A $\mathbb{Z}_q$-Fan theorem

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Abstract

In 1952, Ky Fan proved a combinatorial theorem generalizing the Borsuk-Ulam theorem stating that there is no $\mathbb{Z}_2$-equivariant map from the $d$-dimensional sphere $S^d$ to the $(d-1)$-dimensional sphere $S^{d-1}$. The aim of the present paper is to provide the same kind of combinatorial theorem for Dold’s theorem, which is a generalization of the Borsuk-Ulam theorem when $\mathbb{Z}_2$ is replaced by $\mathbb{Z}_q$, and the spheres replaced by $d$-dimensional $(d-1)$-connected free $\mathbb{Z}_q$-spaces. It provides a combinatorial proof of Dold’s theorem. Moreover, the proof does not work by contradiction.

Key Words: combinatorial proof; Dold’s theorem; Fan’s theorem; labelling; Tucker’s lemma; triangulation.

1 Introduction

Ky Fan gave ([3]) in 1952 a combinatorial generalization of the Borsuk-Ulam theorem:

**Theorem 1 (Fan’s theorem)** Let $T$ be a symmetric triangulation of the $d$-sphere (if $\sigma \in T$ then $-\sigma \in T$) and let $\lambda : V(T) \to \{-1, +1, -2, +2, \ldots, -m, +m\}$ be an antipodal labelling ($\lambda(-v) = -\lambda(v)$) of the vertices of $T$ such that no edge is labelled by $-j, +j$ for some $j$ (there is no antipodal edge). Then we have at least one simplex in $T$ labelled with $-j_0, +j_1, \ldots, (-1)^{d+1}j_d$ where $j_0 < j_1 < \ldots < j_d$.

In combinatorics, a continuous version of Fan’s theorem is used in particular in the study of Kneser graphs (see [6],[10],[11]).

Since there is a generalization of Borsuk-Ulam theorem with other free actions ($\mathbb{Z}_q$-actions) than the central symmetry ($\mathbb{Z}_2$-action), namely Dold’s theorem, a natural question is whether there is a generalization of Fan’s theorem using $q$ “signs” instead of the 2 signs $-,$, $+$ and leading to a purely combinatorial proof of Dold’s theorem.

The present paper gives such a “$\mathbb{Z}_q$-Fan theorem”. An equivariant triangulation $T$ of a free $\mathbb{Z}_q$-space is a triangulation such that if $\sigma \in T$, then $\nu, \sigma \in T$ for all $s \in \mathbb{Z}_q$ ($\nu$ is the homeomorphism corresponding to the action of $s \in \mathbb{Z}_q$ on the $\mathbb{Z}_q$-space).

**Theorem 2 ($\mathbb{Z}_q$-Fan’s theorem)** Let $q$ be an odd positive integer, let $T$ be an equivariant triangulation of a $d$-dimensional $(d-1)$-connected free $\mathbb{Z}_q$-space and let $\lambda : V(T) \to \mathbb{Z}_q \times \{1, 2, \ldots, m\}$ be an equivariant labelling (if $\lambda(v) = (\epsilon, j)$, then $\lambda(\nu, v) = (s + \epsilon, j)$ - counted modulo $q$ - for all $s \in \mathbb{Z}_q$) of the vertices of $T$ such that no edge is labelled by $(\epsilon, j), (\epsilon', j)$, with $\epsilon \neq \epsilon'$, for some $j$. Then we have at least one simplex in $T$ labelled with $(\epsilon_0, j_0), (\epsilon_1, j_1), \ldots, (\epsilon_d, j_d)$ where $\epsilon_i \neq \epsilon_{i+1}$ for all $i \in \{0, 1, \ldots, d-1\}$, and $j_0 < j_1 < \ldots < j_d$. 

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It is not clear whether this theorem is also true for \( q \) even.

The plan is the following: First, we reprove Fan’s theorem with the same kind of techniques we use in the rest of the paper (Section 3). Then, following the same scheme, we prove the \( \mathbb{Z}_q \)-Fan theorem (Section 4). Finally, in the Section 5, we explain how this proof provides a new combinatorial proof of Dold’s theorem, after the one found by Günter M. Ziegler in [12]: the \( \mathbb{Z}_q \)-Tucker lemma. “Combinatorial” means, according to Ziegler, no homology, no continuous map, no approximation. From this point of view our proof has a little advantage: it does not work by contradiction. This provides a new step in the direction of a constructive proof of Dold’s theorem, whose existence is an important question (see the discussion of Mark de Longueville and Rade Zivaljevic in [1]). A constructive proof of Borsuk-Ulam theorem was found by Freund and Todd in 1981 ([4]). Another one, proving also Fan’s theorem (Theorem 1), was proposed by Prescott and Su in 2005 ([9]).

2 Notations

We assume basic knowledge in algebraic topology. A good reference is the book of James Munkres [8].

2.1 General notations

\( \mathbb{Z}_n \) is the set of integers modulo \( n \).

Let \( S \) be a set, and suppose that \( \mathbb{Z}_n \) acts on \( S \). We denote by \( \nu_s \) the action corresponding to \( s \in \mathbb{Z}_n \). We denote \( \nu := \nu_1 \). We have then \( \nu_s = \nu \times \nu \times \ldots \times \nu = \nu^s \) and in particular \( \nu^0 = \text{id} \).

2.2 Simplices, chains and cochains

The definitions of simplices, simplicial complexes, chains and cochains are assumed to be known. We give here some specific or less well-known definitions and notations.

The join of two simplicial complexes \( K \) and \( L \) is denoted by \( K \ast L \) and the join of \( K \) \( k \) times by itself is denoted by \( K^k \).

\( (\mathbb{Z}_q)^m \) is the \((m-1)\)-dimensional simplicial complex whose vertex set is the disjoint union of \( m \) copies of \( \mathbb{Z}_q \) and whose simplices are the subsets of this disjoint union containing at most one vertex of each copy. It is often denoted by \( E_{m-1} \mathbb{Z}_q \) in the literature. A vertex of \( (\mathbb{Z}_q)^m \) is of the form \((\epsilon,j)\), with \( \epsilon \in \mathbb{Z}_q \) and \( j \in \{1,2,\ldots,m\} \).

Let \( c_k \) be a \( k \)-chain and \( c^k \) be a \( k \)-cochain. We denote the value taken by \( c^k \) at \( c_k \) by \( \langle c^k,c_k \rangle \). Moreover, we identify through \( (\langle \ldots \rangle) \) chains and cochains.

Let \( G \) be a group acting on a topological space \( X \). The action of \( G \) on \( X \) is said to be free if every non-trivial element of \( G \) acts without fixed-point. In this case, we also say that \( G \) acts freely on \( X \).

Let \( G \) be a group acting on two sets \( X \) and \( Y \). A map (or a labelling) \( f : X \to Y \) is said to be \( G \)-equivariant if \( f \circ g = g \circ f \) for any \( g \in G \).

2.3 The standard complex

2.3.1 Definition

The standard complex is defined in [5] for instance. Let \( S \) be a set. For \( i = 0,1,2,\ldots \) let \( E_i(S,G) \) be the free module over an abelian group \( G \) generated by \((i+1)\)-tuples \((x_0,\ldots,x_i)\)
with $x_0, \ldots, x_i \in S$. Thus such $(i + 1)$-tuples form a basis of $E_i(S,G)$ over $G$. There is a unique homomorphism

$$\partial : E_{i+1}(S,G) \rightarrow E_i(S,G)$$

such that

$$\partial(x_0, \ldots, x_{i+1}) = \sum_{j=0}^{i+1} (-1)^j (x_0, \ldots, \hat{x}_j, \ldots, x_{i+1}),$$

where the symbol $\hat{x}_j$ means that this term is to be omitted.

An element of $E_i(S,G)$ is an $(i + 1)$-chain and can be written $\sum_k \lambda_k \sigma_k$, where the $\sigma_k$ are $(i + 1)$-tuples of $S$, and the $\lambda_k$ are taken in $G$.

We denote this complex $C(S,G)$ and the corresponding coboundary map $\delta$:

$$\delta(x_0, x_1, \ldots, x_i) = \sum_{a \in S} ((a,x_0, x_1, \ldots, x_i) + \sum_{k=0}^{i-1} (-1)^{k+1}(x_0, x_1, \ldots, x_k, a, x_{k+1}, \ldots, x_i)
+ (-1)^{i+1}(x_0, x_1, \ldots, x_i, a)).$$

A standard complex used throughout the paper is $C(\mathbb{Z}_q, \mathbb{Z}_q)$: $E_i(\mathbb{Z}_q, \mathbb{Z}_q)$ is the free module over $\mathbb{Z}_q$ generated by the elements of $\mathbb{Z}_q^{i+1}$. For instance, $((0,1,0) - (2,2,2) - (0,2,1)) \in C(\mathbb{Z}_3, \mathbb{Z}_3)$ and $\partial((0,1,0) - (2,2,2) + (0,2,1)) = (0,1) - (0,0) + (1,0) - (2,2) + (2,2) - (2,2) + (0,2) - (0,1) + (2,1) + (0,2) + 2(2,2) + (1,0) + 2(0,0) - (0,1,0)$.

As we use in this paper the elements of $C(\mathbb{Z}_q, \mathbb{Z}_q)$ as cochains, we illustrate the action of $\delta$ on one of these elements: for $((0,2) - (0,1)) \in C(\mathbb{Z}_3, \mathbb{Z}_3)$, we have:

$$\delta((0,2) - (0,1)) = \delta(0,2) - \delta(0,1)
= ((0,0,2) + (1,0,2) + (2,0,2) - (0,0,2) - (0,1,2) - (0,0,2) + (0,2,0) + (0,2,1) + (0,2,2)) - ((0,0,1) + (2,0,1) + (1,0,1) - (0,0,1) - (0,2,1) - (0,1,1) + (0,1,0) + (0,1,2) + (0,1,1))
= ((1,0,2) + (2,0,2) + (0,1,2) + (0,2,0) + (0,2,1)) + 2((2,0,1) + (1,0,1) + (0,2,2) + (0,1,0) + (0,1,2))
= (1,0,2) + (2,0,2) + (0,1,2) + (0,2,0) + 2(0,2,1) + 2(2,0,1) + 2(1,0,1) + 2(0,1,0)$.

### 2.3.2 Actions on the standard complex

Moreover, if there is a group $H$ acting on $S$, then $H$ acts also on $C(S,G)$: for $\nu_h$ an action corresponding to an element $h$ of $H$, we extend it as follows: $\nu_{h \#}$ is the unique homomorphism $E_i(S,G) \rightarrow E_i(S,G)$ such that $\nu_{h \#}(x_0, \ldots, x_i) = (\nu_h x_0, \ldots, \nu_h x_i)$. We define $\nu_h$ similarly for cochains.

### 2.3.3 Concatenation

We introduce the following notation: for a $(i + 1)$-tuple $(x_0, x_1, \ldots, x_i) \in S^{i+1}$ and $c_j \in E_j(S,G)$ a $j$-chain, we denote $(x_0, x_1, \ldots, x_i, c_j)$ the $(i+j+1)$-chain $\sum k \lambda_k (x_0, x_1, \ldots, x_i, \sigma_k)$ where the $\sigma_k$ are the $(j+1)$-tuples such that $c_j = \sum_k \lambda_k \sigma_k$.

### 3 Proof of Fan’s theorem

This section is devoted to a new proof of Ky Fan’s theorem (Theorem 1). A simple combinatorial proof can also be found in [7]. In the one presented here, we try to extract the exact mechanism that explains this theorem. We distinguish four steps.

Let $T$ be a symmetric triangulation of the $d$-sphere $S^d$ and let $\lambda : V(T) \rightarrow \{\pm 1, \pm 2, \ldots, \pm m\}$ be an antipodal labelling of the vertices of $T$ such that no edge is labelled by $-j, +j$ for some $j$. $\lambda$ commutes with $\nu$, where $\nu$ is defined for any vertex $v$ of $T$ by $\nu(v) = -v$.  

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In the first step, using the definition of \( \lambda \), we embed \( C(T, \mathbb{Z}_2) \) in the standard complex \( C(\mathbb{Z}_2, \mathbb{Z}_2) \). In the second and third step, we build a sequence \((h_k)_{k \in \{0, 1, \ldots, d\}}\) of \( k \)-chains of \( C(T, \mathbb{Z}_2) \) and a sequence \((e_k)_{k \in \{0, 1, \ldots, d\}}\) of \( k \)-cochains of \( C(\mathbb{Z}_2, \mathbb{Z}_2) \) which satisfy dual relations. Finally, using this duality and an induction, we achieve the proof.

3.1 \( \psi_\# : C(T, \mathbb{Z}_2) \to C(\mathbb{Z}_2, \mathbb{Z}_2) \)

We see \( \lambda \) as a simplicial map going from \( T \) into the \((m - 1)\)-dimensional simplicial complex \( C \), whose simplices are the subsets of \( \{-1, +1, -2, +2, \ldots, -m, +m\} \) containing no pair \( \{-i, +i\} \) for some \( i \in \{1, 2, \ldots, m\} \) (such a complex is the boundary complex of the cross-polytope).

Let \( \phi : x \in \mathbb{Z} \setminus \{0\} \mapsto \phi(x) \in \mathbb{Z}_2 \) where \( \phi(x) = 1 \) if and only if \( x > 0 \). We define then the following chain map \( \phi_\# : C(C, \mathbb{Z}_2) \to C(Z_2, \mathbb{Z}_2) \) for \( \sigma = \{j_0, \ldots, j_k\} \in C \) with \( |j_0| < |j_1| < \ldots < |j_k| \) by \( \phi_\#(\sigma) = (\phi(j_0), \phi(j_1), \ldots, \phi(j_k)) \) (checking that it is a chain map is straightforward).

We define \( \psi_\# := \phi_\# \circ \lambda_\# \). It is a chain map going from the chain complex \( C(T, \mathbb{Z}_2) \) into the standard complex \( C(\mathbb{Z}_2, \mathbb{Z}_2) \). Note that \( \psi_\# \) commutes with \( \nu_\# \) (where \( \nu : a \in \mathbb{Z}_2 \mapsto (a + 1) \in \mathbb{Z}_2 \)).

3.2 the “hemispheres”

It is easy to see that there is a sequence \((h_k)_{k \in \{0, 1, \ldots, d\}}\) of \( k \)-chains in \( C(T, \mathbb{Z}_2) \) such that \( h_0 \) is a vertex and such that

\[
\partial h_{k+1} = (\text{id}_\# + \nu_\#)h_k,
\]

for all \( k \in \{0, 1, \ldots, d-1\} \). These \( k \)-chains can be seen as \( k \)-dimensional hemispheres of \( S^d \). There is an easy construction of them. We can also see their existence through an homology argument: let \( h_0 \) be any vertex; then

\[
\partial (\text{id}_\# + \nu_\#)h_k = (\text{id}_\# + \nu_\#)\partial h_0 = 0
\]

and there exists an \( h_1 \) such that \( \partial h_1 = (\text{id}_\# + \nu_\#)h_0 \) (the 0th homology group of the \( d \)-sphere is 0); finally, if \( h_k \) exists, then

\[
\partial (\text{id}_\# + \nu_\#)h_k = (\text{id}_\# + \nu_\#)\partial h_k = (\text{id}_\# + \nu_\#)(\text{id}_\# + \nu_\#)h_{k-1} = (\text{id}_\# + \nu_\#)^2h_{k-1} = 2\text{id}_\#h_{k-1} = 0;
\]

hence there exists an \( h_{k+1} \) such that \( \partial h_{k+1} = (\text{id}_\# + \nu_\#)h_k \) (the \( k \)th homology group of the \( d \)-sphere is 0 for \( k \leq d - 1 \)).

3.3 the “co-hemispheres”

On the other side, we have for the standard complex \( C(\mathbb{Z}_2, \mathbb{Z}_2) \):

\[
\delta \left(0, 1, 0, 1, \ldots\right) = \left(0, 1, 0, 1, \ldots\right) + \left(1, 0, 1, 0, \ldots\right),
\]

which can be written

\[
\delta e_k = (\text{id}_\# + \nu_\#)e_{k+1}, \tag{2}
\]

where \( e_k = (0, 1, 0, 1, \ldots) \) and where \( \nu : (e_0, e_1, \ldots, e_k) \mapsto (e_0 + 1, e_1 + 1, \ldots, e_k + 1) \) (counted modulo 2). There is an obvious duality between equations (1) and (2). We call the \( e_k \) “co-hemispheres”.

4
3.4 induction

We use now this symmetry to achieve the proof: we prove now the following property by induction on \(k \leq d\):

\[
(e_k, \psi_#(\text{id}_# + \nu_#)h_k) \equiv 1 \mod 2.
\]

It is true for \(k = 0\): \(e_0 = (0)\) and \(\psi_#(\text{id}_# + \nu_#)h_0 = (0) + (1)\).

If it is true for \(k \geq 0\), we have

\[
\langle e_{k+1}, \psi_#(\text{id}_# + \nu_#)h_{k+1} \rangle = \langle (\text{id}_# + \nu_#)e_{k+1}, \psi_#h_{k+1} \rangle = \langle \delta e_k, \psi_#h_{k+1} \rangle = \langle e_k, \psi_#h_{k+1} \rangle \mod 2.
\]

This proves the property. For \(k = d\), it means that there is at least one \(d\)-simplex \(s\) such that \(\psi_#(s) = (0,1,0,1,...)\), which is exactly the statement of the theorem.

4 Proof of \(\mathbb{Z}_q\)-Fan theorem

In this section, we prove Theorem 2. We follow similar four steps.

Let \(q\) be an odd positive integer, let \(T\) be an equivariant triangulation of a \(d\)-dimensional \((d-1)\)-connected free \(\mathbb{Z}_q\)-space and let \(\lambda:V(T) \to \mathbb{Z}_q \times \{1,2,...,m\}\) be an equivariant labelling (if \(\lambda(v) = (e,j)\), then \(\lambda(v) = (e+s,j)\) for all \(s \in \mathbb{Z}_q\) of the vertices of \(T\) such that no edge is labelled by \((e,j),(e',j)\) with \(e \neq e'\), for some \(j\).

In the first step, using the definition of \(\lambda\), we embed \(C(T,\mathbb{Z}_q)\) in the standard complex \(C(\mathbb{Z}_q,\mathbb{Z}_q)\). In the second and third steps, we build a sequence \((h_k)_{k \in \{0,1,...,d\}}\) of \(k\)-chains in \(C(T,\mathbb{Z}_q)\) and a sequence \((e_k)_{k \in \{0,1,...,d\}}\) of \(k\)-cochains in \(C(\mathbb{Z}_q,\mathbb{Z}_q)\) which satisfy dual relations. Finally, using this duality and an induction, we achieve the proof.

4.1 \(\psi_# : C(T,\mathbb{Z}_q) \to C(\mathbb{Z}_q,\mathbb{Z}_q)\)

We see \(\lambda\) as a simplicial map going from \(T\) into the \((m-1)\)-dimensional simplicial complex \((\mathbb{Z}_q)^m\), whose simplices are the subsets of \(\mathbb{Z}_q \times \{1,2,...,m\}\) containing no pair \(\{(e,j),(e',j)\}\) for some \(j \in \{1,2,...,m\}\) and some \(e,e' \in \mathbb{Z}_q\) with \(e \neq e'\).

We define then the following chain map \(\phi_# : C((\mathbb{Z}_q)^m,\mathbb{Z}_q) \to C(\mathbb{Z}_q,\mathbb{Z}_q)\) for \(\sigma = [(e_0,j_0),...,(e_k,j_k)] \in (\mathbb{Z}_q)^m\) with \(j_0 < j_1 < ... < j_k\) by \(\phi_#(\sigma) = (e_0,e_1,...,e_k)\) (checking that it is a chain map is straightforward).

We define \(\psi_# := \phi_# \circ \lambda_#\). It is a chain map going from the chain complex \(C(T,\mathbb{Z}_q)\) into the standard complex \(C(\mathbb{Z}_q,\mathbb{Z}_q)\). Note that \(\psi_#\) commutes with the \(\nu_#\) (where \(\nu : a \in \mathbb{Z}_q \mapsto (a+1) \in \mathbb{Z}_q\)).

4.2 the “hemispheres”

It is not too hard to exhibit a sequence \((h_k)_{k \in \{0,1,...,d\}}\) of \(k\)-chains in \(C(T,\mathbb{Z}_q)\) such that \(h_0\) is a vertex and such that, for \(l\) any integer \(\geq 0\):

\[
\partial h_{2l+1} = (\text{id}_# + \nu_# + ... + \nu_#^{l-1})h_{2l},
\]

\[
\partial h_{2l+2} = (\nu_# - \nu_#^{-1})h_{2l+1}.
\]

(3)

We can also see their existence through an homology argument: let \(h_0\) be any vertex of \(T\); then

\[
\partial (\text{id}_# + \nu_# + ... + \nu_#^{l-1})h_0 = (\text{id}_# + \nu_# + ... + \nu_#^{l-1})\partial h_0 = 0
\]

and there exists an \(h_1\) such that \(\partial h_1 = (\text{id}_# + \nu_# + ... + \nu_#^{l-1})h_0\) (the 0th homology group of \(T\) is \((d-1)\)-connected); finally, if \(h_{2l}\) exists, then

\[
\partial (\text{id}_# + \nu_# + ... + \nu_#^{l-1})h_{2l} = (\text{id}_# + \nu_# + ... + \nu_#^{l-1})\partial h_{2l} = (\text{id}_# + \nu_# + ... + \nu_#^{l-1})(\nu_# - \nu_#^{-1})h_{2l-1} = 0;
\]

(4)
hence there exists an \( h_{2l+1} \) such that \( \partial h_{2l+1} = (\text{id} + \nu_# + \ldots + \nu_{#-1}^q)h_{2l} \),
and if \( h_{2l+1} \) exists, then
\[
\partial(\nu_# - \nu_{#-1})h_{2l+1} = (\nu_# - \nu_{#-1}^q)\partial h_{2l+1} = (\nu_# - \nu_{#-1}^q)(\text{id} + \nu_# + \ldots + \nu_{#-1}^q)h_{2l} = 0;
\]
hence there exists an \( h_{2l+2} \) such that \( \partial h_{2l+2} = (\nu_# - \nu_{#-1}^q)h_{2l+1} \) (the \( k \)th homology group of \( \mathcal{T} \) is 0 for \( k \leq d-1 \): \( \mathcal{T} \) is \((d-1)\)-connected).

### 4.3 the “co-hemispheres”

Our aim is to find a sequence \( (e_k) \) of elements of the standard complex \( C(Z_q, Z_q) \) playing the same role as the \( e_k \) in the proof of Theorem 1 above.

For the proof, it is enough to know that such a sequence exists (the construction of this sequence is given in the Appendix - Lemma 2 - at the end of the paper), which satisfies \( e_0 = (0) \) and, for \( l \) any integer \( \geq 0 \):

\[
\begin{align*}
\delta e_{2l} &= (\nu_# - \nu_{#-1}^q)e_{2l+1}, \\
\delta e_{2l+1} &= (\text{id} + \nu_# + \ldots + \nu_{#-1}^q)e_{2l+2}.
\end{align*}
\]

Again, the \( h_k \) and the \( e_k \) satisfy dual relations. We call the latter “co-hemispheres”.

### 4.4 induction

We use now this symmetry between equations (3) and (4) to achieve the proof: we prove now the following property by induction on \( l \leq d \):

\[
\langle e_{2l}, \psi_#(\text{id} + \nu_# + \ldots + \nu_{#-1}^q)h_{2l} \rangle = (-1)^l \mod q
\]
and
\[
\langle e_{2l+1}, \psi_#((\nu_# - \nu_{#-1}^q)h_{2l+1}) \rangle = (-1)^{l+1} \mod q.
\]

It is true for \( l = 0 \): \( \psi_#((\text{id} + \nu_# + \ldots + \nu_{#-1}^q)h_0) = (0) + (1) + \ldots + (q-1) \) and 
\[
\langle e_0, \psi_#((\text{id} + \nu_# + \ldots + \nu_{#-1}^q)h_{2l}) \rangle = \langle (0), (0) + (1) + \ldots + (q-1) \rangle = 1.
\]

If it is true for \( l \geq 0 \), we have:

\[
\begin{align*}
\langle e_{2l+1}, \psi_#((\nu_# - \nu_{#-1}^q)h_{2l+1}) \rangle &= \langle \nu_{#-1} - \nu_#e_{2l+1}, \psi_#h_{2l+1} \rangle = -\langle \delta e_{2l}, \psi_#h_{2l+1} \rangle \\
&= -\langle e_{2l}, \psi_#\partial h_{2l+1} \rangle = -\langle e_{2l}, \psi_#((\text{id} + \nu_# + \ldots + \nu_{#-1}^q)h_{2l}) \rangle = (-1)^{l+1} \mod q,
\end{align*}
\]
and
\[
\begin{align*}
\langle e_{2l+2}, \psi_#((\text{id} + \nu_# + \ldots + \nu_{#-1}^q)h_{2l+2}) \rangle &= \langle (\text{id} + \nu_# + \ldots + \nu_{#-1}^q)e_{2l+2}, \psi_#h_{2l+2} \rangle = \langle \delta e_{2l+1}, \psi_#h_{2l+2} \rangle \\
&= \langle e_{2l+1}, \psi_#\partial h_{2l+2} \rangle = \langle e_{2l+1}, \psi_#((\nu_# - \nu_{#-1}^q)h_{2l+1}) \rangle = (-1)^{l+1} \mod q.
\end{align*}
\]

This proves the property. For \( k = d \), it means that there is at least one \( d \)-simplex \( \sigma \) such that \( \psi_#(\sigma) = (e_0, e_1, \ldots, e_d) \) with \( e_i \neq e_{i+1} \) for \( i = 0, 1, \ldots, d-1 \) (in the \( e_k \), all \( k + 1 \)-tuples satisfy this property - see Lemma 1 in the Appendix), which is exactly the statement of the theorem.
5 Combinatorial proof of Dold’s theorem

We recall Dold’s theorem (proved by Dold in 1983 [2]):

**Theorem 3 (Dold’s theorem)** Let \( X \) and \( Y \) be two simplicial complexes, which are free \( \mathbb{Z}_n \)-space. If \( f : X \to Y \) is a \( \mathbb{Z}_n \)-equivariant map between free \( \mathbb{Z}_n \)-spaces, then the dimension of \( Y \) is larger than or equal to the connectivity of \( X \).

It is not too hard to give an explicit construction (without using homology arguments) of a sequence \( (h_k)_{k \in \{0, 1, \ldots, d\}} \) of \( k \)-chains in \( C(T, \mathbb{Z}_q) \), where \( T \) is any equivariant triangulation of \( (\mathbb{Z}_p)^{(d+1)} \), such that \( h_0 \) is a vertex and such that, for \( l \) any integer \( \geq 0 \):

\[
\begin{align*}
\partial h_{2l+1} &= (\text{id}_\# + \nu_\# + \ldots + \nu_\#^{q-1})h_{2l}, \\
\partial h_{2l+2} &= (\nu_\# - \nu_\#^{-1})h_{2l+1}.
\end{align*}
\]  

(5)

The proof of Theorem 2 is combinatorial (no homology, no continuous map, no approximation) and does not work by contradiction.

By standard techniques, to prove Theorem 3, it is sufficient to consider the case when \( n = p \) is prime, \( X \) is an equivariant triangulation of \( (\mathbb{Z}_p)^{(d+1)} \) and \( Y := (\mathbb{Z}_p)^d \), and to prove that there is no equivariant simplicial map \( X \to Y \).

Thus Theorem 1 (for \( p = 2 \)) and Theorem 2 (for \( p = q \) odd) together provide a purely combinatorial proof of Theorem 3 without working by contradiction, because they imply that if \( \lambda \) is an equivariant simplicial map \( X \to (\mathbb{Z}_p)^m \) then \( m > d \).

6 Appendix: definition of the \( e_k \) for \( \mathbb{Z}_q \)

6.1 Definitions of \( C \) and \( (e_k) \)

For simplicity, we write \( q = 2r + 1 \). We were not able to find a similar construction for \( q \) even (except of course for \( q = 2 \)).

We define recursively the infinite sequence \( (e_k)_{k \in \mathbb{N}} \) of element of \( C(\mathbb{Z}_q, \mathbb{Z}) \), where \( e_k \in E_k(\mathbb{Z}_q, \mathbb{Z}) \) (we define \( e_k \) with coefficients in \( \mathbb{Z} \), but the relations they will satisfy will be true for coefficients in \( \mathbb{Z}_q \) too).

We first begin with \( e_0 \) and \( e_1 \):

\[ e_0 := (0). \]

\[ e_1 := \sum_{j=0}^{r-1} \sum_{i=0}^j ((2i + 1, 2r - 2j + 2i) - (2r - 2j + 2i, 2i + 1)). \]

We define then the following application \( C : E_k(\mathbb{Z}_q, \mathbb{Z}) \to E_{k+2}(\mathbb{Z}_q, \mathbb{Z}) \) by its value on the natural basis:

\[ C : (a_0, \ldots, a_k) \mapsto (a_0, \ldots, a_k, \nu_\#^a e_1). \]

For \( k \geq 2 \), we can now define the rest of the infinite sequence:

\[ e_k := C(e_{k-2}). \]

This construction implies immediately the following property:

**Lemma 1** Let \( k \geq 0 \), and \( \sigma = (e_0, e_1, \ldots, e_k) \in \mathbb{Z}_q^{k+1} \). If \( \langle e_k, \sigma \rangle \neq 0 \), which means that \( \sigma \) has a non-zero coefficient is the formal sum \( e_k \), then \( e_i \neq e_{i+1} \) for any \( i \in \{0, 1, \ldots, k-1\} \).
6.2 Examples for $q = 3$ and $q = 5$

Let us see for instance what it gives for $q = 3$ and $q = 5$.

For $q = 3$:

- $e_0 = (0)$.
- $e_1 = (1, 2) - (2, 1)$.
- $e_2 = (0, 1, 2) - (0, 2, 1)$.
- $e_3 = (1, 2, 0, 1) - (1, 2, 1, 0) - (2, 1, 2, 0) + (2, 1, 0, 2)$.
- $e_4 = (0, 1, 2, 0, 1) - (0, 1, 2, 1, 0) - (0, 2, 1, 2, 0) + (0, 2, 1, 0, 2)$, and so on.

For $q = 5$:

- $e_0 = (0)$.
- $e_1 = (1, 2) + (3, 4) + (1, 4) - (2, 1) - (4, 3) - (4, 1)$.
- $e_2 = (0, 1, 2) + (0, 3, 4) + (0, 1, 4) - (0, 2, 1) - (0, 4, 3) - (0, 4, 1)$.
- $e_3 = (1, 2, 3, 4) + (1, 2, 0, 1) + (1, 2, 3, 1) - (1, 2, 4, 3) - (1, 2, 1, 0) - (1, 2, 1, 3) + (3, 4, 0, 0) - (3, 4, 0, 3) - (3, 4, 1, 0) - (3, 4, 3, 2) - (3, 4, 3, 0) - (1, 4, 0, 1) - (1, 4, 2, 3) + (1, 4, 0, 3) - (1, 4, 1, 0) - (1, 4, 3, 2) - (1, 4, 3, 0) - (2, 1, 2, 3) - (2, 1, 4, 0) - (2, 1, 1, 4) - (2, 1, 0, 2) - (2, 1, 0, 1) - (2, 1, 1, 0) - (2, 1, 2, 0) - (2, 1, 1, 3) + (1, 2, 3, 1) + (1, 2, 3, 1) - (1, 2, 3, 1) - (0, 1, 2, 4) - (0, 1, 2, 3, 1) - (0, 1, 2, 4, 3) - (0, 1, 2, 1, 0) - (0, 1, 2, 1, 3) + (0, 3, 4, 0, 1) + (0, 3, 4, 2, 3) + (0, 3, 4, 2) + (0, 3, 4, 0, 3) - (0, 3, 4, 1, 0) - (0, 3, 4, 3, 2) - (0, 3, 4, 3, 0) - (0, 1, 4, 0, 1) + (0, 1, 4, 2, 3) + (0, 1, 4, 0, 3) - (0, 1, 4, 1, 0) - (0, 1, 4, 3, 2) - (0, 1, 4, 3, 0) - (0, 2, 1, 2, 3) - (0, 2, 1, 4, 0) - (0, 2, 1, 2, 0) - (0, 2, 1, 0, 2) - (0, 4, 3, 4, 0) - (0, 4, 3, 1, 2) - (0, 4, 3, 2, 0) - (0, 4, 3, 0, 0) - (0, 4, 3, 2, 1) - (0, 4, 3, 2, 1) - (0, 4, 3, 0, 0) - (0, 4, 4, 1, 2, 3) - (0, 4, 4, 1, 2, 0) - (0, 4, 1, 2, 0) + (0, 4, 1, 3, 2) + (0, 4, 1, 0, 4) + (0, 4, 1, 0, 2)$, and so on.

6.3 Induction property of $(e_k)$

We prove now the equations (4).

Lemma 2 For $l \geq 0$, we have:

$$\delta e_{2l} = (\nu^# - \nu^{#-1}) e_{2l+1},$$
$$\delta e_{2l+1} = (\text{id}^# + \nu^# + \ldots + \nu^{#-1}) e_{2l+2}.$$

Proof: We prove first a serie of claims and finally, we prove the equations by induction.

CLAIM 1:

$$\delta ((2) + (4) + \ldots + (2r)) = (\text{id}^# - \nu^#) e_1.$$  (6)

Proof of Claim 1: According to the definition of $e_1$, if $\sigma$ is such that $\langle e_1, \sigma \rangle \neq 0$, then $\sigma$ is of the form $(y, x)$ or $(x, y)$ with $x$ even, $y$ odd and $0 \leq y < x \leq 2r$. Similarly, if $\sigma$ is such that $\langle \nu^# e_1, \sigma \rangle \neq 0$, then $\sigma$ is of the form $(y, x)$ or $(x, y)$ with $x$ even $\geq 2$, $y$ odd and $0 \leq x < y \leq 2r$, or of the form $(0, x)$ or $(x, 0)$ with $x$ even or $0 < x \leq 2r$.

Hence, if $\sigma$ is such that $(\langle \text{id}^# - \nu^# e_1, \sigma \rangle \neq 0$, then $\sigma$ is of the form $(y, x)$ or $(y, x)$ with $x \in X := \{2, 4, \ldots, 2r\}$ and $y \in Y := \{0\} \cup \{1, 3, \ldots, 2r - 1\}$. For $x \in X$ and $y \in Y$, the coefficient of $(y, x)$ in $(\text{id}^# - \nu^# e_1)$ is $-1$ and the coefficient of $(y, x)$ is $+1$. The equality $\delta ((2) + (4) + \ldots + (2r)) = (\text{id}^# - \nu^#) e_1$ follows.

CLAIM 2:

$$\delta e_1 = \sum_{j \in \mathbb{Z}_4} \nu^# j e_2.$$  (7)

Proof of Claim 2: Applying $\delta$ on both sides of equation (6), we get: $\delta e_1 = \nu^# (\delta e_1)$. It implies that $\delta e_1$ can be written $\sum_{j \in \mathbb{Z}_4} \nu^# j (0, h)$, where $h \in E_1(\mathbb{Z}_4, \mathbb{Z})$. As the couples $(x, y)$
in $e_1$ never begin with a 0, we get $(0, e_1)$ while keeping from $\delta e_1$ only the couples beginning with a 0. Hence $h = e_1$, and we have indeed $\delta e_1 = \sum_{j \in \mathbb{Z}_q} \nu^{#j} e_2$, since $e_2 = (0, e_1)$.

**CLAIM 3:** $\nu^# \circ C = C \circ \nu^#$.

**PROOF OF CLAIM 3:** straightforward.

**CLAIM 4:**

$$\delta \circ C = C \circ \delta.$$ (8)

**PROOF OF CLAIM 4:** Let $\sigma = (a_0, \ldots, a_k)$ be a $(k + 1)$-tuple. We have

$$(\delta \circ C)(\sigma) = \delta(\sigma, (\nu^{#a_k} e_1)) = (\delta \circ (\nu^{#a_k} e_1)) + (-1)^{k+1}(\sigma, \delta((\nu^{#a_k} e_1))) = (-1)^{k+1} \sum_{j \in \mathbb{Z}_q} (\sigma, j, (\nu^{#a_k} e_1)),

$$

et

$$(C \circ \delta)(\sigma) = C(\delta \sigma) = (\delta \circ (\nu^{#a_k} e_1)) + (-1)^{k+1} \sum_{j \in \mathbb{Z}_q} (\sigma, j, (\nu^{#j} e_1)) = (-1)^{k+1} \sum_{j \in \mathbb{Z}_q} (\sigma, j, (\nu^{#a_k} e_1)).$$

Hence, $(\delta \circ C)(\sigma) = (C \circ \delta)(\sigma) = (-1)^{k+1} (\sigma, \delta((\nu^{#a_k} e_1))) = (-1)^{k+1} \sum_{j \in \mathbb{Z}_q} (\sigma, j, (\nu^{#j} e_1))$. But, according to equation (7), $\delta((\nu^{#a_k} e_1)) = \sum_{j \in \mathbb{Z}_q} (j, (\nu^{#j} e_1)) = \nu^{#a_k}(\delta e_1) = \sum_{j \in \mathbb{Z}_q} \nu^{#j}(0, e_1) = 0$ (we have $e_2 = (0, e_1)$). Thus $(\delta \circ C)(\sigma) = (C \circ \delta)(\sigma) = 0$.

**Proof of Lemma 2:** By induction on $l$.

For $l = 0$, we have $\delta e_0 = (\nu^# - \nu^{-1}^#) e_1$: indeed, let $c := (2) + (4) + \ldots + (2r)$; according to equation (6), we have $\delta c = (\text{id}^# - \nu^#) e_1$: we have also $\delta(0) + (1) + \ldots + (2r - 1) + (2r) = 0$ (the checking is straightforward); hence, $\delta(0) + \delta c + \delta \nu^{#-1} c = 0$; and thus $\delta(0) = (\nu^# - \nu^{-1}^#) e_1$. Claim 2 is the relation: $\delta e_1 = (\text{id}^# + \nu^# + \ldots + \nu^{#q-1}) e_2$. Lemma 2 is proved for $l = 0$.

Let’s assume that Lemma 2 is proved for $l > 0$. According to Claim 3 and Claim 4, we have then:

$$\delta e_{2l+2} = (\delta \circ C)(e_{2l}) = (C \circ \delta)(e_{2l}) = C((\nu^# - \nu^{-1}^#) e_{2l+1}) = (\nu^# - \nu^{-1}^#) e_{2l+3}$$

and

$$\delta e_{2l+3} = (\delta \circ C)(e_{2l+1}) = (C \circ \delta)(e_{2l+1}) = C(\sum_{j \in \mathbb{Z}_q} \nu^{#j} e_{2l+2}) = \sum_{j \in \mathbb{Z}_q} \nu^{#j} e_{2l+4} = (\text{id}^# + \nu^# + \ldots + \nu^{#q-1}) e_{2l+4}.$$

**References**


