Permanent regime of Frank-Read source for a dislocation phase field model

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Abstract

We study a two-dimensional phase field model describing the motion of dislocations for a Frank-Read source when a shear stress σ is applied. The level sets of the phase describe the dislocation curves and allows their change of topology. The phase solves a nonlinear reaction-diffusion equation of Allen-Cahn type, on a bounded large domain with Neumann boundary conditions. The large domain mimics the whole plane with two holes to which some of the dislocation curves are attached. We show that dislocations are created at the rate ω which is a monotone function of the shear stress σ. Using the time-shifting method, we study the long time behavior of the solution, and this allows us to identify a periodic permanent regime whose periodicity is $1/\omega$. Moreover, using a semi-norm method, we show that the difference between the solution and the periodic permanent regime converges to zero with an exponential decay in time.

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Key words: Frank-Read source; dislocations, phase field model; Allen-Cahn equation, reaction-diffusion equation

1 Introduction

1.1 Setting of the problem

In this paper we investigate a nonlinear diffusion Allen-Cahn type equation with homogenous Neumann condition on a two-dimensional domain $\Omega \subset \mathbb{R}^2$ defined by

$$\Omega = B(0, R_0) \setminus \left(\overline{B(P^+, \varepsilon_0)} \cup \overline{B(P^-, \varepsilon_0)}\right)$$

where $R_0 > 2$, $0 < \varepsilon_0 < 1$ and $P^\pm = (\pm 1, 0)$. We are interested in the existence of solutions $u = u(x,t)$ of the following equation

$$(1.1) \qquad \begin{cases} u_t = \Delta u - V'\left(u + \dfrac{\theta^+(x) - \theta^-(x)}{2\pi}\right) + \sigma & \text{on} \quad \Omega \times I \\ \partial_n u = 0 & \text{on} \quad \partial\Omega \times I, \end{cases}$$

for some time interval $I = \mathbb{R}$ or $I = [t_0, +\infty)$. Here the nonlinearity V is assumed to be smooth and 1-periodic in the following sense

$$(1.2) \qquad V \in C^3(\mathbb{R}) \quad \text{and} \quad V(z+k) = V(z) \quad \text{for} \quad (z,k) \in \mathbb{R} \times \mathbb{Z},$$

and σ is a real number. Here $n = (n_1, n_2)$ denotes the unit outward normal to $\partial\Omega$ and $\partial_n u$ is the normal derivative of u. For $x = (x_1, x_2) \in \Omega$, the angles $\theta^+(x)$ and $\theta^-(x)$ are defined as follows: if we let $Q = (R_0, 0)$, then

$$\theta^+(x) := \angle(\overrightarrow{P^+Q}, \overrightarrow{P^+x}) \quad \text{and} \quad \theta^-(x) := \angle(\overrightarrow{P^-Q}, \overrightarrow{P^-x}),$$

where both angles are defined modulo 2π (see Figure 1). The multivalued functions v and ϕ_0 are defined by

$$v := u + \phi_0 \quad \text{with} \quad \phi_0(x) := \frac{\theta^+(x) - \theta^-(x)}{2\pi} \quad \text{mod} \quad \mathbb{Z}$$

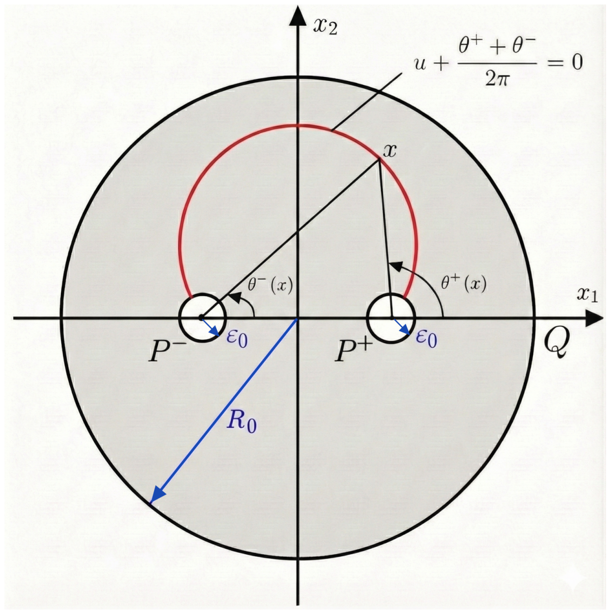

Figure 1: Schematic of the Frank-Read dislocation source.

and v represents a phase transition whose level sets can be seen as dislocation lines. Note that in (1.1), the term involving V' is well defined because of the 1-periodicity of V (see condition (1.2)).

Standard physical model
Dislocations are curve defects in the atomic lattice of a crystal (usually a metal). When a shear stress σ is applied on the crystal, the dislocation curves move and have a dynamics, which is basically modelized by a motion where normal velocity is given by

$$v_n := \kappa + \sigma \tag{1.3}$$

where κ is the curvature of the dislocation line (hence motion by mean curvature + exterior force σ). Physically, a Frank-Read source is a dislocation curve whose extremities are pinned at the two points $P^\pm$ and subjected to a shear stress σ. In the physical setting, the evolution of the dislocation undergoes a change in its topology, thus generating free dislocation loops (surrounding the segment $[P^-, P^+]$ from very far away) while a new dislocation line always remains pinned to the two fixed points which constitute the source. The phenomenon is then repeated periodically in time.

Comparison with our phase field model
Our phase field model (1.1) approximates dislocation dynamics of a Frank-Read source. In our model (1.1), the pinning is modelized by Neumann boundary conditions on the boundary of two discs $B(P^\pm, \varepsilon)$ with ε very small. It can be directly noticed that our domain Ω (say for large radius R_0) and the imposed Neumann condition capture the essential properties of the natural domain of evolution $\mathbb{R}^2 \setminus \{P^\pm\}$ of Frank-Read sources. The domain Ω has indeed the advantage of being bounded, without singularities and free from prescribed boundary values.

It is very classical that the solution to an appropriate rescaling of the Allen-Cahn equation with small parameter δ, converges to a geometric motion with normal velocity given by (1.3) (see for instance [18] for stationary problems and [1] for evolution problems). Hence, in some rough sense, we can think to our model as a model approximating the physical model of a Frank-Read source.

We aim to construct permanent regime solutions of (1.1). This permanent regime solution will be obtained by considering the long time behavior of classical solutions of the Cauchy problem (1.1) for $I := [0, +\infty)$, for some given initial data.

The basic building block is the following result.

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Theorem 1.1 (Basic existence and uniqueness result)
Let $\sigma \in \mathbb{R}$ and assume that V satisfies (1.2). Assume that the initial data u_0 satisfies the following compatibility condition

$$\partial_n u_0 = 0 \quad on \quad \partial\Omega \quad with \quad u_0 \in W^{3,\infty}(\Omega) \tag{1.4}$$

Then there exists a unique function $u : \overline{\Omega} \times [0, +\infty) \to \mathbb{R}$ in the Hölder class for all $\alpha \in (0, 1)$ $u \in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(\overline{\Omega} \times [0, +\infty))$, solution of (1.1) with initial data $u(\cdot, 0) = u_0$.

By the space $C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}$, we mean that $w := D^2u, u_t$ satisfies the following Hölder estimate

$$|w(x,t) - w(y,s)| \leq C \left(|x-y|^\alpha + |t-s|^{\frac{\alpha}{2}}\right)$$

locally on compact sets of $\overline{\Omega} \times [0, +\infty)$, but not necessarily globally. This result follows from standard parabolic theory (see [15, 16]). We will still indicate the sketch of the proof in Subsection 3.2. Built on this very basic result, we will prove several qualitative results below, using two methods (the time shifting method and the semi-norm method).

1.2 Brief review of the literature

Up to our knowledge, there has been no rigorous study of the permanent regime for the dynamics of dislocations in the specific geometry of the Frank-Read source. There are, however, some work dealing with simpler configurations or other sort of results. In this respect, the study of spirals is particularly interesting in the sense that with appropriate gluing conditions, the movement of a Frank-Read source could possibly be obtained by the study of the movement of two spirals attached to the points P^+ and P^-.

The motion of spiral dislocations has been studied with the help of a level-set formulation by Ohtsuka [22] and by Goto, Nakagawa and Ohtsuka [10], who obtained the existence of solutions of the level-set approach, and the uniqueness of the evolution of an initial curve. See also the work of Ohtsuka, Tsai and Giga [23] for numerical simulations, and the work of Lou [17] for an example of homogenization of a curve moving in some oscillating annulus. Smereka [26] introduced another original level-set formulation to simulate the evolution of dislocations on a wider array of geometrical configurations, including the Frank-Read source. Giga, Ishimura and Koshaka [8] showed the existence of spiral-shaped solutions for the motion of dislocations on a compact annulus using a parametrization of the spiral dislocation curves.

Forcadel, Imbert and Monneau [4, 5] also used a parametrization of spirals to prove existence and uniqueness results, this time in an unbounded domain with a singularity at the origin. These articles describe well the problems encountered with the singularity of the attachment point and with the unbounded character of the domain. See also Hudson, Rindler and Rydell [11] for numerical simulations.

Karma and Plapp [13] introduced a phase field formulation to study and simulate the formation of spiral dislocations. Phase field approaches were then also treated by Rodney, Le Bouar and Finel [25], Koslowski, Cuitino and Ortiz [14], Wan, Jin, Cuitino and Khachaturyan [28], Xiang, Cheng, Srolovitz and E [27]. In [20], Ogiwara and Nakamura used such a phase field approach for a theoretical study of the motion of spiral dislocations, in the case of a bounded domain with no singularity, and proved the existence of spiral travelling wave solutions, which exhibit a permanent regime behavior. This result has been generalized in Ogiwara and Nakamura [21]. The setting of our paper shares some partial similarities with [20, 21], even if the solutions we construct for Frank-Read sources are less constrained than spirals. The use of Neumann boundary conditions is consistent with this approach, we also refer to Giga and Sato ([6] and [7]) for other studies of dislocation dynamics with Neumann boundary conditions.

1.3 Main results

The first result of this paper is about qualitative properties of the linear growth of solutions of (1.1). More precisely, using the time shifting method, we show

Theorem 1.2 (Linear growth ω in time)
Let $\sigma \in \mathbb{R}$ and assume that V satisfies (1.2). Let u be the solution of (1.1) with time interval $I := [0, +\infty)$ and initial data $u(\cdot, 0) = u_0$ satisfying (1.4).
i) (Linear growth)
Then there exists a unique $\omega = \omega(\sigma) \in \mathbb{R}$, independent on the initial data u_0, called the growth speed such that

$$|u(x,t) - \omega t| \leq C \quad for\ all \quad (x,t) \in \overline{\Omega} \times I \tag{1.5}$$

where $C > 0$ is a constant independent of σ, but depending on u_0 only through $|u_0|_{L^\infty(\Omega)}$.
ii) (Monotonicity)
Moreover, the function $\sigma \mapsto \omega(\sigma)$ is continuous and nondecreasing.
iii) (Special case of even V)
Furthermore if V is even, i.e. $V(-z) = V(z)$, then $\omega(-\sigma) = -\omega(\sigma)$.

From a physical point of view, the quantity ω can be interpreted as the production rate of dislocations by a Frank-Read source. The above theorem shows that this production rate is a monotone function of the shear stress σ applied to the crystal.

The next result examines the long time behavior of the solution of (1.1) in order to construct a permanent regime solution of (1.1).

Theorem 1.3 (Existence and uniqueness of a periodic permanent regime)
Assume (1.2), fix $\sigma \in \mathbb{R}$ and consider the global time interval $I := \mathbb{R}$. Let ω be the growth rate obtained in Theorem 1.2. Then we have

$$\sigma\omega \geq 0 \quad \text{and} \quad |\omega| \leq |\sigma|. \tag{1.6}$$

i) (Existence and uniqueness for $\omega \neq 0$)
If $\omega \neq 0$, then there exists a solution u_ω of (1.1) satisfying (1.5) on $\overline{\Omega} \times \mathbb{R}$, and

$$u_\omega(x, t + T_\omega) = u_\omega(x, t) + 1 \quad \text{for all} \quad x \in \Omega,\, t > 0, \tag{1.7}$$

with $T_\omega := \dfrac{1}{\omega}$. Moreover, the solution u_ω is unique up to time translations.
ii) (Existence for $\omega = 0$)
If $\omega = 0$, then there exists a stationary solution u_ω of (1.1).
iii) (Monotonicity and symmetries)
The map $t \mapsto \omega \cdot u_\omega(x, t)$ is nondecreasing in time, and we have the symmetry $u_\omega(-x_1, x_2, t) = u_\omega(x_1, x_2, t)$. Moreover $k + u_\omega$ is also a solution of (1.1) for any $k \in \mathbb{Z}$.
iv) (Special case of even V)
Moreover, if V is even, then for any $\sigma \in \mathbb{R}$, we can choose permanent regime solutions such that

$$u_{\omega(-\sigma)}(x, t) = -u_{\omega(\sigma)}(-x, t).$$

Remark 1.4 *When $\omega = 0$, we do not know if the stationary solution is unique (up to addition of integers), especially when the function $\sigma \mapsto \omega(\sigma)$ vanishes on some non-trivial interval in σ.*

Remark 1.5 *Relation (1.6) can be interpreted saying that the answer ω of the system is always lower that the forcing term σ, and also has the same sign as σ. This result is remarkable, and is probably indirectly due to the fact that the right hand side of the PDE makes the derivative of the potential V appear.*

From a physical point of view, the existence of a stationary solution corresponds to a situation where the applied stress is not high enough to overcome the effect of the curvature and then not able to put the dislocation in motion.

Using the semi-norm method, we get the following result.

Theorem 1.6 (Exponential asymptotic when $\omega \neq 0$)
Assume (1.2) and let $\sigma \in \mathbb{R}$ be such that $\omega \neq 0$. Let u be the solution described in Theorem 1.2 with initial data $u(\cdot, 0) = u_0$ satisfying (1.4). Then there exists $\rho > 0$ (independent on u_0, but depending on σ) such that the following holds true. Given the solution u_ω in Theorem 1.3, then there exists some $\tau_* \in \mathbb{R}$ and $C > 0$ (both depending on the initial data u_0 and on σ) such that

$$\|u(\cdot, t) - u_\omega(\cdot, t + \tau_*)\|_{L^\infty(\Omega)} \leq Ce^{-\rho t} \quad \text{for all} \quad t \geq 0. \tag{1.8}$$

1.4 Organization of the paper

This paper is organized as follows. In Section 2, we recall classical parabolic tools. In Section 3, we study the solution u of the Cauchy problem (1.1), namely, we give the sketch of the proof of Theorem 1.1 based on a control of the space oscillations of the solution. In Section 4, we show the time linear growth of the solution (Theorem 1.2). In Section 5, we study the long time behavior of u and thus prove Theorem 1.3 showing the existence of a periodic permanent regime solution u_ω of (1.1). Finally, Section 6 is devoted to the study of the asymptotics with the proof of Theorem 1.6.

2 Some tools for parabolic PDEs

In this whole section, we assume that $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a smooth bounded and connected open set. We now state some classical results that are frequently used throughout this paper. The standard model equation that closely resembles our problem is given by the classical heat equation

$$\begin{cases} u_t &= \Delta u + f \qquad \text{on} \qquad \Omega \times (0,T) \\ \partial_n u &= 0 \qquad\qquad \text{on} \qquad \partial\Omega \times (0,T) \\ u &= \varphi \qquad\qquad \text{on} \qquad \overline{\Omega} \times \{0\} \end{cases} \tag{2.9}$$

where f and φ are given functions. We assume enough regularity on f, φ, and a compatibility boundary condition (for instance we can assume $\partial_n \varphi = 0$ on $\partial\Omega$) to ensure the unique solvability of the Neumann boundary value problem (2.9) in the parabolic Sobolev space $W_p^{2,1}(\Omega \times (0,T))$ with $p > 1$. The compatibility condition is necessary only when $p > N + 2$ where N is the spatial dimension; since $N = 2$, this applies for $p > 4$. To simplify notation we write

$$Q_T = \Omega \times (0,T) \quad \text{and} \quad Q_{T_1,T_2} = \Omega \times (T_1, T_2).$$

The following results concern global and local a priori estimates for solutions of (2.9).

Theorem 2.1 **([15, Theorem 9.1], Global a priori estimates in parabolic Sobolev spaces)** *Let $p > 1$, $f \in L^p(Q_T)$ and $\varphi \in W_p^{2-\frac{2}{p}}(\Omega)$. The compatibility condition $\partial_n \varphi = 0$ on $\partial\Omega$ is required if $p > 4$. Assume that the solution $u \in W_p^{2,1}(Q_T)$ satisfies (2.9). Then there exists a constant $C = C(\Omega, T, p) > 0$ such that the solution u satisfies the following a priori estimate:*

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C \left(\|f\|_{L_p(Q_T)} + \|\varphi\|_{W_p^{2-2/p}(\Omega)} \right).$$

Theorem 2.2 **([16, Theorem 7.22], Interior estimate)** *Let $p > 1$ and let $f \in L^p(Q_{T_1,T_2})$. If $u \in W_p^{2,1}(Q_{T_1,T_2})$ solves the first two lines of (2.9) on Q_{T_1,T_2}. Then for every $0 < \varepsilon < T_2 - T_1$, there exists $C = C(\Omega, T, p, \varepsilon) > 0$ such that*

$$\|u\|_{W_p^{2,1}(Q_{T_1+\varepsilon,T_2})} \leq C \left(\|u\|_{L_p(Q_{T_1,T_2})} + \|f\|_{L_p(Q_{T_1,T_2})} \right). \tag{2.10}$$

We also recall the following parabolic Sobolev embeddings.

Theorem 2.3 **([15, Lemma 3.3], Parabolic Sobolev embeddings in space dimension $N = 2$)** *Let $Q_{T_1,T_2} = \Omega \times (T_1, T_2)$. The following embeddings hold for any $u \in W_p^{2,1}(Q_{T_1,T_2})$:*

(i) (L^q Embedding, $p \leq 2$): If $1 < p \leq 2$, then $u \in L^q(Q_{T_1,T_2})$ for all $q > p$ such that

$$\begin{cases} q \leq p^*, \quad \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{2} & \text{if} \quad p < 2, \\ q < +\infty & \text{if} \quad p = 2, \end{cases}$$

and there exists a constant C, independent on u, such that

$$\|u\|_{L^q(Q_{T_1,T_2})} \leq C\|u\|_{W_p^{2,1}(Q_{T_1,T_2})}. \tag{2.11}$$

(ii) (Hölder Embedding, $p > 2$): If $p > 2$, then $u \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q_{T_1,T_2}})$ for all $\alpha \in (0,1)$ such that

$$\alpha < \alpha^*, \quad \text{where} \quad \alpha^* = \min\left(2 - \frac{4}{p}, 1\right)$$

and there exists a constant C, independent on u, such that

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q_{T_1,T_2}})} \leq C\|u\|_{W_p^{2,1}(Q_{T_1,T_2})}. \tag{2.12}$$

Now, given $T > 0$, let us consider the following problem

$$\begin{cases} v_t &= \Delta v + kv \qquad \text{on} \qquad \Omega \times (0,T) \\ \partial_n v &= 0 \qquad\qquad \text{on} \qquad \partial\Omega \times (0,T) \end{cases} \tag{2.13}$$

We recall that $v \in C_{x,t}^{2,1}(\overline{\Omega} \times [0,T])$ is a subsolution of problem (2.9) if it satisfies

$$\begin{cases} v_t &\leq \Delta v + kv \qquad \text{on} \qquad \Omega \times (0,T) \\ \partial_n v &\leq 0 \qquad\qquad \text{on} \qquad \partial\Omega \times (0,T) \end{cases}$$

Theorem 2.4 (Strong maximum principle)
Let $T > 0$ and a function $k \in C^0(\overline{\Omega} \times (0,T])$. Let $v \in C^{2,1}_{x,t}(\overline{\Omega} \times (0,T])$ be a subsolution of (2.13) in the sense just above, such that

$$v \leq 0 \quad \text{on} \quad \overline{\Omega} \times (0,T)$$

Assume that

$$v = 0 \quad \text{at some point} \quad P_0 \in \overline{\Omega} \times \{T\}$$

then

$$v \equiv 0 \quad \text{on} \quad \overline{\Omega} \times (0,T]$$

The strong maximum principle is very classical, and we refer to [24] (see Theorem 7 on page 174 in Chapter 3, Section 3 of [24] in the edition of 1984). The reader may also consult Theorem 2.7 and Lemma 2.8 in [16], and also [3, 9] for the elliptic counterpart.

3 Study of the Cauchy problem

In a first subsection, we show first a following lemma which provides a uniform control in time of the space oscillation of some a priori given smooth solution of (1.1). In a second subsection, we use this a priori estimate to show the existence of a global solution.

3.1 Control of the space oscillations

Lemma 3.1 (A priori bound: control of space oscillations)
Let $\sigma \in \mathbb{R}$ and assume that V satisfies (1.2). Let u be a smooth solution of (1.1) with $I = [0, +\infty)$ and initial data $u(\cdot, 0) = u_0$ satisfying (1.4). Then there exists an integer constant $N_0 \in \mathbb{N}$ (independent of σ, but depending on the initial data u_0) such that

$$\max_{\overline{\Omega}} u(\cdot,t) - \min_{\overline{\Omega}} u(\cdot,t) \leq N_0 \quad \text{for all} \quad t \geq 0. \tag{3.14}$$

Proof of Lemma 3.1.
Step 1: preliminaries
Let

$$\bar{u}(x,t) = u(x,t) - \langle u(\cdot,t)\rangle \quad \text{with} \quad \langle u(\cdot,t)\rangle := \frac{1}{|\Omega|}\int_\Omega u(y,t)dy$$

and $|\Omega|$ denotes the area of Ω. The function $\bar{u}$ satisfies

$$\begin{cases} \bar{u}_t = \Delta \bar{u} + h, & \text{in} \quad \Omega \times (0,+\infty) \\ \partial_n \bar{u} = 0, & \text{on} \quad \partial\Omega \times (0,+\infty) \\ \bar{u} = \bar{u}_0 := u_0 - \langle u_0 \rangle, & \text{on} \quad \Omega \times \{0\} \end{cases} \tag{3.15}$$

where

$$h(x,t) = f(x,t) - \langle f(\cdot,t)\rangle \quad \text{with} \quad f := -V'(u+\phi_0) + \sigma \tag{3.16}$$

which satisfies $\langle h(\cdot,t)\rangle = 0$. Note that for all $t \geq 0$, we have

$$\|h(\cdot,t)\|_{L^\infty(\Omega)} \leq 2L \quad \text{where} \quad L := \max_{\mathbb{R}} |V'|. \tag{3.17}$$

The goal is now to bound $\bar{u}$ in L^∞. To this end, we introduce the (unbounded) linear operator

$$\begin{array}{llll} A: & D(A) \subset L^2_m(\Omega) & \to & L^2_m(\Omega) \\ & v & \mapsto & Av := -\Delta v \end{array}$$

with Hilbert spaces

$$\begin{cases} D(A) := X_m(\Omega) := \left\{ v \in H^2(\Omega), \quad \int_\Omega v \; dx = 0, \quad \partial_n v = 0 \quad \text{on} \quad \partial\Omega \right\}, \\ L^2_m(\Omega) := \left\{ v \in L^2(\Omega), \quad \int_\Omega v \; dx = 0 \right\} \end{cases}$$

with the scalar products induced respectively by $H^2(\Omega)$ and by $L^2(\Omega)$. Moreover, notice that A enjoys the following properties

$$\left\{\begin{array}{lll} (Av,v)_{L^2_m(\Omega)} \geq 0 & \text{for all} \quad v\in D(A) & (\textbf{monotonicity})\\ \text{for any } f\in L^2_m(\Omega), \text{ there exists } v\in D(A) \text{ such that } u+Au=f & & (\textbf{maximality}) \end{array}\right.$$

Here the maximality property follows from Lax-Milgram theorem and classical elliptic estimates. We skip the details here, but give below in Step 2 some details for a similar and more delicate reasoning. Then from Brezis [2] (Theorem VII.4 on page 105), Hille-Yosida theorem implies the existence of a contraction semigroup $\{e^{tA}\}_{t\in[0,\infty)}$ in the Hilbert space $L^2_m(\Omega)$. Moreover Duhamel formula gives

$$\begin{array}{lll} \bar u(\cdot,t) & = e^{-tA}\bar u_0 & +\displaystyle\int_0^t e^{-(t-s)A}h(\cdot,s)ds \\ & =: \tilde u_0(t) & +\bar{\bar u}(\cdot,t) \qquad\qquad\qquad \text{with} \quad \tilde u_0(\cdot,t):=e^{-tA}\bar u_0. \end{array}$$

Step 2: decomposition on a basis of eigenvectors of A

We now want to decompose $h(\cdot,s)$ onto a basis of eigenvectors of A. To justify this decomposition, let us consider the following Hilbert space

$$Y := \overline{(D(A))}^{H^1(\Omega)}$$

embedded with the scalar product inherited from $H^1(\Omega)$. Then it is easy to see that the symmetric bilinear form

$$\begin{array}{llll} a: & D(A)\times D(A) & \to & \mathbb{R}\\ & (v,w) & \mapsto & a(v,w):=(Av,w)_{L^2}=\displaystyle\int_\Omega \nabla v\cdot\nabla w\; dx=(v,Aw)_{L^2} \end{array}$$

extends by continuity to $a: Y\times Y\to\mathbb{R}$, and defines a scalar product equivalent to $(\cdot,\cdot)_Y$ with for some $\alpha>0$

$$\alpha\cdot(v,v)_Y\leq a(v,v) \quad \text{for all} \quad v\in Y \tag{3.18}$$

as follows from Poincaré-Wirtinger inequality on the bounded open set Ω. Then Lax-Milgram theorem shows that $A^{-1}: L^2_m(\Omega)\to Y$ is well-defined. Moreover standard elliptic theory implies that

$$A^{-1}: L^2_m(\Omega)\to X_m(\Omega):=D(A)$$

is a bounded operator. The compactness of the elliptic Sobolev injection $X_m(\Omega)\hookrightarrow L^2_m(\Omega)$, implies that $A^{-1}: L^2_m(\Omega)\to L^2_m(\Omega)$ is a compact operator. Now for $f,g\in D(A)\subset L^2_m(\Omega)$, we have for $v:=A^{-1}f$ and $w:=A^{-1}g$

$$(A^{-1}f,g)_{L^2}=(v,Aw)_{L^2}=(Av,w)_{L^2}=(f,A^{-1}g)_{L^2}$$

This extends by continuity for $f,g\in L^2_m(\Omega)$, and then shows that A^{-1} is self-adjoint. Moreover, it is known (and easy to recover) that $L^2_m(\Omega)$ is a separable Hilbert space. Therefore Theorem VI.11 in [2] implies classically that the compact self-adjoint operator A admits an (orthonormal) Hilbert basis of eigenvectors $(e_n)_{n\geq 1}$ with

$$Ae_n=\lambda_n e_n, \quad 0<\lambda_n\leq\lambda_{n+1}\to+\infty$$

where the positivity of λ_n follows from the fact that (using (3.18))

$$0<\alpha\cdot(e_n,e_n)_Y\leq a(e_n,e_n)=(Ae_n,e_n)_{L^2}=\lambda_n(e_n,e_n)_{L^2}.$$

Moreover the dimension of any eigenspace (for $\lambda_n\neq 0$) is finite from Fredholm alternative (see Theorem VI.6 in [2]). Finally Lemma VI.2 in [2] on the spectrum of compact operators A^{-1}, implies that $\lambda_n\to+\infty$, as it is classical. Hence we get

$$e^{-(t-s)A}h(\cdot,s)=\sum_{n\geq 1}e^{-(t-s)\lambda_n}h_n(s)e_n$$

with

$$h(\cdot,s):=\sum_{n\geq 1}h_n(s)e_n \quad \text{and} \quad \sum_{n\geq 1}|h_n(s)|^2=\|h(\cdot,s)\|^2_{L^2}\leq|\Omega|\cdot\|h(\cdot,s)\|^2_{L^\infty}\leq|\Omega|(2L)^2.$$

Step 3: Uniform bounds in $L^2_m(\Omega)$
We now compute

$$
\begin{aligned}
\|\bar{\tilde{u}}(\cdot,t)\|^2_{L^2} &= \sum_{n\geq 1}\left|\int_0^t e^{-(t-s)\lambda_n} h_n(s)\ ds\right|^2\\
&\leq \sum_{n\geq 1}\left(\int_0^t e^{-(t-s)\lambda_n}\ ds\right)\cdot\left(\int_0^t e^{-(t-s)\lambda_n}\ h_n^2(s)\ ds\right)\\
&\leq \sum_{n\geq 1}\left(\int_0^t e^{-(t-s)\lambda_1}\ ds\right)\cdot\left(\int_0^t e^{-(t-s)\lambda_1}\ h_n^2(s)\ ds\right)\\
&\leq (\lambda_1)^{-1}\sum_{n\geq 1}\int_0^t e^{-(t-s)\lambda_1}\ h_n^2(s)\ ds\\
&= (\lambda_1)^{-1}\int_0^t e^{-(t-s)\lambda_1}\ \|h(\cdot,s)\|^2_{L^2(\Omega)}\ ds\\
&\leq (\lambda_1)^{-1}|\Omega|(2L)^2\int_0^t e^{-(t-s)\lambda_1}\ ds\\
&\leq (\lambda_1)^{-2}|\Omega|(2L)^2
\end{aligned}
$$

i.e.

$$\|\bar{\tilde{u}}(\cdot,t)\|_{L^2}\leq\sqrt{|\Omega|}\cdot\frac{2L}{\lambda_1}\quad\text{for all}\quad t\geq 0.$$

Moreover, we also have (from the classical energy estimate for Heat equation with Neumann boundary equation) $\|\tilde{u}_0(\cdot,t)\|_{L^2(\Omega)}\leq\|\bar{u}_0\|_{L^2(\Omega)}$. Therefore, we get

$$\|\bar{u}(\cdot,t)\|_{L^2}\leq\sqrt{|\Omega|}\cdot\frac{2L}{\lambda_1}+\|\bar{u}_0\|_{L^2(\Omega)}\quad\text{for all}\quad t\geq 0. \tag{3.19}$$

Step 4: Bounds in $L^\infty(\Omega)$
Step 4.1: initial regularity
Since h is smooth enough, it follows from the regularity of parabolic equations (See Theorems 2.1 and 2.3) that for some time $T_0>0$ (independent of σ, but depending on the initial data u_0), we have:

$$\|\bar{u}\|_{C_t^{\frac{\alpha}{2}}(\overline{\Omega}\times[0,2T_0)}+\|\bar{u}\|_{L^\infty(\Omega\times[0,2T_0])}\leq C_0, \tag{3.20}$$

where the constant $C_0>0$ is independent of σ, but depending on T_0, L, Ω and u_0.

Step 4.2: interior in time estimate
The global boundedness of h together with the uniform-in-time L^2 bound (3.19) of $\bar{u}$ enables us, using interior L^p estimates for parabolic equations, to establish an estimate similar to (3.20) for the time interval $(2T,3T)$. In what follows, the constants $C, C_1>0$ (independent of σ and u_0, but depending on T_0,Ω and constants p,q,α) may vary from line to line. As a first step, we apply the interior L^p estimate with $p=2$ (see Theorem 2.2) for strong solutions of parabolic equations and we obtain

$$\|\bar{u}\|_{W_2^{2,1}(\Omega\times(T_0,3T_0))}\leq C\big(\|\bar{u}\|_{L^2(\Omega\times(0,3T_0))}+\|h\|_{L^2(\Omega\times(0,3T_0))}\big). \tag{3.21}$$

By the parabolic Sobolev embedding (Theorem 2.3), we get that for all $q>2$, there exists $C>0$ (depending in particular on q) such that

$$\|\bar{u}\|_{L^q(\Omega\times(T_0,3T_0))}\leq C\|\bar{u}\|_{W_2^{2,1}(\Omega\times(T_0,3T_0))}. \tag{3.22}$$

Hence, by reapplying the interior L^p estimate with $p=q$ and reducing the time interval, we get

$$\|\bar{u}\|_{W_q^{2,1}(\Omega\times(2T_0,3T_0))}\leq C\big(\|\bar{u}\|_{L^q(\Omega\times(T_0,3T_0))}+\|h\|_{L^q(\Omega\times(T_0,3T_0))}\big).$$

From (3.21) and (3.22), the final estimate leads to

$$\|\bar{u}\|_{W_q^{2,1}(\Omega\times(2T_0,3T_0))}\leq C\big(\|\bar{u}\|_{L^2(\Omega\times(0,3T_0))}+\|h\|_{L^q(\Omega\times(0,3T_0))}\big).$$

This, together with the uniform L^∞ bound (3.17) on h and the uniform L^2_x bound (3.19) on $\bar{u}(\cdot,t)$ infers that

$$\|\bar{u}\|_{W_q^{2,1}(\Omega\times(2T_0,3T_0))}\leq C_1.$$

Since $q > 2$, we may rely on the parabolic Sobolev embedding in Hölder spaces to finally conclude that

$$\|\bar{u}\|_{C_t^{\frac{\alpha}{2}}(\overline{\Omega}\times[2T_0,3T_0])} + \|\bar{u}\|_{L^\infty(\Omega\times(2T_0,3T_0))} \leq C_1.$$

Step 4.3: propagation of the interior in time estimate
For any $a \geq 2T_0$, and assuming the control of the norm $||\bar{u}||_{L^2(\Omega\times(a-2T_0,a+T_0))}$, we want to get a control of $\bar{u}$ in better norms on the time interval $(a, a+T_0)$. The whole process of Step 4.2 can be easily extended, replacing $\Omega\times(2T_0, 3T_0)$ by $\Omega\times(a, a+T_0)$ for any $a \geq 2T_0$. We get

$$\|\bar{u}\|_{C_t^{\frac{\alpha}{2}}(\overline{\Omega}\times[a,a+T_0])} + \|\bar{u}\|_{L^\infty(\Omega\times(a,a+T_0))} \leq C_1. \tag{3.23}$$

with C_1 independent of σ and the initial data u_0.

Step 4.4: conclusion
We can finally conclude from (3.20) and (3.23) that $\|\bar{u}(\cdot,t)\|_{L^\infty(\overline{\Omega})} \leq C_0$, where C_0 is independent of σ and therefore

$$\max_{\overline{\Omega}} u(\cdot,t) - \min_{\overline{\Omega}} u(\cdot,t) \leq 2\|\bar{u}(\cdot,t)\|_{L^\infty(\overline{\Omega})} \leq N_0 \quad \text{for all} \quad t \geq 0,$$

which shows the required control on space oscillations. □

3.2 Sketch of the proof of Theorem 1.1: Cauchy problem

Sketch of the proof of Theorem 1.1.
Short time existence of a solution of (1.1) with initial data $u_0 \in W^{3,\infty}(\Omega)$ satisfying $\partial_n u_0 = 0$ on $\partial\Omega$, can easily be obtained by a fixed point theorem, and classical parabolic theory, on a short time interval $[0, T_0]$ for $T_0 > 0$ small enough (depending on the size of the initial data u_0). We can obviously repeat the argument on further intervals of length $T_k > 0$ as time goes on. To get global solutions, we must show that

$$\sum_k T_k = +\infty. \tag{3.24}$$

The difficulty here comes from the fact that the size of the solution u can increase as the time increases, which may reduce the size of T_k, and possibly not satisfy (3.24). Here we circumvent this difficulty, showing that even if u may increase, the quantity

$$\bar{u}(\cdot,t) := u(\cdot,t) - m(t) \quad \text{with} \quad m(t) := \frac{1}{|\Omega|}\int_\Omega u(y,t)dy$$

stays bounded in $L^\infty(\Omega)$ uniformly in time $t \in [0,+\infty)$. Moreover, we have $m'(t) =< u_t >=< f >$ with $f = -V'(\dots) + \sigma$ defined in (3.16). Therefore

$$|m'(t))| \leq \|f\|_{L^\infty(\Omega\times[0,+\infty))} \leq L + |\sigma| \tag{3.25}$$

and m is Lipschitz continuous, uniformly in time. Now, those two uniform bounds on $\bar{u}$ and on m' are sufficient to show that the equation for $\bar{u}$ has nice enough estimates such that there exists some $\delta > 0$ such that for all $k \geq 0$, we have $T_k \geq \delta > 0$. This implies the existence of a global solution and ends the proof of the theorem.

4 Proof of Theorem 1.2: linear growth

We are now ready to present the proof of our first qualitative result.

Proof of Theorem 1.2.
We use the time-shifting method introduced in [12].
For $T > 0$, we define the following approximations of the mean time slope of the solution u,

$$\lambda^+(T) = \sup_{t\geq 0} \frac{u(0,t+T) - u(0,t)}{T} \quad \text{and} \quad \lambda^-(T) = \inf_{t\geq 0} \frac{u(0,t+T) - u(0,t)}{T}.$$

We will show that $\omega = \lim_{T\to\infty} \lambda^+(T) = \lim_{T\to\infty} \lambda^-(T)$ and the proof is divided into several steps.

Step 1: Uniform bound on $\lambda^{\pm}(T)$ for $T \geq 1$
It is evident from the definition that $\lambda^- \leq \lambda^+$. Fix $t_0 \geq 0$. Since (1.1) is invariant by time translations and addition of integers we know that

$$\underline{u}(x,t) = u(x, t_0 + t) - k_0, \quad \text{with} \quad k_0 := \lfloor \min_{\overline{\Omega}} u(\cdot, t_0) \rfloor$$

is a solution of (1.1). Using (3.14), we compute

$$\begin{aligned} \underline{u}(x,0) \quad &= u(x,t_0) - k_0 \\ &\leq \max_{\overline{\Omega}} u(\cdot, t_0) - k_0 \\ &\leq \min_{\overline{\Omega}} u(\cdot, t_0) - k_0 + N_0 \\ &\leq 1 + N_0 =: A. \end{aligned}$$

This suggests to set

$$\overline{u}(x,t) = A + Bt,$$

for some constant $B \in \mathbb{R}$ to fix later. Then

$$\overline{u}_t = B \geq \Delta \overline{u} - V'\left(\overline{u} + \phi_0\right) + \sigma, \quad \text{if} \quad B := \|V'\|_{L^\infty} + \sigma$$

which shows that $\overline{u}$ is a supersolution of (1.1) with $\overline{u}(\cdot, 0) \geq \underline{u}(\cdot, 0)$. From the comparison principle, we deduce

$$\overline{u}(\cdot, t) \geq \underline{u}(\cdot, t) \quad \text{for all} \quad t \geq 0.$$

At $x = 0$, we deduce

$$u(0, t_0 + t) - u(0, t_0) \leq \underline{u}(0,t) \leq \overline{u}(0,t) = A + Bt$$

Hence for $T \geq 1$, we get $\lambda^+(T) \leq C_1 := A + B$. We get a similar estimate for $\lambda^-(T)$ and deduce that

$$-C_1 \leq \lambda^-(T) \leq \lambda^+(T) \leq C_1 \quad \text{for} \quad T \geq 1.$$

Step 2: Upper bound of $\lambda^+(T) - \lambda^-(T)$
Let $T > 0$ and assume that $\lambda^+(T), \lambda^-(T)$ are attained at some points $t^+, t^- \geq 0$ respectively. Let

$$\begin{aligned} k := \lfloor u(0,t^+) - u(0,t^-) \rfloor + 1 \quad &\geq u(0,t^+) - u(0,t^-) \\ &\geq u(x,t^+) - u(x,t^-) - 2N_0 \end{aligned}$$

where in the last inequality we have used Lemma 3.1. We set

$$v(x,t) := u(x, t + t^- - t^+) + k + 2N_0$$

Then $v(x,t^+) = u(x,t^-) + k + 2N_0 \geq u(x,t^+)$. The comparison principle gives

$$u(x,t) \leq v(x,t) \quad \text{for all } t \geq t^+,$$

and in particular for $t = t^+ + T$, we get

$$u(x, t^+ + T) \leq u(x, t^- + T) + k + 2N_0$$

Taking $x = 0$ and substracting $u(0,t^+)$, we get

$$\begin{aligned} u(0, t^+ + T) - u(0,t^+) \quad &\leq u(0, t^- + T) - u(0,t^+) + k + 2N_0 \\ &\leq u(0, t^- + T) - u(0,t^-) + 1 + 2N_0 \end{aligned}$$

where we have used the definition of k in the last inequality. Dividing by T, we get

$$0 \leq \lambda^+(T) - \lambda^-(T) \leq \frac{1 + 2N_0}{T}. \tag{4.26}$$

The cases when $\lambda^{\pm}$ are not attained can be proved using quasi-optimizers in the inf/sup definition of $\lambda^{\pm}$.

Step 3: Estimate of the variations of λ^+ and λ^-
Step 3.1: uniform time continuity of u
We write

$$u = \bar{u} + m \quad \text{with} \quad m(t) :=< u(\cdot, t) >:= |\Omega|^{-1} \int_\Omega u(\cdot, t) dx$$

Recall from (3.25), that we know that m is uniformly Lipschitz continuous. Now because $\bar{u}$ satisfies the short time estimate (3.20) up to time $2T_0$, and the uniform Hölder estimate (3.23) above time $2T_0$, we deduce that both $\bar{u}$ and u are uniformly continuous in time.

Step 3.2: core of the argument

Let $n \in \mathbb{N}^*$ and assume that $\lambda^+(nT)$ is attained at $t^+ \geq 0$, i.e. that $\lambda^+(nT) = \dfrac{u(0,t^+ + nT) - u(0,t^+)}{nT}$. Clearly

$$nT\lambda^+(nT) = u(0,t^+ + nT) - u(0,t^+) \leq nT\lambda^+(T),$$

and hence

$$\lambda^+(nT) \leq \lambda^+(T). \tag{4.27}$$

The above computations can be easily adjusted to the case where the supremum is not attained. Arguing in a similar manner we get the inequality

$$\lambda^-(nT) \geq \lambda^-(T). \tag{4.28}$$

Now, let $T_1, T_2 > 0$ such that $\dfrac{T_1}{T_2} = \dfrac{p}{q} \in \mathbb{Q}$. From (4.27), (4.28) and (4.26) we get

$$\lambda^+(T_1) \geq \lambda^+(qT_1) = \lambda^+(pT_2) \geq \lambda^-(pT_2) \geq \lambda^-(T_2) \geq \lambda^+(T_2) - \frac{1+2N_0}{T_2},$$

so

$$\lambda^+(T_2) - \lambda^+(T_1) \leq \frac{1+2N_0}{T_2}.$$

Exchanging T_1 and T_2, we get

$$|\lambda^+(T_2) - \lambda^+(T_1)| \leq (1+2N_0)\max\left(\frac{1}{T_1}, \frac{1}{T_2}\right). \tag{4.29}$$

Moreover from Step 3.1, we deduce that $T \mapsto \lambda^+(T)$ is continuous. This implies that (4.29) remains valid for all $T_1, T_2 > 0$. The above method can be adapted to get a similar estimate on λ^-, and gives

$$|\lambda^-(T_2) - \lambda^-(T_1)| \leq (1+2N_0)\max\left(\frac{1}{T_1}, \frac{1}{T_2}\right) \quad \text{for all} \quad T_1, T_2 > 0. \tag{4.30}$$

Step 4: Limit of $\lambda^{\pm}$

If T_n is a sequence such that $T_n \to \infty$ then both sequences $\lambda^{\pm}(T_n)$ are Cauchy thanks to (4.29) and (4.30). Consequently, both functions $\lambda^{\pm}(T)$ have limits as $T \to \infty$. Moreover, from (4.26), these two limits are equal. We may then define

$$\omega := \lim_{T\to\infty} \lambda^+(T) = \lim_{T\to\infty} \lambda^-(T).$$

Moreover from (4.29) and (4.30) with $T_1 = T > 0$ and $T_2 \to \infty$, we deduce that

$$|\lambda^+(T) - \omega| \leq \frac{1+2N_0}{T} \quad \text{and} \quad |\lambda^-(T) - \omega| \leq \frac{1+2N_0}{T}.$$

Therefore, we deduce for all $T > 0$

$$\omega T - (1+2N_0) \leq u(0,T) - u(0,0) \leq \omega T + (1+2N_0).$$

Together with (3.14), this leads to

$$\omega T - (1+3N_0) \leq u(x,T) - u(0,0) \leq \omega T + (1+3N_0).$$

We finally get for any $T > 0$

$$|u(x,T) - \omega T| \leq 1 + 3N_0 + |u(0,0)| =: C$$

and this shows (1.5) with constant C.

Step 5: Monotonicity of ω
Let $\sigma_1 \le \sigma_2$ and let u_1, u_2 be solutions of (1.1) corresponding to σ_1 and σ_2 respectively, with the same initial data $u_1(\cdot, 0) = 0 = u_2(\cdot, 0)$. Since $\sigma_1 \le \sigma_2$, then u_2 is a supersolution of

$$\begin{cases} u_t &= \Delta u - V'(u + \phi_0) + \sigma_1 \qquad \text{on} \quad \Omega \times (0, \infty) \\ \partial_n u &= 0 \qquad\qquad\qquad\qquad\quad \text{on} \quad \partial\Omega \times (0, \infty), \end{cases}$$

while u_1 is a solution and hence a subsolution. Eventually, by the comparison principle, we get

$$u_1 \le u_2. \tag{4.31}$$

Note also that, thanks to (1.5), we may find $C > 0$ such that $|u_1 - \omega_1 t| \le C$ and $|u_2 - \omega_2 t| \le C$. Assume by contradiction that $\omega_1 > \omega_2$. Then for $t^* > \dfrac{2C}{\omega_1 - \omega_2}$, and using the above inequalities, we obtain

$$u_1(x, t^*) \ge \omega_1 t^* - C > \omega_2 t^* + C \ge u_2(x, t^*).$$

This is in contradiction with (4.31). Therefore we conclude that $\omega_1 \le \omega_2$.

Step 6: Continuity of the map $\sigma \mapsto \omega(\sigma)$
The proof is classical and follows from the uniqueness of the solution and the fact that the constant C in inequality (1.5) is independent of σ. Indeed consider a sequence $\sigma_n \to \sigma_\infty \in \mathbb{R}$, and with zero initial data, the associated solutions $u_n \to u_\infty$ satisfying

$$|u_n(x, t) - t\omega_n| \le C \quad \text{with} \quad \omega_n = \omega(\sigma_n).$$

Because of the monotonicity of the map $\sigma \mapsto \omega(\sigma)$, the sequence ω_n is then bounded, and up to the extraction of a subsequence, we deduce that $\omega_n \to \omega_\infty \in \mathbb{R}$ with $|u_\infty(x, t) - t\omega_\infty| \le C$. The uniqueness of the solution u_∞ implies $\omega_\infty = \omega(\sigma_\infty)$, and then shows that $\omega(\sigma_n) \to \omega(\sigma_\infty)$, which shows the continuity of the map ω.

Step 7: Uniqueness of ω with respect to the initial data u_0
We claim that the growth speed ω is the same for any initial data u_0 satisfying (1.4) Indeed, if u_1, u_2 are two solutions of (1.1) with initial data $u_1^0 = u_1(\cdot, 0)$ and $u_2^0 = u_2(\cdot, 0)$ respectively then there exist ω_1, ω_2, C_1 and C_2 such that for all $(x, t) \in \overline{\Omega} \times [0, \infty[$ we have

$$|u_1(x, t) - \omega_1 t| \le C_1 \quad \text{and} \quad |u_2(x, t) - \omega_2 t| \le C_2. \tag{4.32}$$

Since u_1^0 and u_2^0 are bounded then there exists an integer $N \in \mathbb{N}$ such that $|u_2^0 - u_1^0| \le N$. By the comparison principle, the same inequality remains valid for all times $t \ge 0$,

$$|u_2 - u_1| \le N$$

because $u_1 \pm N$ is still a solution of (1.1). However, we have $(\omega_2 - \omega_1)t = (\omega_2 - \omega_1)t - (u_2 - u_1) + (u_2 - u_1)$ and then from (4.32), we deduce that

$$|(\omega_2 - \omega_1)t| \le C_1 + C_2 + N$$

which implies $\omega_2 = \omega_1$ for large t.

Step 8: Control of the constant C in (1.5) by $\|u_0\|_{L^\infty(\Omega)}$
Let us call C_0 the constant for zero initial data such that

$$|u(\cdot, t) - \omega t| \le C_0$$

where C_0 is known not to depend on σ. Now for another initial data u_0 and its associated solution u_{u_0}, we set the integer part $N_* := \lceil \|u_0\|_{L^\infty(\Omega)} \rceil$. Then $-N_* \le u_0 \le N_*$ implies by the comparison principle that

$$-N_* + u \le u_{u_0} \le N_* + u.$$

Therefore

$$|u_{u_0}(\cdot, t) - \omega t| \le N_* + C_0 =: C_0'$$

which shows that C_0' is also independent on σ, and only depends on $\|u_0\|_{L^\infty(\Omega)}$.

Step 9: Proof when V is even
Consider the central symmetry $S:\mathbb{R}^2\to\mathbb{R}^2$ with $S(x):=-x$. We notice that

$$\begin{cases} S(P^{\pm})=P^{\mp} \\ S(\Omega)=\Omega \\ \theta^{\pm}(S(x))=\theta^{\mp}(x)+\pi \mod (2\pi) \\ V'(-z)=-V'(z) \end{cases}$$

Recall that u solves $u_t=\Delta u - V'(u+\phi_0)+\sigma$ with zero initial data. Then it is easy to check that

$$v(x,t):=-u(S(x),t) \tag{4.33}$$

does not affect the Neumann boundary condition nor the zero initial data and solves

$$v_t=\Delta v - V'(v+\phi_0)-\sigma.$$

Hence v solves Cauchy problem (1.1), but with σ changed in $-\sigma$. From (1.5), we have

$$|v(\cdot,t)-t\omega(-\sigma)|\le C,\quad |u(\cdot,t)-t\omega(\sigma)|\le C$$

Then (4.33) shows that $\omega(-\sigma)=-\omega(\sigma)$, and ends the proof of the theorem. □

5 Proof of Theorem 1.3: existence of a permanent regime

The solution u obtained in Theorem 1.2 on $\overline{\Omega}\times[0,+\infty)$ will serve as a base for the existence result of Theorem 1.3 on $\overline{\Omega}\times\mathbb{R}$. Indeed, in the case where $\omega\neq 0$, the solution u_ω is constructed by examining time translations of u, while for $\omega=0$, the solution u_ω is constructed by examining the long time behavior of u. This will be made clear in the following proof.

Proof of Theorem 1.3. We only consider $\omega>0$ if $\omega\neq 0$. The case $\omega<0$ is treated similarly. The proof is divided into five steps. The first four steps deal with $\omega\neq 0$, and the final step deals with $\omega=0$. Let u be the solution obtained in Theorem 1.2 defined on $\overline{\Omega}\times[0,+\infty)$.

Step 1: Case $\omega>0$ and existence of a global solution of (1.1) satisfying (1.5) on $\overline{\Omega}\times\mathbb{R}$
Let u be the solution to Cauchy problem (1.1) on $\Omega\times(0,+\infty)$ with zero initial data. Consider the function w defined by

$$w(x,t):=u(x,t)-\omega t,\quad \text{with}\quad |w(x,t)|\le C \quad \text{for all}\quad (x,t)\in\overline{\Omega}\times[0,+\infty)$$

then w satisfies for $t_0=0$

$$\left\{\begin{array}{lll} w_t=\Delta w+f_\omega & \text{on} & \Omega\times(t_0,\infty) \\ \partial_n w=0 & \text{on} & \partial\Omega\times(t_0,\infty), \end{array}\right. \quad\Bigg|\quad \text{with}\quad f_\omega:=-V'(w+\phi_0+\omega t)-\omega+\sigma. \tag{5.34}$$

Setting

$$w_n(x,t):=w\left(x,t+\frac{n}{\omega}\right)\quad\text{with}\quad |w_n(x,t)|\le C\quad\text{for all}\quad (x,t)\in\overline{\Omega}\times[-\frac{n}{\omega},+\infty)$$

we see that w_n is solution of (5.34) with $t_0=-\dfrac{n}{\omega}$.

Let $M>0$. Then for n large enough, we have the bound $|w_n|_{L^\infty(\Omega\times(-2M,2M))}\le C$ which is independent on n. Therefore interior parabolic estimates and standard parabolic regularity theory imply that the sequence w_n is bounded in $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[-M,M])$ for some $\alpha\in(0,1)$. This shows that w_n is pre-compact in $C^{2+\beta,1+\frac{\beta}{2}}(\overline{\Omega}\times[-M,M])$ for $0<\beta<\alpha$. Now considering a sequence $M\to+\infty$, and using a diagonal extraction procedure, we ensure the existence of

$$w_\infty\in C^{2+\beta,1+\frac{\beta}{2}}_{\text{loc}}(\overline{\Omega}\times\mathbb{R})\quad\text{with}\quad |w_\infty|_{L^\infty(\Omega\times\mathbb{R})}\le C$$

such that $\lim\limits_{n\to\infty} w_n=w_\infty$ in $C^{2+\beta,1+\frac{\beta}{2}}_{\text{loc}}(\overline{\Omega}\times\mathbb{R})$, up to the extraction of a subsequence. Moreover w_∞ is still solution of (5.34), but now for $t_0=-\infty$.

Now, setting

$$u_n(x,t):=w_n(x,t)+\omega t=u\left(x,t+\frac{n}{\omega}\right)-n$$

we see that u_n is a solution of (1.1) on $\Omega \times (-\frac{n}{\omega}, +\infty)$ and converges, in $C^{2+\beta,1+\frac{\beta}{2}}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R})$, to $u_\infty = w_\infty + \omega t$, which satisfies

$$|u_\infty(x,t) - \omega t| \leq C. \tag{5.35}$$

where u_∞ is a global solution of (1.1) on $\Omega \times \mathbb{R}$.

Step 2: Existence of a permanent regime solution
In this step we show the existence of the solution u_ω by relying on u_∞ given in the first step. For the clarity of presentation we denote u_∞ by u. Define

$$T_\omega := \sup\{\tau \in [0,+\infty), \quad u(x,t+\tau) \leq u(x,t) + 1 \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times \mathbb{R}\}.$$

Since u is Lipschitz continuous in time and verifies (5.35), we infer that T is positive and finite with

$$u(x,t+T_\omega) \leq u(x,t) + 1 \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{5.36}$$

From the definition of T_ω, we know that for every $n \in \mathbb{N}\setminus\{0\}$ there exists $(x_n, t_n) \in \overline{\Omega} \times \mathbb{R}$ such that

$$u(x_n, t_n + T_\omega) + \frac{1}{n} \geq u(x_n, t_n) + 1. \tag{5.37}$$

Note that x_n converges (up to a subsequence) to $x_\infty \in \overline{\Omega}$, and $t_n \to t_\infty \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Case 1: $t_\infty \in \mathbb{R}$.
Then by passing to the limit $n \to \infty$ in (5.37) and by using (5.36), we get

$$u(x_\infty, t_\infty + T_\omega) = u(x_\infty, t_\infty) + 1. \tag{5.38}$$

This shows that we have equality at (x_∞, t_∞) in the inequality (5.36). Because both functions $u(\cdot, \cdot + T_\omega)$ and $u+1$ are both global solutions of (1.1), we deduce from the strong maximum principle (Theorem 2.4) (in its strong comparison version) that we have

$$u(x,t+T_\omega) = u(x,t) + 1 \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times \mathbb{R}$$

which shows (1.7) with $u_\omega := u$. Moreover (5.35) implies that $T_\omega = \dfrac{1}{\omega}$.

Case 2: $|t_\infty| = \infty$.
We aim to drag the contact point close to $\{t = 0\}$. To this end, we set

$$u_n(x,t) := u(x, t + \frac{m_n}{\omega}) - m_n, \quad t_n =: \frac{m_n}{\omega} + \tau_n \quad \text{with} \quad m_n \in \mathbb{Z}, \quad \tau_n \in [0, \frac{1}{\omega})$$

Then up to the extraction of a subsequence, we have $\tau_n \to \tau_\infty \in [0, \frac{1}{\omega}]$ and $u_n \to u_\infty$ which satisfies (5.36) and $u_\infty(x_\infty, \tau_\infty + T) = u_\infty(x_\infty, \tau_\infty) + 1$. As in Case 1, we deduce that

$$u_\infty(x, t + T_\omega) = u_\infty(x,t) + 1 \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times \mathbb{R}$$

and (5.35) is still true and again implies $T_\omega = \omega^{-1}$. This gives again (1.7) but with $u_\omega := u_\infty$.

Step 3: Monotonicity
Define

$$\lambda_* := \inf\{\lambda \in [0,+\infty) : u_\omega(x, t+\lambda') \geq u_\omega(x,t) \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times \mathbb{R}, \quad \lambda' \in [\lambda, +\infty)\}.$$

From (5.35), we deduce that

$$\begin{aligned} u_\omega(x,t+\lambda') - u_\omega(x,t) \quad &= \omega\lambda' + \{u_\omega(x,t+\lambda') - \omega(t+\lambda')\} - \{u_\omega(x,t) - \omega t\} \\ &\geq \omega\lambda' - 2C \end{aligned}$$

and then λ_* is bounded. Moreover we have $u_\omega(x, t+\lambda_*) \geq u_\omega(x,t)$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. Recall that $T_\omega = \omega^{-1}$. From the T_ω-periodicity of $t \mapsto u_\omega(x,t) - \omega t$, we deduce from the definition of λ_*, the existence of a contact point $(x_*, t_*) \in \overline{\Omega} \times [0, T_\omega)$ such that $u_\omega(x_*, t_* + \lambda_*) = u_\omega(x_*, t_*)$. Then the strong comparison principle implies

$$u_\omega(x, t + \lambda_*) = u_\omega(x,t).$$

Consequently $u_\omega(x,t+N\lambda_*) = u_\omega(x,t)$ for all $N\in\mathbb{N}$, and this implies for $(x,t)=(0,0)$

$$|u_\omega(0,0)-\omega N\lambda_*| = |u_\omega(0,N\lambda_*)-\omega N\lambda_*| \le C$$

where the last inequality follows from (5.35). The limit $N\to+\infty$ implies that $\lambda_*=0$ which shows the time monotonicity of u_ω.

Step 4: Uniqueness up to time translations
Let u_ω^1, u_ω^2 be two global solutions of (1.1), (1.5) on $\overline{\Omega}\times\mathbb{R}$ satisfying periodicity condition (1.7). Since the function $u_\omega^2-u_\omega^1$ is periodic in time with period T_ω, then there exists an integer $N\in\mathbb{N}$ such that

$$u_\omega^1 + N > u_\omega^2.$$

Let

$$\beta_* = \sup\{\beta\ge 0 : u_\omega^1(x,t)+N \ge u_\omega^2(x,t+\beta)\}.$$

Similar arguments as in Step 3 infer that

$$u_\omega^1(x,t)+N = u_\omega^2(x,t+\beta_*),$$

and so, owing to (1.7), we may write

$$u_\omega^1(x,t) = u_\omega^2(x,t+\beta_*-NT_\omega)$$

which shows the uniqueness of the solution, up to time translations.

Step 5: Case $\omega=0$ and existence of a stationary solution
Associate to the dissipative PDE (1.1), let us introduce the following energy defined for $v\in H^1(\Omega)$ as

$$E(v) = \int_\Omega \left\{\frac{1}{2}|\nabla v|^2 + V\left(v+\phi_0\right) - \sigma v\right\}.$$

Because $u\in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[0,+\infty))$ solves equation (1.1), as it is very classical, we can multiply by u_t and integrate over Ω, to deduce that

$$\int_\Omega |u_t(\cdot,t)|^2 + \frac{d}{dt}E(u(\cdot,t)) = 0, \tag{5.39}$$

Because of (1.5) and $\omega=0$, we have

$$|u(x,t)|\le C \quad \text{for all} \quad (x,t)\in\overline{\Omega}\times[0,+\infty) \tag{5.40}$$

Therefore the energy $E(u(\cdot,t))$ is nonincreasing and bounded from below, hence converging to a constant. Using (5.40), we deduce from classical parabolic estimates that $|u|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[0,+\infty))}\le C_2$ for some $\alpha\in(0,1)$. Now setting $u_n(x,t):=u(x,t+n)$, up to the extraction of a subsequence, we have $u_n\to u_\infty$ in $C_{loc}^{2+\beta,1+\frac{\beta}{2}}(\overline{\Omega}\times\mathbb{R})$ for any $\beta\in(0,\alpha)$. Therefore

$$E(u_\infty(\cdot,t)) = E_\infty := \lim_{s\to+\infty} E(u(\cdot,s))$$

and because u_∞ is still solution of (1.1), it also satisfies (5.39), which implies

$$\int_\Omega \frac{1}{2}|(u_\infty)_t(\cdot,t)|^2 = 0$$

Therefore u_∞ is a stationary solution of (1.1), as desired.

Step 6: proof of (1.6) and bounds on $\omega(\sigma)$
Given $\sigma\in\mathbb{R}$, consider $\omega=\omega(\sigma)$ and the permanent regime solution $u:=u_\omega$ which is T_ω-time periodic with $T_\omega=\omega^{-1}$ when $\omega\neq0$, and independent on time, when $\omega=0$. Then u satisfies again the energy equality (5.39). Now writing $w:=u-\omega t$, where the functions w and w_t are T_ω-time periodic, we get after integration in time over

$[0, T_\omega]$

$$\begin{aligned} 0 &= \left\{ \int_{\Omega\times[0,T_\omega]} |u_t(\cdot,t)|^2 \right\} + E(u(\cdot,T_\omega)) - E(u(\cdot,0)) \\ &= \left\{ \int_{\Omega\times[0,T_\omega]} |w_t+\omega|^2 \right\} + \int_\Omega \{-\sigma(\omega T_\omega)\} \\ &= \left\{ \int_{\Omega\times[0,T_\omega]} |w_t|^2 + \omega^2 + 2\omega w_t \right\} + \int_{\Omega\times[0,T_\omega]} \{-\sigma\omega\} \\ &= \int_{\Omega\times[0,T_\omega]} \left\{ |w_t|^2 + \omega^2 - \sigma\omega \right\} \end{aligned}$$

This implies that

$$\sigma\omega \geq \omega^2$$

which shows (1.6). Notice that it remains true when $\omega = 0$.

Step 7: proof of iv) when V is even
The result follows as in Step 9 of the proof of Theorem 1.2. □

6 Proof of Theorem 1.6: asymptotic convergence

Proof of Theorem 1.6.
We work with $\omega \neq 0$ and assume without loss of generality that $\omega > 0$ (the case $\omega < 0$ is similar). We call u_ω the permanent regime solution given by Theorem 1.3, and u be any solution given by Theorem 1.2. Starting from Step 3, we will use the semi-norm method.

Step 1: bound on the time derivative of u_ω
From Theorem 1.3, we know that

$$0 \leq \partial_t u_\omega \quad \text{and} \quad u_\omega(x, t+T_\omega) = 1 + u_\omega(x,t) \quad \text{with} \quad T_\omega = \omega^{-1}. \tag{6.41}$$

Because u_ω solves the parabolic PDE (1.1) on $\overline{\Omega}\times\mathbb{R}$, we deduce that $w := \partial_t u_\omega \geq 0$ also solves the following parabolic PDE

$$\begin{cases} w_t = \Delta w - wV''(u_\omega + \phi_0) & \text{in} \quad \Omega\times\mathbb{R}, \\ \partial_n w = 0 & \text{on} \quad \partial\Omega\times\mathbb{R}. \end{cases} \tag{6.42}$$

Because $V \in C^3$ (from assumption (1.2)), we see that the strong comparison principle shows that either $w \equiv 0$ (which is impossible from (6.41)), or $w > 0$. Because $u_\omega(x,t) - \omega t$ is T-periodic, we deduce that w is also T-periodic, and this implies the existence of some constant $c_1 > 0$ such that

$$0 < c_1 \leq w = \partial_t u_\omega. \tag{6.43}$$

Step 2: long time convergence
We now set

$$u_n(x,t) := u(x, t + n\omega^{-1}) - n.$$

From the proof of Theorem 1.3, we know that up to a subsequence, we have for some $\beta \in (0,1)$

$$u_{n_j} \to u^1_\infty \quad \text{in } C^{2+\beta, 1+\frac{\beta}{2}}_{loc}(\overline{\Omega}\times\mathbb{R}) \quad \text{with} \quad |u^1_\infty(x,t) - \omega t| \leq C \quad \text{for all} \quad (x,t) \in \overline{\Omega}\times\mathbb{R}.$$

Then an argument similar to Step 4 of the proof of Theorem 1.3 gives the existence of some $\tau_1 \in \mathbb{R}$ such that

$$u^1_\infty = u^{\tau_1}_\omega \quad \text{with} \quad u^{\tau_1}_\omega := u_\omega(\cdot, \cdot + \tau_1).$$

Similarly for a second subsequence, we get the existence of some $\tau_2 \in \mathbb{R}$ such that

$$u_{n_k} \to u^2_\infty = u^{\tau_2}_\omega \quad \text{in } C^{2+\beta, 1+\frac{\beta}{2}}_{loc}(\overline{\Omega}\times\mathbb{R}).$$

Hence for any $\varepsilon > 0$ and $M > 0$, there exists j_ε and k_ε large enough such that

$$\begin{cases} |u_{n_{j_\varepsilon}} - u_\omega^{\tau_1}|_{L^\infty(\Omega\times[-M,M])} \le \varepsilon, \\ |u_{n_{k_\varepsilon}} - u_\omega^{\tau_2}|_{L^\infty(\Omega\times[-M,M])} \le \varepsilon. \end{cases}$$

Then using time derivative estimate (6.43), we get

$$\begin{cases} u_{n_{j_\varepsilon}} \le \varepsilon + u_\omega^{\tau_1} \le u_\omega^{\tau_1}(\cdot, \cdot + \varepsilon c_1^{-1}), \\ u_\omega^{\tau_2}(\cdot, \cdot - \varepsilon c_1^{-1}) \le -\varepsilon + u_\omega^{\tau_2} \le u_{n_{k_\varepsilon}} \end{cases}$$

Without loss of generality, assume that $0 \le N := n_{k_\varepsilon} - n_{j_\varepsilon}$. Then the relation $u_{n_{j_\varepsilon}}(\cdot, \cdot + NT_\omega) - N = u_{n_{k_\varepsilon}}$ implies

$$\begin{aligned} u_\omega^{\tau_2}(\cdot, \cdot - \varepsilon c_1^{-1}) &\le u_{n_{k_\varepsilon}} \\ &= u_{n_{j_\varepsilon}}(\cdot, \cdot + NT_\omega) - N \\ &\le u_\omega^{\tau_1}(\cdot, \cdot + NT_\omega + \varepsilon c_1^{-1}) - N \\ &= u_\omega^{\tau_1}(\cdot, \cdot + \varepsilon c_1^{-1}). \end{aligned}$$

This implies

$$\tau_2 - \varepsilon c_1^{-1} \le \tau_1 + \varepsilon c_1^{-1},$$

i.e. $\tau_2 - \tau_1 \le 2\varepsilon c_1^{-1}$. We also get similarly the upper bound on $\tau_1 - \tau_2$, and deduce that

$$|\tau_2 - \tau_1| \le 2\varepsilon c_1^{-1}.$$

Because $\varepsilon > 0$ is arbitrarily small, we deduce that $\tau_1 = \tau_2$. Therefore $\tau_* := \tau_1 = \tau_2$ is independent on the choice of the convergence subsequence. We conclude to the convergence of the full sequence

$$|u(\cdot, t) - u_\omega^{\tau_*}(\cdot, t)|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to +\infty. \tag{6.44}$$

Step 3: exponential convergence
We use the semi-norm method introduced in [19].
Step 3.1: preliminaries
For $t \ge 1$ and $\tau \in \mathbb{R}$, we set

$$M(t) := \sup_{t_0 \ge t} N(t_0) \quad \text{with} \quad N(t_0) := \inf_{\tau\in\mathbb{R}} \left\{ \sup_{\overline{\Omega}\times[t_0-1,t_0+1]} |u - u_\omega^\tau| \right\} \quad \text{and} \quad u_\omega^\tau := u_\omega(\cdot, \cdot + \tau)$$

where the infimum and suppremum in $N(t_0)$ are reached. Here $N(t_0)$ is a semi-norm. From (6.44), we know that $M(t) \to 0$ as $t \to +\infty$. We want to show that

$$\text{there exists } \mu \in (0,1) \text{ and } R > 0 \text{ such that} \quad M(t+R) \le \mu M(t) \quad \text{for all} \quad t \ge 1 \tag{6.45}$$

Step 3.2: beginning of the proof by contradiction
Assume that (6.45) is false. Then there exists sequences $1 > \mu_n \to 1$ and $R_n \to +\infty$, $t_n \ge 1$, such that

$$0 \longleftarrow \varepsilon_n := M(t_n + R_n) > \mu_n M(t_n). \tag{6.46}$$

Because $N(t_0) \to 0$ as $t_0 \to +\infty$, we know that there exists some $s_n \ge t_n + R_n$ and $\tau_n \in \mathbb{R}$ such that the suppremum defining $M(t_n + R_n)$ is reached at time s_n with

$$\varepsilon_n = M(t_n + R_n) = N(s_n) = \sup_{\overline{\Omega}\times[s_n-1,s_n+1]} |u - u_\omega^{\tau_n}| \tag{6.47}$$

From (6.44), we know that $\tau_n \to \tau_*$. We also write $s_n = m_n\omega^{-1} + r_n$ with $m_n \in \mathbb{Z}$ and $r_n \in [0, \omega^{-1})$, and set

$$w_n := \varepsilon_n^{-1}(u_n - u_\omega^{\tau_n}) \quad \text{with} \quad u_n(x,t) := u(x, t + m_n\omega^{-1}) - m_n.$$

Then we can rewrite (6.47) and (6.46) respectively as

$$\inf_{\tau\in\mathbb{R}} \left\{ \sup_{\overline{\Omega}\times[r_n-1,r_n+1]} \left|w_n - \varepsilon_n^{-1}(u_\omega^\tau - u_\omega^{\tau_n})\right| \right\} = \sup_{\overline{\Omega}\times[r_n-1,r_n+1]} |w_n| = 1 \tag{6.48}$$

and

$$\text{(6.49)} \qquad \sup_{t_0\in[r_n-R_n,+\infty)} \inf_{\tau\in\mathbb{R}} \left\{ \sup_{\overline{\Omega}\times[t_0-1,t_0+1]} \left| w_n - \varepsilon_n^{-1}(u_\omega^\tau - u_\omega^{\tau_n}) \right| \right\} \le \frac{1}{\mu_n} \to 1$$

Moreover w_n solves

$$\left\{ \begin{array}{lll} \partial_t w_n = \Delta w_n + k_n w_n & \text{on} & \Omega\times[r_n-R_n,+\infty) \\ \partial_n w_n = 0 & \text{on} & (\partial\Omega)\times[r_n-R_n,+\infty) \end{array} \right| \quad \text{with} \quad k_n(x,t) := -\left\{ \frac{V'(u_n+\phi_0) - V'(u_\omega^{\tau_n}+\phi_0)}{u_n - u_\omega^{\tau_n}} \right\}.$$

Step 3.3: passing to the limit
We have

$$k_n \to k_\infty = -V''(u_\omega^{\tau_*}+\phi_0) \quad \text{in} \quad L^\infty_{loc}(\overline{\Omega}\times\mathbb{R})$$

Moreover standard parabolic estimates show that w_n is bounded locally uniformly in $W^{2,1}_p$ for any $p>2$. Hence we get $w_n \to w_\infty$ in $L^\infty_{loc}(\overline{\Omega}\times\mathbb{R})$, where w_∞ solves the linearized problem

$$\text{(6.50)} \qquad \left\{ \begin{array}{lll} \partial_t w_\infty = \Delta w_\infty + k_\infty w_\infty & \text{on} & \Omega\times\mathbb{R}, \\ \partial_n w_\infty = 0 & \text{on} & (\partial\Omega)\times\mathbb{R}. \end{array} \right.$$

Notice that the following function $w_0 := \partial_t u_\omega^{\tau_*} \ge c_1 > 0$ is solution of (6.50). Now (6.48) and (6.49) give at the limit with $r_n \to r_\infty \in [0,\omega^{-1}]$

$$\text{(6.51)} \qquad 1 = \inf_{\lambda\in\mathbb{R}} \left\{ \sup_{\overline{\Omega}\times[r_\infty-1,r_\infty+1]} |w_\infty - \lambda w_0| \right\} = \sup_{\overline{\Omega}\times[r_\infty-1,r_\infty+1]} |w_\infty|$$

and

$$\text{(6.52)} \qquad \inf_{\lambda\in\mathbb{R}} \left\{ \sup_{\overline{\Omega}\times[t_0-1,t_0+1]} |w_\infty - \lambda w_0| \right\} \le 1 \quad \text{for all} \quad t_0\in\mathbb{R}.$$

Step 3.4: uniform bound on w_∞
Notice that we have

$$\text{(6.53)} \qquad 0 < c_1 \le w_0 \le c_2$$

where c_1 comes from the bound from below (6.43) on $\partial_t u_\omega$. From (6.52), we deduce that for any $t_0\in\mathbb{R}$, there exists some $\lambda_{t_0}\in\mathbb{R}$ such that

$$\text{(6.54)} \qquad \sup_{\overline{\Omega}\times[t_0-1,t_0+1]} |w_\infty - \lambda_{t_0} w_0| \le 1 \le c_1^{-1} w_0(\cdot,t_0)$$

and then

$$\text{(6.55)} \qquad \left\{\lambda_{t_0} - c_1^{-1}\right\} w_0(\cdot,t_0) \le w_\infty(\cdot,t_0) \le \left\{\lambda_{t_0} + c_1^{-1}\right\} w_0(\cdot,t_0).$$

Then the comparison principle for PDE (6.50) implies

$$\text{(6.56)} \qquad \left\{\lambda_{t_0} - c_1^{-1}\right\} w_0(\cdot,t) \le w_\infty(\cdot,t) \le \left\{\lambda_{t_0} + c_1^{-1}\right\} w_0(\cdot,t) \quad \text{for all} \quad t\ge t_0.$$

Because (6.55) also holds for t, we get

$$\left\{\lambda_t - c_1^{-1}\right\} w_0(\cdot,t) \le w_\infty(\cdot,t) \le \left\{\lambda_t + c_1^{-1}\right\} w_0(\cdot,t).$$

Using (6.56), and dividing by $w_0(\cdot,t)$, we deduce $|\lambda_t - \lambda_{t_0}| \le 2c_1^{-1}$. Moreover relation (6.51) shows that we can take $\lambda_{|t=r_\infty} = 0$, which gives $|\lambda_{t_0}| \le 2c_1^{-1}$. Therefore (6.54) implies the following bound

$$|w_\infty|_{L^\infty(\overline{\Omega}\times\mathbb{R})} \le 1 + 2c_2c_1^{-1}$$

Step 3.5: touching w_∞ from above
Recall that $w_0 \ge c_1 > 0$ and define

$$\lambda_* := \inf\left\{\lambda\in\mathbb{R}, \quad w_\infty \le \lambda' w_0 \quad \text{for all} \quad \lambda'\ge\lambda\right\}$$

Then $\lambda_* \in \mathbb{R}$. Moreover, there exists a sequence of points $(x_\varepsilon, t_\varepsilon) \in \overline{\Omega} \times \mathbb{R}$ such that

$$(w_\infty - \{\lambda_* - \varepsilon\} w_0)(x_\varepsilon, t_\varepsilon) > 0 \quad \text{with} \quad w_\infty \le \lambda_* w_0$$

We write $t_\varepsilon = m_\varepsilon \omega^{-1} + r_\varepsilon$ with $m_\varepsilon \in \mathbb{Z}$, $r_\varepsilon \in [0, \omega^{-1})$, and set

$$w_{\infty,\varepsilon}(\cdot,\cdot) := w_\infty(\cdot, \cdot + m_\varepsilon \omega^{-1}) \quad \text{and} \quad w_0(\cdot, \cdot + m_\varepsilon \omega^{-1}) \equiv w_0,$$

where we have used the periodicity of w_0. Up to extract a convergent subsequence, we get $(x_\varepsilon, r_\varepsilon) \to (x_0, r_0) \in \overline{\Omega} \times [0, \omega^{-1}]$, and

$$\left\{ \begin{array}{ll} w_{\infty,\varepsilon} \to w_{\infty,0} & \text{locally uniformly on } \overline{\Omega} \times \mathbb{R}, \text{ with } w_{\infty,0} \text{ solution of (6.50)}, \\ w_{\infty,0} \le \lambda_* w_0 & \text{with equality at } (x_0, r_0) \end{array} \right.$$

Again the strong comparison principle implies $w_{\infty,0} = \lambda_* w_0$. We have moreover $m_\varepsilon \to m_0 \in \mathbb{Z} \cup \{-\infty, +\infty\}$, and we distinguish three cases.

Case A: $m_0 \in \mathbb{Z}$.
This implies that $w_{\infty,0}(\cdot,\cdot) = w_\infty(\cdot, \cdot + m_0 \omega^{-1}) = \lambda_* w_0$, and the periodicity of w_0 implies $w_\infty = \lambda_* w_0$.

Case B: $m_0 = -\infty$.
Then $w_\infty(\cdot, \cdot + m_\varepsilon \omega^{-1}) = w_{\infty,\varepsilon} \to \lambda_* w_0$ locally uniformly. Hence for any $\delta > 0$, we can find ε_δ small enough such that for all $\varepsilon \in (0, \varepsilon_\delta)$ we have

$$\{\lambda_* - \delta\} w_0(\cdot, 0) \le w_\infty(\cdot, m_\varepsilon \omega^{-1}) \le \{\lambda_* + \delta\} w_0(\cdot, 0)$$

The comparison principle implies (using also the periodicity of w_0) that

$$\{\lambda_* - \delta\} w_0 \le w_\infty \le \{\lambda_* + \delta\} w_0 \quad \text{on} \quad [m_\varepsilon \omega^{-1}, +\infty)$$

In the limit $\varepsilon \to 0$ and then $\delta \to 0$, we recover $w_\infty = \lambda_* w_0$.

Case C: $m_0 = +\infty$ **(the only remaining possibility)**.
This means that for all $t_0 \in \mathbb{R}$, we have

$$0 \ge \sup_{\overline{\Omega} \times (-\infty, t_0]} (w_\infty - \lambda_* w_0) = -\eta(t_0) < 0 \quad \text{with} \quad \eta(t_0) \to 0 \quad \text{as} \quad t_0 \to +\infty$$

Then fix $t_0 \le 0$. We get with $\eta_0 := \eta(0)$

$$(w_\infty - \lambda_* w_0)(\cdot, t_0) \le -\eta_0 < 0$$

Hence (6.53) implies $(w_\infty - \lambda_* w_0)(\cdot, t_0) \le -\eta_0 c_2^{-1} w_0(\cdot, t_0)$, i.e.

$$w_\infty(\cdot, t_0) \le \left\{\lambda_* - \eta_0 c_2^{-1}\right\} w_0(\cdot, t_0)$$

and the comparison principle implies

$$w_\infty \le \left\{\lambda_* - \eta_0 c_2^{-1}\right\} w_0 \quad \text{on} \quad \overline{\Omega} \times [t_0, +\infty)$$

In the limit $t_0 \to -\infty$, we recover

$$w_\infty \le \left\{\lambda_* - \eta_0 c_2^{-1}\right\} w_0 \quad \text{on} \quad \overline{\Omega} \times \mathbb{R}$$

This leads to a contradiction with the definition of λ_*, and then rules out Case C.

Conclusion
We deduce that in all cases, we get $w_\infty = \lambda_* w_0$, which is in contradiction with (6.51). Therefore we conclude that (6.45) is true.

Step 3.6: consequences of (6.45)
From (6.45), we deduce the existence of some $\rho > 0$ and a constant $C > 0$ such that

$$M(t) \le C e^{-\rho t} \quad \text{for all} \quad t \ge 1$$

This implies for all $t_0 \ge 1$ that there exists some $\tau_{t_0} \in \mathbb{R}$ such that

$$\sup_{\overline{\Omega} \times [t_0-1, t_0+1]} |u - u_\omega^{\tau_{t_0}}| = \inf_{\tau \in \mathbb{R}} \left\{ \sup_{\overline{\Omega} \times [t_0-1, t_0+1]} |u - u_\omega^{\tau}| \right\} = N(t_0) \le C e^{-\rho t_0} \tag{6.57}$$

Hence (6.53) implies

$$c_1|\tau_{(t_0+1)} - \tau_{t_0}| \leq \sup_{\overline{\Omega}\times[t_0,t_0+1]} |u_\omega^{\tau_{(t_0+1)}} - u_\omega^{\tau_{t_0}}| \leq N(t_0) + N(t_0+1) \leq 2Ce^{-\rho t_0}$$

We also know that $\tau_{t_0} \to \tau_*$ as $t_0 \to +\infty$. This gives by a telescopic/geometric sum

$$c_1|\tau_* - \tau_{t_0}| \leq \frac{2C}{1-e^{-\rho}} e^{-\rho t_0}$$

Again (6.53) implies with (6.57) that for all $t_0 \geq 1$

$$\begin{aligned}\sup_{\overline{\Omega}\times[t_0-1,t_0+1]} |u - u_\omega^{\tau_*}| &\leq \sup_{\overline{\Omega}\times[t_0-1,t_0+1]} |u - u_\omega^{\tau_{t_0}}| + \sup_{\overline{\Omega}\times[t_0-1,t_0+1]} |u_\omega^{\tau_{t_0}} - u_\omega^{\tau_*}| \\ &\leq Ce^{-\rho t_0} + c_2|\tau_* - \tau_{t_0}| \\ &\leq Ce^{-\rho t_0}\left\{1 + \frac{c_2}{c_1}\cdot\frac{2}{1-e^{-\rho}}\right\}\end{aligned}$$

which implies (1.8) and ends the proof of the theorem. □

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