Uniqueness of equilibrium

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Frédéric Meunier and Thomas Pradeau, The uniqueness property for networks with several origin-destination pairs, *European Journal of Operational Research*, 2014.

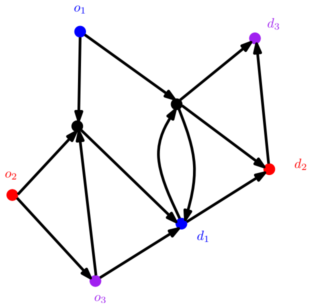
Model

$D = (V, A)$ a directed graph.

$\mathcal{L} \subseteq V^2$ a set of origin-destination pairs.

b^{od} = number of users going from o to d (the demand).

On each arc $a \in A$, there is a continuous cost $c_a(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$.



x_a = number of users choosing arc a.

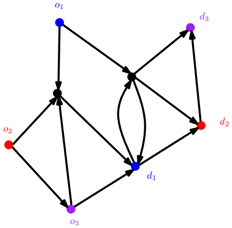
$\sum_{a \in P} c_a(x_a)$ = cost of a path P.

Equilibrium

x_a^{od} = number of users choosing arc a among those going from o to d.

$(x_a^{od})_{a \in A, (o,d) \in \mathcal{L}}$ is an equilibrium if

$$
\begin{array}{ll}
(x_a^{od})_{a \in A} = o\text{-}d \text{ flow of value } b^{od} & (o,d) \in \mathcal{L} \\
x_a = \displaystyle\sum_{(o,d) \in \mathcal{L}} x_a^{od} & a \in A \\
\displaystyle\sum_{a \in P} c_a(x_a) \leq \sum_{a \in Q} c_a(x_a) & P, Q \in \mathcal{P}^{od},\ P \text{ is used},\ (o,d) \in \mathcal{L}
\end{array}
$$

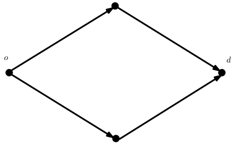


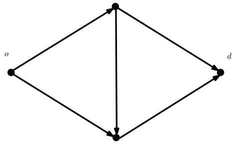
$\mathcal{P}^{od}$ = set of o-d paths

P used if $x_a^{od} > 0$ for all $a \in P$.

Practical interest

- This model = good approximation of what happens in practice
  - ⋆ used in transport engineering, telecoms,...
- Useful since the phenomenons are nonintuitive
  - ⋆ Braess paradox = opening a new road may increase all travel times
  - ⋆ paradox recovered by the model





Existence and uniqueness of the equilibrium

Theorem (Beckman, 1956)

An equilibrium always exists and it is unique when the cost functions $c_a(\cdot)$ are increasing.

"Unique" means there are unique x_a's solutions of the system

$$\begin{cases} (x_a^{od})_{a\in A} = o\text{-}d \text{ flow of value } b^{od} & (o,d)\in\mathcal{L} \\ x_a = \sum_{(o,d)\in\mathcal{L}} x_a^{od} & a\in A \\ \sum_{a\in P} c_a(x_a) \leq \sum_{a\in Q} c_a(x_a) & P,Q \in \mathcal{P}^{od},\ P \text{ is used},\ (o,d)\in\mathcal{L} \end{cases}$$

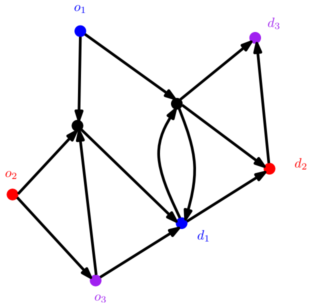
Model – multiclass case

$D = (V, A)$ a directed graph.

$\mathcal{L} \subseteq V^2$ a set of origin-destination pairs.

$b^{od,k}$ = number of class k users going from o to d (the demand).

On each arc $a \in A$ and for each class k, there is a continuous cost $c_a^k(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$.



x_a = number of users choosing arc a.

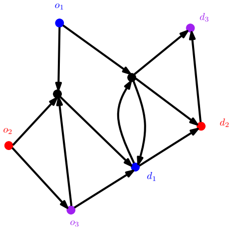
$\sum_{a \in P} c_a^k(x_a)$ = cost of path P experienced by class k.

Equilibrium

$x_a^{od,k}$ = number of class k users choosing arc a among those going from o to d.

$(x_a^{od,k})_{a\in A,(o,d)\in\mathcal{L},k\in K}$ is an equilibrium if

$(x_a^{od,k})_{a\in A} = o\text{-}d$ flow of value $b^{od,k}$	$(o,d)\in\mathcal{L}$, $k\in K$
$x_a = \displaystyle\sum_{(o,d)\in\mathcal{L},\, k\in K} x_a^{od,k}$	$a\in A$
$\displaystyle\sum_{a\in P} c_a^k(x_a) \leq \sum_{a\in Q} c_a^k(x_a)$	$P,Q\in\mathcal{P}^{od}$, P is used, $(o,d)\in\mathcal{L}$, $k\in K$



$\mathcal{P}^{od}$ = set of o-d paths

P used if $x_a^{od,k} > 0$ for all $a\in P$.

The uniqueness issue

Theorem (Schmeidler, 1973)

An equilibrium always exists in the multiclass setting.

There are examples with several equilibria, i.e. several possible x_a's, while all $c_a^k(\cdot)$ are increasing: uniqueness is not automatically ensured. (It contrasts with the monoclass case).

Challenge: Find necessary and/or sufficient conditions ensuring uniqueness.

Uniqueness property

G = undirected graph, $\mathcal{L}$ = collection of o-d pairs.

$(G, \mathcal{L})$ has the uniqueness property (UP) if for any classes, demands $(b^{od,k})$, and increasing costs $(c_a^k(\cdot))$, the equilibrium is unique.

(on the digraph where each edge has been replaced by two opposite arcs)

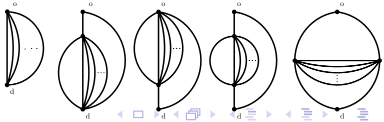
Theorem (Milchtaich, 2007)

There is only one o-d pair:

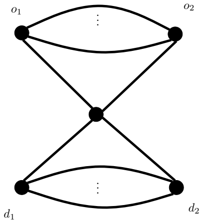
$$(G, \{(o,d)\}) \text{ has the UP} \iff G \text{ is “nearly-parallel”}.$$

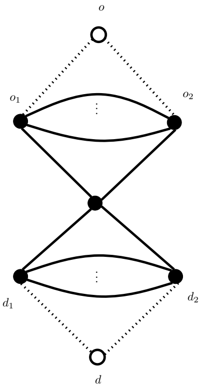
“Nearly-parallel” $=$

combination in series of



Uniqueness for general graphs using Milchtaich's theorem

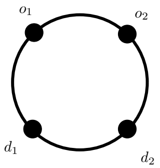


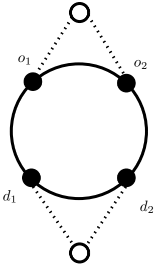


- Add a fictitious origin vertex connected to every origin.
- Add a fictitious destination vertex connected to every destination.

Augmented graph has uniqueness property $\Rightarrow$ Original graph has uniqueness property.

Uniqueness for general graphs using Milchtaich's theorem

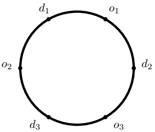


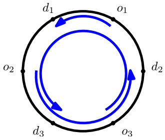


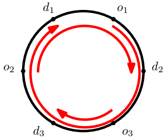
- Add a fictitious origin vertex connected to every origin.
- Add a fictitious destination vertex connected to every destination.

Augmented graph has not the uniqueness property $\Rightarrow$????

Uniqueness property on a cycle

$(G, \mathcal{L}) =$ 



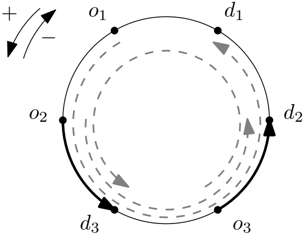


Theorem

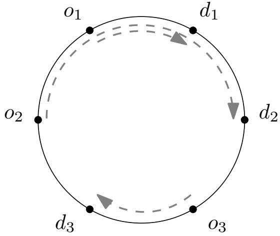
Assume that G is a cycle and let $\mathcal{L}$ be any collection of o-d pairs.

$(G, \mathcal{L})$ has the UP $\iff$ Each arc belongs to at most two o-d paths.

Example not having the uniqueness property



The positive paths.



The negative paths.

The arcs $o_2 d_3$ and $o_3 d_2$ are contained in three *o*-*d* paths.

Proof strategy

Step 1.

Each arc of D belongs to at most two o-d paths
$\Downarrow$
The equilibrium flows are unique whatever are the classes K, *increasing costs* $(c_a^k(\cdot))$, *and demands* $(b^{od,k})$

Step 2.

There is an arc of D belonging to at least three o-d paths
$\Downarrow$
There exist classes K, *increasing costs* $(c_a^k(\cdot))$, *and demands* $(b^{od,k})$ *leading to two equilibria with distinct flows*

Proof by an explicit construction of costs and demands.

Step 1:

Each arc of D belongs to at most two o-d paths

$\Downarrow$

The equilibrium flows are unique whatever are the classes, costs, and demands

- Let $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ be two equilibria. Define $\Delta_{od} = x^{od}_{P^+} - \hat{x}^{od}_{P^+}$.
- Suppose $\Delta_{o_0 d_0} \neq 0$ for some o_0-d_0. There exists an o_1-d_1 s.t. $\Delta_{o_0 d_0}\Delta_{o_1 d_1} < 0$ and $\Delta_{o_0 d_0} + \Delta_{o_1 d_1} < 0$.
- We repeat this argument and get an infinite sequence $|\Delta_{o_0 d_0}| < |\Delta_{o_1 d_1}| < \cdots < |\Delta_{o_j d_j}| < \cdots$.
- Contradiction with finiteness.

Step 2:

There is an arc of D belonging to at least three o-d paths

$\Downarrow$

There exist classes K, increasing costs $(c_a^k(\cdot))$, and demands $(b^{od,k})$ leading to two equilibria with distinct flows

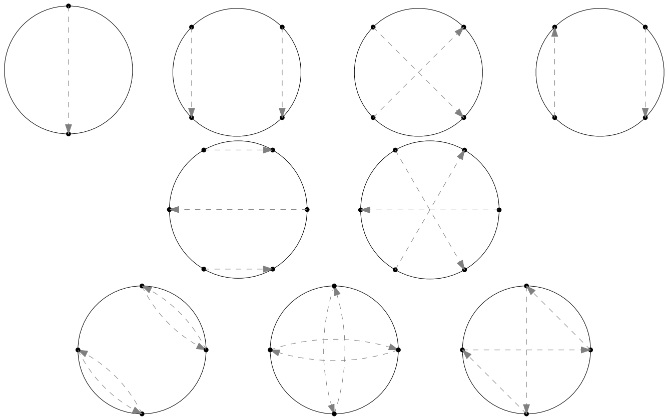
An arc in 3 *o*-*d* paths: explicitly building of cost functions and demands leading to two equilibria with distinct flows.

Some features:

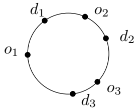
- Three classes.
- Affine cost functions.
- Explicit construction of two equilibria.
- These equilibria are strict and “single-path”.

Structural characterization

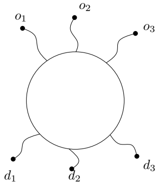
Each arc in at most two *o*-*d* paths $\Leftrightarrow$ $(G, \mathcal{L})$ homeomorphic to a minor of one of



Corollary for general graphs: examples

If 

is in $(G, \mathcal{L})$, G does not have the UP.

If 

is in $(G, \mathcal{L})$, G does not have the UP.

Having a minor without the uniqueness property

A subgraph of $(G, \mathcal{L})$ does not have the UP $\Longrightarrow$ $(G, \mathcal{L})$ does not have the UP.

A minor of $(G, \mathcal{L})$ does not have the UP:

- If the contractions involve only bridges, G does not have the UP.
- If the counterexamples are obtained via "single-path" flows, G does not have the UP.
- And in general, open question.

Strong uniqueness property

G has the strong uniqueness property (SUP) $=$ $(G, \mathcal{L})$ has the UP for any collection of *o-d* pairs $\mathcal{L}$

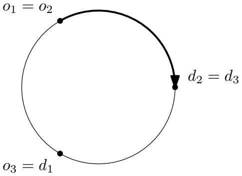
Theorem

G has the SUP $\iff$ *No cycles of length* 3 *or more.*

Graph having the SUP are thus graphs obtained from a forest by replicating some edges.

Proof

(⇒)
The graph



has one arc in three *o*-*d* paths: no UP.

(⇐) results from two easy statements:

- A graph with two vertices and parallel edges has the SUP.
- Glueing two graphs on a vertex maintains the SUP.

□

Open questions

Generalization
UP for general graphs?

Minor
What if a minor does not have UP?

Question
What if *one-way* edges are allowed?

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Generalization
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Thank you