathsf{L})$, we denote by $\mathsf{K} * \mathsf{L}$ the *join* of these two complexes, which is the simplicial complex having $V(\mathsf{K}) \uplus V(\mathsf{L})$ as vertex set and

$$\{F \uplus G : F \in \mathsf{K}, G \in \mathsf{L}\}$$

as set of faces. We define also K^{*n} to be the join of n disjoint copies of K.

A sequence $(j_1, j_2, \ldots, j_m)$ of elements of Z_p is said to be *alternating* if any two consecutive terms are different. Let $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n$. We denote by $\operatorname{alt}(X)$ the size of the longest alternating subsequence of non-zero terms in X. For instance (assume $p = 5$) $\operatorname{alt}(\omega^2, \omega^3, 0, \omega^3, \omega^5, 0, 0, \omega^2) = 4$ and $\operatorname{alt}(\omega^1, \omega^4, \omega^4, \omega^4, 0, 0, \omega^4) = 2$.

Any element $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n$ can alternatively and without further mention be denoted by a p-tuple $(X_1, \ldots, X_p)$ where $X_j := \{i \in [n] : x_i = \omega^j\}$. Note that the X_j are then necessarily disjoint. For two elements $X, Y \in (Z_p \cup \{0\})^n$, we denote by $X \subseteq Y$ the fact that for all $j \in [p]$ we have $X_j \subseteq Y_j$. When $X \subseteq Y$, note that the sequence of non-zero terms in $(x_1, \ldots, x_n)$ is a subsequence of $(y_1, \ldots, y_n)$.

The proof of Theorem 1 makes use of a variant of the Z_p-Tucker lemma by Ziegler [Zie02].

Lemma 1 (Z_p-Tucker lemma)**.** *Let p be a prime, $n, m \geq 1$, $\alpha \leq m$ and let*

$$\begin{array}{rccc} \lambda : & (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} & \longrightarrow & Z_p \times [m] \\ & X & \longmapsto & (\lambda_1(X), \lambda_2(X)) \end{array}$$

be a Z_p-equivariant map satisfying the following properties:

- *for all $X^{(1)} \subseteq X^{(2)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$, if $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \leq \alpha$, then $\lambda_1(X^{(1)}) = \lambda_1(X^{(2)})$;*
- *for all $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$, if $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = \lambda_2(X^{(p)}) \geq \alpha + 1$, then the $\lambda_1(X^{(i)})$ are not pairwise distinct for $i = 1, \ldots, p$.*

Then $\alpha + (m - \alpha)(p - 1) \geq n$.

We can alternatively say that $X \mapsto \lambda(X) = (\lambda_1(X), \lambda_2(X))$ is a Z_p-equivariant simplicial map from $\operatorname{sd}\left(Z_p^{*n}\right)$ to $\left(Z_p^{*\alpha}\right) * \left((\sigma^{p-1}_{p-2})^{*(m-\alpha)}\right)$, where $\operatorname{sd}(\mathsf{K})$ denotes the first barycentric subdivision of a simplicial complex K.

Proof of the Z_p-Tucker lemma. According to Dold's theorem [Dol83, Mat03], if such a map λ exists, the dimension of $\left(Z_p^{*\alpha}\right) * \left((\sigma^{p-1}_{p-2})^{*(m-\alpha)}\right)$ is strictly larger than the connectivity of Z_p^{*n}, that is $\alpha + (m - \alpha)(p - 1) - 1 > n - 2$. □

It is also possible to give a purely combinatorial proof of this lemma through the generalized Ky Fan theorem from [HSSZ09].

3. Almost stable Kneser hypergraphs

Proof of Theorem 1. We follow the scheme used by Ziegler in [Zie02]. We endow $2^{[n]}$ with an arbitrary linear order $\preceq$.

Assume that $KG^p\binom{[n]}{k}^{\sim}_{2\text{-stab}}$ is properly colored with C colors $\{1, \ldots, C\}$. For $S \in V(n, k, 2)$, we denote by $c(S)$ its color. Let $\alpha = p(k-1)$ and $m = p(k-1) + C$.

Let $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. We can write alternatively $X = (X_1, \ldots, X_p)$.

- if $\mathrm{alt}(X) \leq p(k-1)$, let j be the index of the X_j containing the smallest integer (ω^j is then the first non-zero term in $(x_1, \ldots, x_n)$), and define
$$\lambda(X) := (j, \mathrm{alt}(X)).$$
- if $\mathrm{alt}(X) \geq p(k-1)+1$: in the longest alternating subsequence of non-zero terms of X, at least one of the elements of Z_p appears at least k times; hence, in at least one of the X_j there is an element S of $V(n,k,2)$; choose the smallest such S (according to $\preceq$). Let j be such that $S \subseteq X_j$ and define
$$\lambda(X) := (j, c(S) + p(k-1)).$$

λ is a Z_p-equivariant map from $(Z_p \cup \{0\})^n \setminus \{(0,\ldots,0)\}$ to $Z_p \times [m]$.

Let $X^{(1)} \subseteq X^{(2)} \in (Z_p \cup \{0\})^n \setminus \{(0,\ldots,0)\}$. If $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \leq \alpha$, then the longest alternating subsequences of non-zero terms of $X^{(1)}$ and $X^{(2)}$ have the same size. Clearly, the first non-zero terms of $X^{(1)}$ and $X^{(2)}$ are equal.

Let $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \in (Z_p \cup \{0\})^n \setminus \{(0,\ldots,0)\}$. If $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = \lambda_2(X^{(p)}) \geq \alpha+1$, then for each $i \in [p]$ there is $S_i \in V(n,k,2)$ and $j_i \in [p]$ such that we have $S_i \subseteq X^{(i)}_{j_i}$ and $\lambda_2(X^{(i)}) = c(S_i) + p(k-1)$. If all $\lambda_1(X^{(i)})$ would be distinct, then it would mean that all j_i would be distinct, which implies that the S_i would be disjoint but colored with the same color, which is impossible since c is a proper coloring.

We can thus apply the Z_p-Tucker lemma (Lemma 1) and conclude that $n \leq p(k-1) + C(p-1)$, that is
$$C \geq \left\lceil \frac{n-(k-1)p}{p-1} \right\rceil.$$
□

To prove Corollary 1, we prove the following lemma, both statement and proof of which are inspired by Lemma 3.3 of [ADŁ09].

Lemma 2. *Let r_1, r_2, s_1, s_2 be non-negative integers ≥ 1, and define $r = r_1 r_2$ and $s = s_1 s_2$.*

Assume that for $i = 1, 2$ we have $\chi\left(KG^{r_i}\binom{[n]}{k}^{\sim}_{s_i\text{-stab}}\right) = \left\lceil \frac{n-(k-1)r_i}{r_i-1} \right\rceil$ for all integers n and k such that $n \geq r_i k$.

Then we have $\chi\left(KG^{r}\binom{[n]}{k}^{\sim}_{s\text{-stab}}\right) = \left\lceil \frac{n-(k-1)r}{r-1} \right\rceil$ for all integers n and k such that $n \geq rk$.

Proof. Let $n \geq (t-1)(r-1) + rk$. We have to prove that $\chi\left(KG^{r}\binom{[n]}{k}^{\sim}_{s\text{-stab}}\right) > t$. For a contradiction, assume that $KG^r\binom{[n]}{k}_{s\text{-stab}}$ is properly colored with t colors. For $S \in V(n,k,s)$, we denote by $c(S)$ its color. We wish to prove that there are $S_1, \ldots, S_r$ disjoint elements of $V(n,k,s)$ with $c(S_1) = \ldots = c(S_r)$.

Take $A \in V(n, n_1, s_2)$, where $n_1 := r_1 k + (t-1)(r_1-1)$. Denote $a_1 < \ldots < a_{n_1}$ the elements of A and define $h : V(n_1, k, s_1) \to [t]$ as follows: let $B \in V(n_1, k, s_1)$; the k-subset $S = \{a_i : i \in B\} \subseteq [n]$ is an element of $V(n,k,s)$, and gets as such a color $c(S)$; define $h(B)$ to be this $c(S)$. Since $n_1 = r_1 k + (t-1)(r_1-1)$, there are $B_1, \ldots, B_{r_1}$ disjoint elements of $V(n_1, k, s_1)$ having the same color by h. Define $\tilde{h}(A)$ to be this common color.

Make the same definition for all $A \in V(n, n_1, s_2)$. The map $\tilde{h}$ is a coloring of $KG^{r_2}\binom{[n]}{n_1}^{\sim}_{s_2\text{-stab}}$ with t colors. Now, note that
$$(t-1)(r-1) + rk = (t-1)(r_1 r_2 - r_2 + r_2 - 1) + r_1 r_2 k = (t-1)(r_2-1) + r_2((t-1)(r_1-1) + r_1 k)$$
and thus that $n \geq (t-1)(r_2-1) + r_2 n_1$. Hence, there are $A_1, \ldots, A_{r_2}$ disjoint elements of $V(n, n_1, s_2)$ with the same color. Each of the A_i gets its color from r_1 disjoint elements of $V(n,k,s)$, whence there are $r_1 r_2$ disjoint elements of $V(n,k,s)$ having the same color by the map c. □

Proof of Corollary 1. Direct consequence of Theorem 1 and Lemma 2. □

4. Short combinatorial proof of Schrijver's theorem

Recall that Schrijver's theorem is

Theorem 2. *Let $n \geq 2k$. $\chi\left(KG\binom{[n]}{k}_{\text{2-stab}}\right) = n - 2k + 2$.*

When specialized for $p = 2$, Theorem 1 does not imply Schrijver's theorem since the vertex set is allowed to contain subsets with 1 and n together. However, by a slight modification of the proof, we can get a short combinatorial proof of Schrijver's theorem. Alternative proofs of this kind – but not that short – have been proposed in [Meu08, Zie02]

For a positive integer n, we write $\{+,-,0\}^n$ for the set of all *signed subsets* of $[n]$, that is, the family of all pairs (X^+, X^-) of disjoint subsets of $[n]$. Indeed, for $X \in \{+,-,0\}^n$, we can define $X^+ := \{i \in [n] : X_i = +\}$ and analogously X^-.

We define $X \subseteq Y$ if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$.

By $\text{alt}(X)$ we denote the length of the longest alternating subsequence of non-zero signs in X. For instance: $\text{alt}(+0--+0-) = 4$, while $\text{alt}(--++--+0+-) = 5$.

The proof makes use of the following well-known lemma see [Mat03, Tuc46, Zie02] (which is a special case of Lemma 1 for $p = 2$).

Lemma 3 (Tucker's lemma)**.** *Let $\lambda : \{-,0,+\}^n \setminus \{(0,0,\ldots,0)\} \to \{-1,+1,\ldots,-(n-1),+(n-1)\}$ be a map such that $\lambda(-X) = -\lambda(X)$. Then there exist A, B in $\{-,0,+\}^n$ such that $A \subseteq B$ and $\lambda(A) = -\lambda(B)$.*

Proof of Schrijver's theorem. The inequality $\chi\left(KG^2\binom{[n]}{k}_{\text{2-stab}}\right) \leq n - 2k + 2$ is easy to prove (with an explicit coloring [Kne55, Mat03] – see also Proposition 2 below). So, to obtain a combinatorial proof, it is sufficient to prove the reverse inequality.

Let us assume that there is a proper coloring c of $KG^2\binom{[n]}{k}_{\text{2-stab}}$ with $n-2k+1$ colors. We define the following map λ on $\{-,0,+\}^n \setminus \{(0,0,\ldots,0)\}$.

- if $\text{alt}(X) \leq 2k-1$, we define $\lambda(X) = \pm\,\text{alt}(X)$, where the sign is determined by the first sign of the longest alternating subsequence of X (which is actually the first non zero term of X).
- if $\text{alt}(X) \geq 2k$, then X^+ and X^- both contain a stable subset of $[n]$ of size k. Among all stable subsets of size k included in X^- and X^+, select the one having the smallest color. Call it S. Then define $\lambda(X) = \pm(c(S) + 2k - 1)$ where the sign indicates which of X^- or X^+ the subset S has been taken from. Note that $c(S) \leq n - 2k$.

The fact that for any $X \in \{-,0,+\}^n \setminus \{(0,0,\ldots,0)\}$ we have $\lambda(-X) = -\lambda(X)$ is obvious. λ takes its values in $\{-1,+1,\ldots,-(n-1),+(n-1)\}$. Now let us take A and B as in Tucker's lemma, with $A \subseteq B$ and $\lambda(A) = -\lambda(B)$. We cannot have $\text{alt}(A) \leq 2k-1$ since otherwise we will have a longest alternating subsequence in B containg the one of A, of same length but with a different sign. Hence $\text{alt}(A) \geq 2k$. Assume w.l.o.g. that $\lambda(A)$ is defined by a stable subset $S_A \subseteq A^-$. Then the stable subset S_B defining $\lambda(B)$ is such that $S_B \subseteq B^+$, which implies that $S_A \cap S_B = \emptyset$. We have moreover $c(S_A) = |\lambda(A)| = |\lambda(B)| = c(S_B)$, but this contradicts the fact that c is a proper coloring of $KG^2\binom{[n]}{k}_{\text{2-stab}}$. □

5. And when the stability is larger than the uniformity ?

It seems (among other things, through computational tests – see Conclusion – and Proposition 1) that Conjecture 1 can be generalized as follows.

Conjecture 2. *Let n,k,r,s be non-negative integers such that $n \geq sk$ and $s \geq r$. Then*

$$\chi\left(KG^r\binom{[n]}{k}_{s\text{-stab}}\right) = \left\lceil \frac{n-(k-1)s}{r-1} \right\rceil.$$

Conjecture 1 is the particular case when $s = r$. If c is a proper coloring of the Kneser hypergraph $KG^r\binom{[n]}{k}_{s\text{-stab}}$, then $X \mapsto \left\lceil \frac{1}{\rho}c(X) \right\rceil$ is a proper coloring of $KG^{\rho(r-1)+1}\binom{[n]}{k}_{s\text{-stab}}$, whence we have

(1) $$\chi\left(KG^{\rho(r-1)+1}\binom{[n]}{k}_{s\text{-stab}}\right) \leq \left\lceil \frac{1}{\rho}\chi\left(KG^r\binom{[n]}{k}_{s\text{-stab}}\right) \right\rceil.$$

We prove the easy part of the equality of Conjecture 2.

Proposition 2. *Let n,k,r,s be non-negative integers such that $n \geq sk$ and $s \geq r$. Then*

$$\chi\left(KG^r\binom{[n]}{k}_{s\text{-stab}}\right) \leq \left\lceil \frac{n-(k-1)s}{r-1} \right\rceil.$$

Proof. According to Inequality (1), it is enough to check the inequality for $r = 2$. We give the usual explicit coloring (see [Kne55, Erd76, Zie02]): for S an s-stable k-subset of $[n]$, we define its colors by

$$c(S) := \min(\min(S), n-(k-1)s).$$

This coloring uses at most $n-(k-1)s$ colors, and is proper: if A and B are two disjoint s-stable k-subsets of $[n]$ having the same color by c, then, necessarily, they both get the color $n-(k-1)s$ and they both have all elements $\geq n-(k-1)s$; but there is only one s-stable k-subset of $[n]$ having all its elements $\geq n-(k-1)s$, namely $\{n-(k-1)s, n-(k-2)s, \ldots, n-s, n\}$; a contradiction. □

Inequality (1) implies that if Conjecture 1 is true, then Conjecture 2 is also true for Kneser hypergraphs $KG^r\binom{[n]}{k}_{s\text{-stab}}$ when we have simultaneously $s \equiv 1 \bmod (r-1)$ and $n-(k-1)s \equiv \beta \bmod (s-1)$ for some $\beta \in [r-1]$. Indeed, put $s := (r-1)\rho+1$; if ρ divides $\chi := \chi\left(KG^r\binom{[n]}{k}_{s\text{-stab}}\right)$, there is nothing to prove; if not, write $\chi = \rho q + v$, where q and v are integers, and $v \in [\rho-1]$ and write $n-(k-1)s = (s-1)u+\beta$, with integer u; Inequality (1) implies that $q \geq u$; hence

$$\chi \geq \frac{(r-1)\rho u + (r-1)v}{r-1} \geq \frac{(s-1)u+\beta}{r-1} = \frac{n-(k-1)s}{s-1}$$

since $v \geq 1$ (used for the central inequality).

A lemma similar to Lemma 2 holds. It implies that it is enough to prove the cases

- $r = s$ and
- r and s coprime

to prove Conjecture 2.

Lemma 4. *If Conjecture 1 holds for r' (and all n and k such that $n \geq r'k$) and Conjecture 2 holds for r'' and s'' such that $s'' \geq r''$ (and all n and k such that $n \geq s''k$), then Conjecture 2 holds for $r = r'r''$ and $s = r's''$.*

Again, the proof follows a very similar scheme as the proof of Lemma 3.3 of [ADŁ09].

Proof of Lemma 4. Let $n \geq t(r-1)+s(k-1)+1$. We have to prove that $\chi\left(KG^r\binom{[n]}{k}_{s\text{-stab}}\right) > t$. For a contradiction, we assume that $KG^r\binom{[n]}{k}_{s\text{-stab}}$ is properly colored with t colors by $c : S \in V(n,k,s) \mapsto c(S) \in \{1,\ldots,t\}$. We will prove that there are $S_1,\ldots,S_r$ disjoint s-stable k-subsets of $[n]$ with $c(S_1) = \ldots = c(S_r)$.

Now, take A an r'-stable n'-subset of $[n]$, where $n' := t(r''-1)+s''(k-1)+1$. Denote $a_1 < \ldots < a_{n'}$ its elements and define $h(B)$ for any s''-stable k-subset B of $[n']$ as follows: the

k-subset $S = \{a_i : i \in B\} \subseteq [n]$ is an s-stable k-subset of $[n]$, and gets as such a color $c(S)$; define $h(B)$ to be this $c(S)$. Since $n' = t(r-1) + s''(k-1) + 1$, there are $B_1, \ldots, B_{r''}$ disjoint s''-stable k-subsets of $[n']$ having the same color by h. Define $\tilde{h}(A)$ to be this common color.

Make the same definition for all r'-stable n'-subsets A of $[n]$. The map $\tilde{h}$ is a coloring of $KG^{r'}\binom{[n]}{n'}_{r'\text{-stab}}$ with t colors. Now, note that

$$t(r-1)+s(k-1)+1 = t(r'r''-r'+r'-1)+r's''(k-1)+1 = r'(t(r''-1)+s''(k-1)+1)+(t-1)(r'-1)$$

and thus that $n \geq (t-1)(r'-1) + r'n'$. Hence, there are $A_1, \ldots, A_{r'}$ disjoint r'-stable n'-subsets with the same color (assuming that Conjecture 1 is true). Each of the A_i gets its color from r'' disjoint s''-stable k-subsets, whence there are $r'r''$ disjoint $s''r'$-stable k-subsets of $[n]$ having the same color by the map c. □

We prove now Proposition 1, which is the particular case when $n = ks+1$ and $r = 2$. The proof is quite natural and does not use any advanced tools from topology.

Proof of Proposition 1. Proposition 2 reduces the proof of the simple checking that s colors are not enough. Assume for a contradiction that $KG^2\binom{[ks+1]}{k}_{s\text{-stab}}$ is properly colored with colors $1, 2, \ldots, s$.

Without loss of generality, we can assume that the subset $A_{1,1} := \{1, s+1, 2s+1, \ldots, (k-1)s+1\}$ is colored with color 1, the subset $A_{1,2} := \{2, s+2, 2s+2, \ldots, (k-1)s+2\}$ with color 2, ..., $A_{1,s} := \{s, 2s, \ldots, ks\}$ with color s, that is, each of the s subsets of the form $\{i, s+i, 2s+i, \ldots, (k-1)s+i\}$ with $i = 1, 2, \ldots, s$, denoted $A_{1,i}$, is colored with color i.

The subset $B := \{s+1, 2s+1, \ldots, ks+1\}$ is disjoint from each of the $A_{1,i}$, except the first one $A_{1,1}$, whence it gets color 1.

Now, we consider the following s subsets: $A_{2,1} := \{1, s+1, 2s+1, \ldots, (k-2)s+1, (k-1)s+2\}$, $A_{2,2} := \{2, s+2, 2s+2, \ldots, (k-2)s+2, (k-1)s+3\}$, ..., $A_{2,s} := \{s, 2s, \ldots, (k-1)s, ks+1\}$. (They differ from the subsets $A_{1,i}$ only by their largest element). $A_{2,s}$ is disjoint from each element of $A_{1,i}$ except for $i = s$, whence it gets color s. The subsets $A_{2,i}$, for $i = 2, \ldots, s-1$, are disjoint from B and $A_{2,s}$, and pairwise disjoint, whence they are colored with colors $2, \ldots, s-1$. The subset A_{21} is disjoint from all $A_{2,i}$ for $i \geq 2$, whence it gets color 1.

Similary, we define $A_{j,i}$ for $j \in [k]$ and $i \in [s]$:

$$A_{j,i} := \{i, s+i, 2s+i, \ldots, (k-j)s+i, (k-j+1)s+i+1, (k-j+2)s+i+1, \ldots, (k-1)s+i+1\}.$$

The subset $A_{j,s}$ is disjoint from each $A_{(j-1),i}$ for $i = 1, \ldots, s-1$. The subsets $A_{j,i}$ for $i = 2, \ldots, s-1$ are disjoint from B. The subset $A_{j,1}$ is disjoint from all $A_{j,i}$ for $i \geq 2$. These three facts combined with an induction on j imply that the color of $A_{j,s}$ is s, the colors of the $A_{j,i}$ for $i = 2, \ldots, s-1$ are $2, \ldots, s-1$ and the color of $A_{j,1}$ is 1.

In particular for $j = k$ and $i = 1$, we get that the color of $A_{k,1}$ is 1. But $A_{k,1}$ and B are disjoint, whence they cannot have the same color; a contradiction. □

6. Concluding remarks

We have seen that one of the main ingredients is the notion of alternating sequence of elements in Z_p. Here, our notion only requires that such an alternating sequence must have $x_i \neq x_{i+1}$. To prove Conjecture 1, we probably need something stronger. For example, a sequence is said to be alternating if any p consecutive terms are all distinct. However, all our attempts to get something through this approach have failed.

Recall that Alon, Drewnowski and Łuczak [ADŁ09] proved Conjecture 1 when r is a power of 2. With the help of a computer and `lpsolve`, we have checked that Conjecture 1 is moreover true for

- $n \leq 9$, $k = 2$, $r = 3$.
- $n \leq 12$, $k = 3$, $r = 3$.
- $n \leq 14$, $k = 4$, $r = 3$.

- $n \leq 13$, $k = 2$, $r = 5$.
- $n \leq 16$, $k = 3$, $r = 5$.
- $n \leq 21$, $k = 4$, $r = 5$.

With the same approach, Conjecture 2 has been checked for

- $n \leq 9$, $k = 2$, $r = 2$, $s = 3$.
- $n \leq 10$, $k = 2$, $r = 2$, $s = 4$.
- $n \leq 11$, $k = 3$, $r = 2$, $s = 3$.
- $n \leq 13$, $k = 3$, $r = 2$, $s = 4$.
- $n \leq 14$, $k = 4$, $r = 2$, $s = 3$.
- $n \leq 17$, $k = 4$, $r = 2$, $s = 4$.
- $n \leq 11$, $k = 2$, $r = 3$, $s = 4$.
- $n \leq 14$, $k = 3$, $r = 3$, $s = 4$.
- $n \leq 12$, $k = 2$, $r = 3$, $s = 5$.
- $n \leq 13$, $k = 2$, $r = 4$, $s = 5$.

References

[ADŁ09] N. Alon, L. Drewnowski, and T Łuczak, *Stable Kneser hypergraphs and ideals in* $\mathbb{N}$ *with the Nikodým property*, Proceedings of the American Mathematical Society **137** (2009), 467–471.

[AFL86] N. Alon, P. Frankl, and L. Lovász, *The chromatic number of Kneser hypergraphs*, Transactions Amer. Math. Soc. **298** (1986), 359–370.

[BdL03] A. Björner and M. de Longueville, *Neighborhood complexes of stable Kneser graphs*, Combinatorica **23** (2003), 23–34.

[Dol83] A. Dold, *Simple proofs of some Borsuk-Ulam results*, Contemp. Math. **19** (1983), 65–69.

[Erd76] P. Erdős, *Problems and results in combinatorial analysis*, Colloquio Internazionale sulle Teorie Combinatorie (Rome 1973), Vol. II, No. 17 in Atti dei Convegni Lincei, 1976, pp. 3–17.

[HSSZ09] B. Hanke, R. Sanyal, C. Schultz, and G. Ziegler, *Combinatorial stokes formulas via minimal resolutions*, Journal of Combinatorial Theory, series A **116** (2009), 404–420.

[Kne55] M. Kneser, *Aufgabe 360*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 2. Abteilung, vol. 50, 1955, p. 27.

[Lov78] L. Lovász, *Kneser's conjecture, chromatic number and homotopy*, Journal of Combinatorial Theory, Series A **25** (1978), 319–324.

[Mat03] J. Matoušek, *Using the Borsuk-Ulam theorem*, Springer Verlag, Berlin–Heidelberg–New York, 2003.

[Mat04] ———, *A combinatorial proof of Kneser's conjecture*, Combinatorica **24** (2004), 163–170.

[Meu08] F. Meunier, *Combinatorial Stokes formulae*, European Journal of Combinatorics **29** (2008), 286–297.

[Sch78] A. Schrijver, *Vertex-critical subgraphs of Kneser graphs*, Nieuw Arch. Wiskd., III. Ser. **26** (1978), 454–461.

[Tuc46] A. W. Tucker, *Some topological properties of disk and sphere*, Proceedings of the First Canadian Mathematical Congress, Montreal 1945, 1946, pp. 285–309.

[Zie02] G. Ziegler, *Generalized Kneser coloring theorems with combinatorial proofs*, Invent. Math. **147** (2002), 671–691.

Université Paris Est, CERMICS, ENPC, 6-8 avenue Blaise Pascal, Cité Descartes Champs-sur-Marne, 77455 Marne-la-Vallée cedex 2, France.

E-mail address: frederic.meunier@enpc.fr