A $\mathbb{Z}_q$-Fan theorem

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Abstract

In 1952, Ky Fan proved a combinatorial theorem generalizing the Borsuk-Ulam theorem stating that there is no $\mathbb{Z}_2$-equivariant map from the d-dimensional sphere S^d to the $(d-1)$-dimensional sphere S^{d-1}. The aim of the present paper is to provide the same kind of combinatorial theorem for Dold's theorem, which is a generalization of the Borsuk-Ulam theorem when $\mathbb{Z}_2$ is replaced by $\mathbb{Z}_q$, and the spheres replaced by d-dimensional $(d-1)$-connected free $\mathbb{Z}_q$-spaces. It provides a combinatorial proof of Dold's theorem. Moreover, the proof does not work by contradiction.

Key Words: combinatorial proof; Dold's theorem; Fan's theorem; labelling; Tucker's lemma; triangulation.

1 Introduction

Ky Fan gave ([3]) in 1952 a combinatorial generalization of the Borsuk-Ulam theorem:

Theorem 1 (Fan's theorem) *Let T be a symmetric triangulation of the d-sphere (if $\sigma \in \mathsf{T}$ then $-\sigma \in \mathsf{T}$) and let $\lambda : V(\mathsf{T}) \to \{-1,+1,-2,+2,\ldots,-m,+m\}$ be an antipodal labelling ($\lambda(-v) = -\lambda(v)$) of the vertices of T such that no edge is labelled by $-j$,$+j$ for some j (there is no antipodal edge). Then we have at least one simplex in T labelled with $-j_0, +j_1, \ldots, (-1)^{d+1}j_d$ where $j_0 < j_1 < \ldots < j_d$.*

In combinatorics, a continuous version of Fan's theorem is used in particular in the study of Kneser graphs (see [6],[10],[11]).

Since there is a generalization of Borsuk-Ulam theorem with other free actions ($\mathbb{Z}_q$-actions) than the central symmetry ($\mathbb{Z}_2$-action), namely Dold's theorem, a natural question is whether there is a generalization of Fan's theorem using q "signs" instead of the 2 signs $-,+$ and leading to a purely combinatorial proof of Dold's theorem.

The present paper gives such a "$\mathbb{Z}_q$-Fan theorem". An *equivariant triangulation* T of a free $\mathbb{Z}_q$-space is a triangulation such that if $\sigma \in \mathsf{T}$, then $\nu_s\sigma \in \mathsf{T}$ for all $s \in \mathbb{Z}_q$ (ν_s is the homeomorphism corresponding to the action of $s \in \mathbb{Z}_q$ on the $\mathbb{Z}_q$-space).

Theorem 2 ($\mathbb{Z}_q$-Fan's theorem) *Let q be an odd positive integer, let T be an equivariant triangulation of a d-dimensional $(d-1)$-connected free $\mathbb{Z}_q$-space and let $\lambda : V(\mathsf{T}) \to \mathbb{Z}_q \times \{1,2,\ldots,m\}$ be a equivariant labelling (if $\lambda(v) = (\epsilon, j)$, then $\lambda(\nu_s v) = (s+\epsilon, j)$ - counted modulo q - for all $s \in \mathbb{Z}_q$) of the vertices of T such that no edge is labelled by (ϵ, j),(ϵ', j), with $\epsilon \neq \epsilon'$, for some j. Then we have at least one simplex in T labelled with $(\epsilon_0, j_0), (\epsilon_1, j_1), \ldots, (\epsilon_d, j_d)$ where $\epsilon_i \neq \epsilon_{i+1}$ for all $i \in \{0,1,\ldots,d-1\}$, and $j_0 < j_1 < \ldots < j_d$.*

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It is not clear whether this theorem is also true for q even.

The plan is the following: First, we reprove Fan's theorem with the same kind of technics we use in the rest of the paper (Section 3). Then, following the same scheme, we prove the $\mathbb{Z}_q$-Fan theorem (Section 4). Finally, in the Section 5, we explain how this proof provides a new combinatorial proof of Dold's theorem, after the one found by Günter M. Ziegler in [12]: the $\mathbb{Z}_p$-Tucker lemma. "Combinatorial" means, according to Ziegler, no homology, no continuous map, no approximation. From this point of view our proof has a little advantage: it does not work by contradiction. This provides a new step in the direction of a constructive proof of Dold's theorem, whose existence is an important question (see the discussion of Mark de Longueville and Rade Zivaljevic in [1]). A constructive proof of Borsuk-Ulam theorem was found by Freund and Todd in 1981 ([4]). Another one, proving also Fan's theorem (Theorem 1), was proposed by Prescott and Su in 2005 ([9]).

2 Notations

We assume basic knowledge in algebraic topology. A good reference is the book of James Munkres [8].

2.1 General notations

$\mathbb{Z}_n$ is the set of integers modulo n.

Let S be a set, and suppose that $\mathbb{Z}_n$ acts on S. We denote by ν_s the action corresponding to $s \in \mathbb{Z}_n$. We denote $\nu := \nu_1$. We have then $\nu_s = \underbrace{\nu \times \nu \times \ldots \times \nu}_{s \text{ terms}} = \nu^s$ and in particular $\nu^0 = \text{id}$.

2.2 Simplices, chains and cochains

The definitions of simplices, simplicial complexes, chains and cochains are assumed to be known. We give here some specific or less well-known definitions and notations.

The join of two simplicial complexes K and L is denoted by $\mathsf{K} * \mathsf{L}$ and the join of K k times by itself is denoted by K^{*k}.

$(\mathbb{Z}_q)^{*m}$ is the $(m-1)$-dimensional simplicial complex whose vertex set is the disjoint union of m copies of $\mathbb{Z}_q$ and whose simplices are the subsets of this disjoint union containing at most one vertex of each copy. It is often denoted by $E_{m-1}\mathbb{Z}_q$ in the literature. A vertex of $(\mathbb{Z}_q)^{*m}$ is of the form (ϵ, j), with $\epsilon \in \mathbb{Z}_q$ and $j \in \{1, 2, \ldots, m\}$.

Let c_k be a k-chain and c^k be a k-cochain. We denote the value taken by c^k at c_k by $\langle c^k, c_k \rangle$. Moreover, we identify through $\langle ., . \rangle$ chains and cochains.

Let G be a group acting on a topological space X. The action of G on X is said to be free if every non-trivial element of G acts without fixed-point. In this case, we also say that G acts freely on X.

Let G be a group acting on two sets X and Y. A map (or a labelling) $f : X \to Y$ is said to be *G-equivariant* if $f \circ g = g \circ f$ for any $g \in G$.

2.3 The standard complex

2.3.1 Definition

The *standard complex* is defined in [5] for instance. Let S be a set. For $i = 0, 1, 2. \ldots$ let $E_i(S, G)$ be the free module over an abelian group G generated by $(i+1)$-tuples $(x_0, \ldots, x_i)$

with $x_0, \ldots, x_i \in S$. Thus such $(i+1)$-tuples form a basis of $E_i(S,G)$ over G. There is a unique homomorphism

$$\partial : E_{i+1}(S,G) \to E_i(S,G)$$

such that

$$\partial(x_0, \ldots, x_{i+1}) = \sum_{j=0}^{i+1} (-1)^j (x_0, \ldots, \hat{x}_j, \ldots, x_{i+1}),$$

where the symbol $\hat{x}_j$ means that this term is to be omitted.

An element of $E_i(S,G)$ is an $(i+1)$-chain and can be written $\sum_k \lambda_k \sigma_k$, where the σ_k are $(i+1)$-tuples of S, and the λ_k are taken in G.

We denote this complex $\mathcal{C}(S,G)$ and the corresponding coboundary map δ:

$$\delta(x_0, x_1, \ldots, x_i) = \sum_{a \in S} \Big((a, x_0, x_1, \ldots, x_i) + \sum_{k=0}^{i-1} (-1)^{k+1} (x_0, x_1, \ldots, x_k, a, x_{k+1}, \ldots, x_i) + (-1)^{i+1} (x_0, x_1, \ldots, x_i, a) \Big).$$

A standard complex used throughout the paper is $\mathcal{C}(\mathbb{Z}_q, \mathbb{Z}_q)$: $E_i(\mathbb{Z}_q, \mathbb{Z}_q)$ is the free module over $\mathbb{Z}_q$ generated by the elements of $\mathbb{Z}_q^{i+1}$. For instance, $\big((0,1,0) - (2,2,2) - (0,2,1)\big) \in \mathcal{C}(\mathbb{Z}_3, \mathbb{Z}_3)$ and $\partial\big((0,1,0) - (2,2,2) + (0,2,1)\big) = (0,1) - (0,0) + (1,0) - (2,2) + (2,2) - (2,2) + (0,2) - (0,1) + (2,1) = (2,1) + (0,2) + 2(2,2) + (1,0) + 2(0,0)$.

As we use in this paper the elements of $\mathcal{C}(\mathbb{Z}_q, \mathbb{Z}_q)$ as cochains, we illustrate the action of δ on one of these elements: for $\big((0,2) - (0,1)\big) \in \mathcal{C}(\mathbb{Z}_3, \mathbb{Z}_3)$, we have:
$\delta\big((0,2) - (0,1)\big) = \delta(0,2) - \delta(0,1)$
$= \big((0,0,2) + (1,0,2) + (2,0,2) - (0,0,2) - (0,1,2) - (0,2,2) + (0,2,0) + (0,2,1) + (0,2,2)\big) - \big((0,0,1) + (2,0,1) + (1,0,1) - (0,0,1) - (0,2,1) - (0,1,1) + (0,1,0) + (0,1,2) + (0,1,1)\big)$
$= \big((1,0,2) + (2,0,2) + 2(0,1,2) + (0,2,0) + (0,2,1)\big) + 2\big((2,0,1) + (1,0,1) + 2(0,2,1) + (0,1,0) + (0,1,2)\big)$
$= (1,0,2) + (2,0,2) + (0,1,2) + (0,2,0) + 2(0,2,1) + 2(2,0,1) + 2(1,0,1) + 2(0,1,0)$.

2.3.2 Actions on the standard complex

Moreover, if there is a group H acting on S, then H acts also on $\mathcal{C}(S,G)$: for ν_h an action corresponding to an element h of H, we extend it as follows: $\nu_{h\#}$ is the unique homomorphism $E_i(S,G) \to E_i(S,G)$ such that $\nu_{h\#}(x_0, \ldots, x_i) = (\nu_h x_0, \ldots, \nu_h x_i)$. We define $\nu_h^{\#}$ similarly for cochains.

2.3.3 Concatenation

We introduce the following notation: for a $(i+1)$-tuple $(x_0, x_1, \ldots, x_i) \in S^{i+1}$ and $c_j \in E_j(S,G)$ a j-chain, we denote $(x_0, x_1, \ldots, x_i, c_j)$ the $(i+j+1)$-chain $\sum_k \lambda_k (x_0, x_1, \ldots, x_i, \sigma_k)$ where the σ_k are the $(j+1)$-tuples such that $c_j = \sum_k \lambda_k \sigma_k$.

3 Proof of Fan's theorem

This section is devoted to a new proof of Ky Fan's theorem (Theorem 1). A simple combinatorial proof can also be found in [7]. In the one presented here, we try to extract the exact mechanism that explains this theorem. We distinguish four steps.

Let T be a symmetric triangulation of the d-sphere S^d and let $\lambda : V(\mathsf{T}) \to \{\pm 1, \pm 2, \ldots, \pm m\}$ be an antipodal labelling of the vertices of T such that no edge is labelled by $-j$,$+j$ for some j. λ commutes with ν, where ν is defined for any vertex v of T by $\nu(v) = -v$.

In the first step, using the definition of λ, we embed $\mathcal{C}(\mathsf{T}, \mathbb{Z}_2)$ in the standard complex $\mathcal{C}(\mathbb{Z}_2, \mathbb{Z}_2)$. In the second and third step, we build a sequence $(h_k)_{k\in\{0,1,\ldots,d\}}$ of k-chains of $\mathcal{C}(\mathsf{T}, \mathbb{Z}_2)$ and a sequence $(e_k)_{k\in\{0,1,\ldots,d\}}$ of k-cochains of $\mathcal{C}(\mathbb{Z}_2, \mathbb{Z}_2)$ which satisfy dual relations. Finally, using this duality and an induction, we achieve the proof.

3.1 $\psi_\# : \mathcal{C}(\mathsf{T}, \mathbb{Z}_2) \to \mathcal{C}(\mathbb{Z}_2, \mathbb{Z}_2)$

We see λ as a simplicial map going from T into the $(m-1)$-dimensional simplicial complex C, whose simplices are the subsets of $\{-1, +1, -2, +2, \ldots, -m, +m\}$ containing no pair $\{-i, +i\}$ for some $i \in \{1, 2, \ldots, m\}$ (such a complex is the *boundary complex of the cross-polytope*).

Let $\phi : x \in \mathbb{Z}\backslash\{0\} \mapsto \phi(x) \in \mathbb{Z}_2$ where $\phi(x) = 1$ if and only if $x > 0$. We define then the following chain map $\phi_\# : \mathcal{C}(\mathsf{C}, \mathbb{Z}_2) \to \mathcal{C}(\mathbb{Z}_2, \mathbb{Z}_2)$ for $\sigma = \{j_0, \ldots, j_k\} \in \mathsf{C}$ with $|j_0| < |j_1| < \ldots < |j_k|$ by $\phi_\#(\sigma) = (\phi(j_0), \phi(j_1), \ldots, \phi(j_k))$ (checking that it is a chain map is straightforward).

We define $\psi_\# := \phi_\# \circ \lambda_\#$. It is a chain map going from the chain complex $\mathcal{C}(\mathsf{T}, \mathbb{Z}_2)$ into the standard complex $\mathcal{C}(\mathbb{Z}_2, \mathbb{Z}_2)$. Note that $\psi_\#$ commutes with $\nu_\#$ (where $\nu : a \in \mathbb{Z}_2 \mapsto (a+1) \in \mathbb{Z}_2$).

3.2 the "hemispheres"

It is easy to see that there is a sequence $(h_k)_{k\in\{0,1,\ldots,d\}}$ of k-chains in $\mathcal{C}(\mathsf{T}, \mathbb{Z}_2)$ such that h_0 is a vertex and such that

$$\partial h_{k+1} = (\mathrm{id}_\# + \nu_\#)h_k, \tag{1}$$

for all $k \in \{0, 1, \ldots, d-1\}$. These k-chains can be seen as k-dimensional hemispheres of S^d. There is an easy construction of them. We can also see their existence through an homology argument: let h_0 be any vertex; then

$$\partial(\mathrm{id}_\# + \nu_\#)h_k = (\mathrm{id}_\# + \nu_\#)\partial h_0 = 0$$

and there exists an h_1 such that $\partial h_1 = (\mathrm{id}_\# + \nu_\#)h_0$ (the 0th homology group of the d-sphere is 0); finally, if h_k exists, then

$$\partial(\mathrm{id}_\# + \nu_\#)h_k = (\mathrm{id}_\# + \nu_\#)\partial h_k = (\mathrm{id}_\# + \nu_\#)(\mathrm{id}_\# + \nu_\#)h_{k-1} = (\mathrm{id}_\# + \nu_\#^2)h_{k-1} = 2\mathrm{id}_\# h_{k-1} = 0;$$

hence there exists an h_{k+1} such that $\partial h_{k+1} = (\mathrm{id}_\# + \nu_\#)h_k$ (the kth homology group of the d-sphere is 0 for $k \leq d-1$).

3.3 the "co-hemispheres"

On the other side, we have for the standard complex $\mathcal{C}(\mathbb{Z}_2, \mathbb{Z}_2)$:

$$\delta \underbrace{(0,1,0,1,\ldots)}_{k \text{ terms}} = \underbrace{(0,1,0,1,\ldots)}_{k+1 \text{ terms}} + \underbrace{(1,0,1,0,\ldots)}_{k+1 \text{ terms}},$$

which can be written

$$\delta e_k = (\mathrm{id}^\# + \nu^\#)e_{k+1}, \tag{2}$$

where $e_k = \underbrace{(0,1,0,1,\ldots)}_{k \text{ terms}}$ and where $\nu : (\epsilon_0, \epsilon_1, \ldots, \epsilon_k) \mapsto (\epsilon_0 + 1, \epsilon_1 + 1, \ldots, \epsilon_k + 1)$ (counted modulo 2). There is an obvious duality between equations (1) and (2). We call the e_k "co-hemispheres".

3.4 induction

We use now this symmetry to achieve the proof: we prove now the following property by induction on $k \leq d$:

$$\langle e_k, \psi_{\#}\big((\mathrm{id}_{\#} + \nu_{\#})h_k\big)\rangle = 1 \bmod 2.$$

It is true for $k = 0$: $e_0 = (0)$ and $\psi_{\#}\big((\mathrm{id}_{\#} + \nu_{\#})h_0\big) = (0) + (1)$.

If it is true for $k \geq 0$, we have

$$\langle e_{k+1}, \psi_{\#}\big((\mathrm{id}_{\#} + \nu_{\#})h_{k+1}\big)\rangle = \langle(\mathrm{id}^{\#} + \nu^{\#})e_{k+1}, \psi_{\#}h_{k+1}\rangle = \langle \delta e_k, \psi_{\#}h_{k+1}\rangle$$

$$= \langle e_k, \psi_{\#}\partial h_{k+1}\rangle = \langle e_k, \psi_{\#}\big((\mathrm{id}_{\#} + \nu_{\#})h_k\big)\rangle = 1 \bmod 2.$$

This proves the property. For $k = d$, it means that there is at least one d-simplex σ such that $\psi_{\#}(\sigma) = (0, 1, 0, 1, \ldots)$, which is exactly the statement of the theorem. ∎

4 Proof of $\mathbb{Z}_q$-Fan theorem

In this section, we prove Theorem 2. We follow similar four steps.

Let q be an odd positive integer, let T be an equivariant triangulation of a d-dimensional $(d-1)$-connected free $\mathbb{Z}_q$-space and let $\lambda : V(\mathsf{T}) \to \mathbb{Z}_q \times \{1, 2, \ldots, m\}$ be an equivariant labelling (if $\lambda(v) = (\epsilon, j)$, then $\lambda(\nu_s v) = (s + \epsilon, j)$ for all $s \in \mathbb{Z}_q$) of the vertices of T such that no edge is labelled by (ϵ, j),(ϵ', j), with $\epsilon \neq \epsilon'$, for some j.

In the first step, using the definition of λ, we embed $\mathcal{C}(\mathsf{T}, \mathbb{Z}_2)$ in the standard complex $\mathcal{C}(\mathbb{Z}_q, \mathbb{Z}_q)$. In the second and third steps, we build a sequence $(h_k)_{k\in\{0,1,\ldots,d\}}$ of k-chains in $\mathcal{C}(\mathsf{T}, \mathbb{Z}_q)$ and a sequence $(e_k)_{k\in\{0,1,\ldots,d\}}$ of k-cochains in $\mathcal{C}(\mathbb{Z}_q, \mathbb{Z}_q)$ which satisfy dual relations. Finally, using this duality and an induction, we achieve the proof.

4.1 $\psi_{\#} : \mathcal{C}(\mathsf{T}, \mathbb{Z}_q) \to \mathcal{C}(\mathbb{Z}_q, \mathbb{Z}_q)$

We see λ as a simplicial map going from T into the $(m-1)$-dimensional simplicial complex $(\mathbb{Z}_q)^{*m}$, whose simplices are the subsets of $\mathbb{Z}_q \times \{1, 2, \ldots, m\}$ containing no pair $\{(\epsilon, j), (\epsilon', j)\}$ for some $j \in \{1, 2, \ldots, m\}$ and some $\epsilon, \epsilon' \in \mathbb{Z}_q$ with $\epsilon \neq \epsilon'$.

We define then the following chain map $\phi_{\#} : \mathcal{C}((\mathbb{Z}_q)^{*m}, \mathbb{Z}_q) \to \mathcal{C}(\mathbb{Z}_q, \mathbb{Z}_q)$ for $\sigma = [(\epsilon_0, j_0), \ldots, (\epsilon_k, j_k)] \in (\mathbb{Z}_q)^{*m}$ with $j_0 < j_1 < \ldots < j_k$ by $\phi_{\#}(\sigma) = (\epsilon_0, \epsilon_1, \ldots, \epsilon_k)$ (checking that it is a chain map is straightforward).

We define $\psi_{\#} := \phi_{\#} \circ \lambda_{\#}$. It is a chain map going from the chain complex $\mathcal{C}(\mathsf{T}, \mathbb{Z}_q)$ into the standard complex $\mathcal{C}(\mathbb{Z}_q, \mathbb{Z}_q)$. Note that $\psi_{\#}$ commutes with the $\nu_{\#}$ (where $\nu : a \in \mathbb{Z}_q \mapsto (a+1) \in \mathbb{Z}_q$).

4.2 the "hemispheres"

It is not too hard to exhibit a sequence $(h_k)_{k\in\{0,1,\ldots,d\}}$ of k-chains in $\mathcal{C}(\mathsf{T}, \mathbb{Z}_q)$ such that h_0 is a vertex and such that, for l any integer ≥ 0:

$$\begin{aligned} \partial h_{2l+1} &= (\mathrm{id}_{\#} + \nu_{\#} + \ldots + \nu_{\#}^{q-1})h_{2l}, \\ \partial h_{2l+2} &= (\nu_{\#} - \nu_{\#}^{-1})h_{2l+1}. \end{aligned} \tag{3}$$

We can also see their existence through an homology argument: let h_0 be any vertex of T; then

$$\partial(\mathrm{id}_{\#} + \nu_{\#} + \ldots + \nu_{\#}^{q-1})h_0 = (\mathrm{id}_{\#} + \nu_{\#} + \ldots + \nu_{\#}^{q-1})\partial h_0 = 0$$

and there exists an h_1 such that $\partial h_1 = (\mathrm{id}_{\#} + \nu_{\#} + \ldots + \nu_{\#}^{q-1})h_0$ (the 0th homology group of T is 0: T is $(d-1)$-connected); finally, if h_{2l} exists, then

$$\partial(\mathrm{id}_{\#}+\nu_{\#}+\ldots+\nu_{\#}^{q-1})h_{2l} = (\mathrm{id}_{\#}+\nu_{\#}+\ldots+\nu_{\#}^{q-1})\partial h_{2l} = (\mathrm{id}_{\#}+\nu_{\#}+\ldots+\nu_{\#}^{q-1})(\nu_{\#}-\nu_{\#}^{-1})h_{2l-1} = 0;$$

hence there exists an h_{2l+1} such that $\partial h_{2l+1} = (\mathrm{id}_\# + \nu_\# + \ldots + \nu_\#^{q-1})h_{2l}$,
and if h_{2l+1} exists, then

$$\partial(\nu_\# - \nu_\#^{-1})h_{2l+1} = (\nu_\# - \nu_\#^{-1})\partial h_{2l+1} = (\nu_\# - \nu_\#^{-1})(\mathrm{id}_\# + \nu_\# + \ldots + \nu_\#^{q-1})h_{2l} = 0;$$

hence there exists an h_{2l+2} such that $\partial h_{2l+2} = (\nu_\# - \nu_\#^{-1})h_{2l+1}$ (the kth homology group of T is 0 for $k \leq d-1$: T is $(d-1)$-connected).

4.3 the "co-hemispheres"

Our aim is to find a sequence (e_k) of elements of the standard complex $\mathcal{C}(\mathbb{Z}_q, \mathbb{Z}_q)$ playing the same role than the e_k in the proof of Theorem 1 above.

For the proof, it is enough to know that such a sequence exists (the construction of this sequence is given in the Appendix - Lemma 2 - at the end of the paper), which satisfies $e_0 = (0)$ and, for l any integer ≥ 0:

$$\begin{array}{l} \delta e_{2l} = (\nu^\# - \nu^{\#-1})e_{2l+1}, \\ \delta e_{2l+1} = (\mathrm{id}^\# + \nu^\# + \ldots + \nu^{\# q-1})e_{2l+2}. \end{array} \tag{4}$$

Again, the h_k and the e_k satisfy dual relations. We call the latter "co-hemispheres".

4.4 induction

We use now this symmetry between equations (3) and (4) to achieve the proof: we prove now the following property by induction on $l \leq d$:

$$\langle e_{2l}, \psi_\#\big((\mathrm{id}_\# + \nu_\# + \ldots + \nu_\#^{q-1})h_{2l}\big)\rangle = (-1)^l \bmod q$$

and

$$\langle e_{2l+1}, \psi_\#\big((\nu_\# - \nu_\#^{-1})h_{2l+1}\big)\rangle = (-1)^{l+1} \bmod q.$$

It is true for $l = 0$: $\psi_\#\big((\mathrm{id}_\# + \nu_\# + \ldots + \nu_\#^{q-1})h_0\big) = (0) + (1) + \ldots + (q-1)$ and $\langle e_0, \psi_\#\big((\mathrm{id}_\# + \nu_\# + \ldots + \nu_\#^{q-1})h_{2l}\big)\rangle = \langle (0), (0) + (1) + \ldots + (q-1)\rangle = 1$.

If it is true for $l \geq 0$, we have:

$$\langle e_{2l+1}, \psi_\#\big((\nu_\# - \nu_\#^{-1})h_{2l+1}\big)\rangle = \langle(\nu^{\#-1} - \nu^\#)e_{2l+1}, \psi_\# h_{2l+1}\rangle = -\langle \delta e_{2l}, \psi_\# h_{2l+1}\rangle$$

$$= -\langle e_{2l}, \psi_\# \partial h_{2l+1}\rangle = -\langle e_{2l}, \psi_\#\big((\mathrm{id}_\# + \nu_\# + \ldots + \nu_\#^{q-1})h_{2l}\big)\rangle = (-1)^{l+1} \bmod q,$$

and

$$\langle e_{2l+2}, \psi_\#\big((\mathrm{id}_\# + \nu_\# + \ldots + \nu_\#^{q-1})h_{2l+2}\big)\rangle = \langle(\mathrm{id}^\# + \nu^\# + \ldots + \nu^{\# q-1})e_{2l+2}, \psi_\# h_{2l+2}\rangle = \langle \delta e_{2l+1}, \psi_\# h_{2l+2}\rangle$$

$$= \langle e_{2l+1}, \psi_\# \partial h_{2l+2}\rangle = \langle e_{2l+1}, \psi_\#\big((\nu_\# - \nu_\#^{-1})h_{2l+1}\big)\rangle = (-1)^{l+1} \bmod q.$$

This proves the property. For $k = d$, it means that there is at least one d-simplex σ such that $\psi_\#(\sigma) = (\epsilon_0, \epsilon_1, \ldots, \epsilon_d)$ with $\epsilon_i \neq \epsilon_{i+1}$ for $i = 0, 1, \ldots, d-1$ (in the e_k, all $k+1$-tuples satisfy this property - see Lemma 1 in the Appendix), which is exactly the statement of the theorem. ∎

5 Combinatorial proof of Dold's theorem

We recall Dold's theorem (proved by Dold in 1983 [2]):

Theorem 3 (Dold's theorem) *Let X and Y be two simplicial complexes, which are free $\mathbb{Z}_n$-space. If $f : \mathsf{X} \to \mathsf{Y}$ is a $\mathbb{Z}_n$-equivariant map between free $\mathbb{Z}_n$-spaces, then the dimension of Y is larger than or equal to the connectivity of X.*

It is not too hard to give an explicit construction (without using homology arguments) of a sequence $(h_k)_{k\in\{0,1,\ldots,d\}}$ of k-chains in $\mathcal{C}(\mathsf{T}, \mathbb{Z}_q)$, where T is any equivariant triangulation of $(\mathbb{Z}_p)^{*(d+1)}$, such that h_0 is a vertex and such that, for l any integer ≥ 0:

$$\begin{aligned} \partial h_{2l+1} &= (\mathrm{id}_\# + \nu_\# + \ldots + \nu_\#^{q-1})h_{2l}, \\ \partial h_{2l+2} &= (\nu_\# - \nu_\#^{-1})h_{2l+1}. \end{aligned} \tag{5}$$

The proof of Theorem 2 is combinatorial (no homology, no continuous map, no approximation) and does not work by contradiction.

By standard technics, to prove Theorem 3, it is sufficient to consider the case when $n = p$ is prime, X is an equivariant triangulation of $(\mathbb{Z}_p)^{*(d+1)}$ and $\mathsf{Y} := (\mathbb{Z}_p)^{*d}$, and to prove that there is no equivariant simplicial map $\mathsf{X} \to \mathsf{Y}$.

Thus Theorem 1 (for $p = 2$) and Theorem 2 (for $p = q$ odd) together provide a purely combinatorial proof of Theorem 3 without working by contradiction, because they imply that if λ is a equivariant simplicial map $\mathsf{X} \to (\mathbb{Z}_p)^{*m}$ then $m > d$.

6 Appendix: definition of the e_k for $\mathbb{Z}_q$

6.1 Definitions of C and (e_k)

For simplicity, we write $q = 2r + 1$. We were not able to find a similar construction for q even (except of course for $q = 2$).

We define recursively the infinite sequence $(e_k)_{k\in\mathbb{N}}$ of element of $\mathcal{C}(\mathbb{Z}_q, \mathbb{Z})$, where $e_k \in E_k(\mathbb{Z}_q, \mathbb{Z})$ (we define e_k with coefficients in $\mathbb{Z}$, but the relations they will satisfy will be true for coefficients in $\mathbb{Z}_q$ too).

We first begin with e_0 and e_1:

$$e_0 := (0).$$

$$e_1 := \sum_{j=0}^{r-1} \sum_{i=0}^{j} \big((2i+1, 2r-2j+2i) - (2r-2j+2i, 2i+1)\big).$$

We define then the following application $C : E_k(\mathbb{Z}_q, \mathbb{Z}) \to E_{k+2}(\mathbb{Z}_q, \mathbb{Z})$ by its value on the natural basis:

$$C : (a_0, \ldots, a_k) \mapsto (a_0, \ldots, a_k, \nu_\#^{a_k} e_1).$$

For $k \geq 2$, we can now define the rest of the infinite sequence:

$$e_k := C(e_{k-2}).$$

This construction implies immediately the following property:

Lemma 1 *Let $k \geq 0$, and $\sigma = (\epsilon_0, \epsilon_1, \ldots, \epsilon_k) \in \mathbb{Z}_q^{k+1}$. If $\langle e_k, \sigma\rangle \neq 0$, which means that σ has a non-zero coefficient is the formal sum e_k, then $\epsilon_i \neq \epsilon_{i+1}$ for any $i \in \{0, 1, \ldots, k-1\}$.*

6.2 Examples for $q = 3$ and $q = 5$

Let us see for instance what it gives for $q = 3$ and $q = 5$.

For $q = 3$: $e_0 = (0)$,
$e_1 = (1,2) - (2,1)$,
$e_2 = (0,1,2) - (0,2,1)$,
$e_3 = (1,2,0,1) - (1,2,1,0) - (2,1,2,0) + (2,1,0,2)$,
$e_4 = (0,1,2,0,1) - (0,1,2,1,0) - (0,2,1,2,0) + (0,2,1,0,2)$, and so on.

For $q = 5$: $e_0 = (0)$,
$e_1 = (1,2) + (3,4) + (1,4) - (2,1) - (4,3) - (4,1)$,
$e_2 = (0,1,2) + (0,3,4) + (0,1,4) - (0,2,1) - (0,4,3) - (0,4,1)$,
$e_3 = (1,2,3,4) + (1,2,0,1) + (1,2,3,1) - (1,2,4,3) - (1,2,1,0) - (1,2,1,3) + (3,4,0,1) + (3,4,2,3) + (3,4,0,3) - (3,4,1,0) - (3,4,3,2) - (3,4,3,0) + (1,4,0,1) + (1,4,2,3) + (1,4,0,3) - (1,4,1,0) - (1,4,3,2) - (1,4,3,0) - (2,1,2,3) - (2,1,4,0) - (2,1,2,0) + (2,1,3,2) + (2,1,0,4) + (2,1,0,2) - (4,3,4,0) - (4,3,1,2) - (4,3,4,2) + (4,3,0,4) + (4,3,2,1) + (4,3,2,4) - (4,1,2,3) - (4,1,4,0) - (4,1,2,0) + (4,1,3,2) + (4,1,0,4) + (4,1,0,2)$,
$e_4 = (0,1,2,3,4) + (0,1,2,0,1) + (0,1,2,3,1) - (0,1,2,4,3) - (0,1,2,1,0) - (0,1,2,1,3) + (0,3,4,0,1) + (0,3,4,2,3) + (0,3,4,0,3) - (0,3,4,1,0) - (0,3,4,3,2) - (0,3,4,3,0) + (0,1,4,0,1) + (0,1,4,2,3) + (0,1,4,0,3) - (0,1,4,1,0) - (0,1,4,3,2) - (0,1,4,3,0) - (0,2,1,2,3) - (0,2,1,4,0) - (0,2,1,2,0) + (0,2,1,3,2) + (0,2,1,0,4) + (0,2,1,0,2) - (0,4,3,4,0) - (0,4,3,1,2) - (0,4,3,4,2) + (0,4,3,0,4) + (0,4,3,2,1) + (0,4,3,2,4) - (0,4,1,2,3) - (0,4,1,4,0) - (0,4,1,2,0) + (0,4,1,3,2) + (0,4,1,0,4) + (0,4,1,0,2)$, and so on.

6.3 Induction property of (e_k)

We prove now the equations (4):

Lemma 2 *For $l \geq 0$, we have:*

$$\delta e_{2l} = (\nu^{\#} - \nu^{\#-1}) e_{2l+1},$$
$$\delta e_{2l+1} = (\mathrm{id}^{\#} + \nu^{\#} + \ldots + \nu^{\# q-1}) e_{2l+2}.$$

Proof: We prove first a serie of claims and finally, prove the equations by induction.

CLAIM 1:

$$\delta\big((2) + (4) + \ldots + (2r)\big) = (\mathrm{id}^{\#} - \nu^{\#}) e_1. \tag{6}$$

PROOF OF CLAIM 1: According to the definition of e_1, if a σ is such that $\langle e_1, \sigma\rangle \neq 0$, then σ is of the form (y,x) or (x,y) with x even, y odd and $0 \leq y < x \leq 2r$. Similarly, if σ is such that $\langle \nu^{\#} e_1, \sigma\rangle \neq 0$, then σ is either of the form (y,x) or (x,y) with x even ≥ 2, y odd and $0 \leq x < y \leq 2r$, or of the form $(0,x)$ or $(x,0)$ with x even or $0 < x \leq 2r$.

Hence, if σ is such that $\langle (\mathrm{id}^{\#} - \nu^{\#}) e_1, \sigma\rangle \neq 0$, then σ is of the form (x,y) or (y,x) with $x \in X := \{2,4,\ldots,2r\}$ and $y \in Y := \{0\} \cup \{1,3,\ldots,2r-1\}$. For $x \in X$ and $y \in Y$, the coefficient of (x,y) in $(\mathrm{id}^{\#} - \nu^{\#}) e_1$ is -1 and the coefficient of (y,x) is $+1$. The equality $\delta\big((2) + (4) + \ldots + (2r)\big) = (\mathrm{id}^{\#} - \nu^{\#}) e_1$ follows.

CLAIM 2:

$$\delta e_1 = \sum_{j \in \mathbb{Z}_q} \nu^{\# j} e_2. \tag{7}$$

PROOF OF CLAIM 2: Applying δ on both sides of equation (6), we get: $\delta e_1 = \nu^{\#}(\delta e_1)$. It implies that δe_1 can be written $\sum_{j \in \mathbb{Z}_q} \nu^{\# j}(0,h)$, where $h \in E_1(\mathbb{Z}_q, \mathbb{Z})$. As the couples (x,y)

in e_1 never begin with a 0, we get $(0, e_1)$ while keeping from δe_1 only the couples beginning with a 0. Hence $h = e_1$, and we have indeed $\delta e_1 = \sum_{j\in\mathbb{Z}_q} \nu^{\#j} e_2$, since $e_2 = (0, e_1)$.

CLAIM 3: $\nu^{\#} \circ C = C \circ \nu^{\#}$.

PROOF OF CLAIM 3: straightforward.

CLAIM 4:

$$\delta \circ C = C \circ \delta. \tag{8}$$

PROOF OF CLAIM 4: Let $\sigma = (a_0, \ldots, a_k)$ be a $(k+1)$-tuple. We have

$$\begin{array}{rcl} (\delta \circ C)(\sigma) & = & \delta\big(\sigma, (\nu^{\#a_k} e_1)\big) \\ & = & \big((\delta\sigma), (\nu^{\#a_k} e_1)\big) + (-1)^{k+1}\big(\sigma, \delta(\nu^{\#a_k} e_1)\big) - (-1)^{k+1} \sum_{j\in\mathbb{Z}_q} \big(\sigma, j, (\nu^{\#a_k} e_1)\big), \end{array}$$

et

$$\begin{array}{rcl} (C \circ \delta)(\sigma) & = & C(\delta\sigma) \\ & = & \big((\delta\sigma), (\nu^{\#a_k} e_1)\big) + (-1)^{k+1} \sum_{j\in\mathbb{Z}_q} \big(\sigma, j, (\nu^{\#j} e_1)\big) - (-1)^{k+1} \sum_{j\in\mathbb{Z}_q} \big(\sigma, j, (\nu^{\#a_k} e_1)\big). \end{array}$$

Hence, $(\delta\circ C)(\sigma) - (C\circ\delta)(\sigma) = (-1)^{k+1}\big(\sigma, \delta(\nu^{\#a_k} e_1)\big) - (-1)^{k+1} \sum_{j\in\mathbb{Z}_q} (\sigma, j, \nu^{\#j} e_1)$. But, according to equation (7), $\delta(\nu^{\#a_k} e_1) - \sum_{j\in\mathbb{Z}_q} (j, \nu^{\#j} e_1) = \nu^{\#a_k}(\delta e_1) - \sum_{j\in\mathbb{Z}_q} \nu^{\#j}(0, e_1) = 0$ (we have $e_2 = (0, e_1)$). Thus $(\delta \circ C)(\sigma) - (C \circ \delta)(\sigma) = 0$.

Proof of Lemma 2: By induction on l.

For $l = 0$, we have $\delta e_0 = (\nu^{\#} - \nu^{\#-1})e_1$: indeed, let $c := (2) + (4) + \ldots + (2r)$; according to equation (6), we have $\delta c = (\mathrm{id}^{\#} - \nu^{\#})e_1$; we have also, $\delta\big((0) + (1) + \ldots + (2r-1) + (2r)\big) = 0$ (the checking is straightforward); hence, $\delta(0) + \delta c + \delta\nu^{\#-1} c = 0$; and thus $\delta(0) = (\nu^{\#} - \nu^{\#-1})e_1$. Claim 2 is the relation: $\delta e_1 = (\mathrm{id}^{\#} + \nu^{\#} + \cdots + \nu^{\#q-1})e_2$. Lemma 2 is proved for $l = 0$.

Let's assume that Lemma 2 is proved for $l \geq 0$. According to Claim 3 and Claim 4, we have then:

$$\delta e_{2l+2} = (\delta \circ C)(e_{2l}) = (C \circ \delta)(e_{2l}) = C\big((\nu^{\#} - \nu^{\#-1})e_{2l+1}\big) = (\nu^{\#} - \nu^{\#-1})e_{2l+3}$$

and

$$\delta e_{2l+3} = (\delta\circ C)(e_{2l+1}) = (C\circ\delta)(e_{2l+1}) = C\big(\sum_{j\in\mathbb{Z}_q} \nu^{\#j} e_{2l+2}\big) = \sum_{j\in\mathbb{Z}_q} \nu^{\#j} e_{2l+4} = (\mathrm{id}^{\#} + \nu^{\#} + \ldots + \nu^{\#q-1})e_{2l+4}.$$

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